EULER'S THEOREM AS THE PATH TOWARDS GEOMETRY

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1. Introduction

The roots of this article reside as much in curricular pragmatism as in programmatic, ideologic convictions. By curricular pragmatism we mean the anankian[1] drive for one single course in Mathematics for Architecture Students at the Technion – Israel Institute of Technology, course that would encapsulate as much formative knowledge as deemed feasible. The need for an "Object Oriented", fast approach, compact course, arises from curricular constraints: once the second half of a tandem of three-hours per week, semestrial courses (the first one comprised elements of Matrix Algebra and Introduction to Calculus), it was reduced to a single semestrial course, of two weekly-hours (the Algebra-Calculus half being abandoned completely). Moreover, this reduction in scope was accompanied by an augmentation of the Syllabus: while the previous geometric course comprised only symmetry (albeit treated in some detail[2]), the new Course was envisioned as a comprehensive introduction to incidence and symmetry of geometrical objects, an approach that is commonly hold to represents the corner-stone, the main goal of a Course of this type. (The chosen basic reference text being [Baglivo and Graver, 1983]). Thus, apart and beyond the absolute importance of Euler's Formula relative to the corpus of classical mathematics and the role it played in its development (Betti numbers[3] and Homology in general, on one hand and the Global Gauss-Bonnet Theorem[4] on the other hand being, not the least of its outshoots), its simplicity, yet potency (in the sense of representing an jumping board, an opening towards a variety of subjects belonging to the fields of Topology and Geometry) recommend Euler's Theorem as natural candidate for a cornerstone, a red-thread running along and directing the whole Course.

Since a second purpose of any such course is to help develop geometric intuition and spatial imaginative powers, Euler's Theorem introduces one effortlessly in the realms of geometric creativity, by its natural generalizations into two directions: (a) high genus and non orientable surfaces s.a. the Klein Bottle and The Projective Plane (thus representing an *excursum* in Non-Euclidian geometries and also into patterns and tilings), and (b) Star and Uniform Polyhedra. (We shall indicate how and where these objects and ideas arise.)

Moreover, this approach allows for a perfect integration between the metric and topological aspects, thus presenting the dialectics of Mathematics, represented by the computational-arithmetic/figurative-geomet-
ric dichotomy.[5]

We shall show in what manner the path outlined herein facilitates a potent yet flexible Curriculum: one can put more accent on Graph Theory or, alternatively, on "rigid" geometry, on incidence and topology or rather upon trigonometric computations. Through Graphs one attains such applications as: organization graph of architectural structures (introducing duality and planarity), traversability, routing, connectedness, classification of architectural plans. Even the very approach to the proofs shines light on different aspects of Geometry.

Upon these manifold proofs and the different light they shine upon the kaleidoscopic aspects of Geometry, we shall dwell (alas, briefly!) in the following section.

2. The Mathematics

The five Platonic solids, their role in the development of Geometry and Philosophical ideas of classical Greece (see [Crowe], p. 2) as well as and the inevitable Keplerian cosmogonical vision[6] with its sense of wonder (see [Crowe], p. 4) provide the best introduction into the realm of Polyhedra. Therefore the proof of the existence of only five Platonic Solids provides an excellent motivation for introducing Euler's Theorem.[7]

Our proof of choice, the first one to present our students, is basically Euler's original one, based upon first triangulating the polyhedron, and then one by one removing the resulting triangles, while showing that the number $V - E + F$ remains invariant during this process. As a special didactic trick, we present Gamow's variant of this proof, (see [Gamow, 1988])[8]. In Gamow's presentation the polyhedron's edges are viewed as dams, the polyhedron as an isle, and the exterior of the resulting map as the sea. As a special flourish, we present this as a last defense against a Spanish Armada invading the Netherlands, sometimes in a mythical 16-th century (see Figure 1). The advantage of this proof resides in it simplicity and in the fact that it introduces the extremely important notion of triangulation. But it provides us with even more depth: since an essential step in the proof is the removal of the upper face f_0 (see Figure 1) and the projection of the reminder of the polyhedron on the plane[8], this allows us to discuss the Stereographic Projection. (Considering projections can also lead to Steiner's proof of Euler's Theorem[9] – see [Sommerville, 1942, p. 142]). Moreover, faces as homeomorphic images of the disk and the topological concept of map (as opposed to that of a mere graph embedding) come under study naturally. From this point on, the possibilities are practically unlimited. The most direct road leads to counting the regular tilings of the plane and to that of Archimedian solids and semi-regular tilings of the plane.

But the main advantage lies, first and foremost, in the fact that one can introduce the notion of graph in a light, natural manner, and with them a variety of problems of great significance in the theoretical setting and of vast practical importance, such as planarity and duality (via "the 5 brothers problem" (see [Bagivo and Gramer, 1988]) and the "gas-light-water graph" (see [Baglivo and Graver, 1988], [Tietze, 1965])) and thus to the Theorems of Kuratowsky and Whitney (see [Baglivo and Graver, 1988], [Tietze, 1965]). Trees are the simplest truly interesting graphs and they provide us with a third Proof of Euler's Theorem: Von Staudt's proof[10]

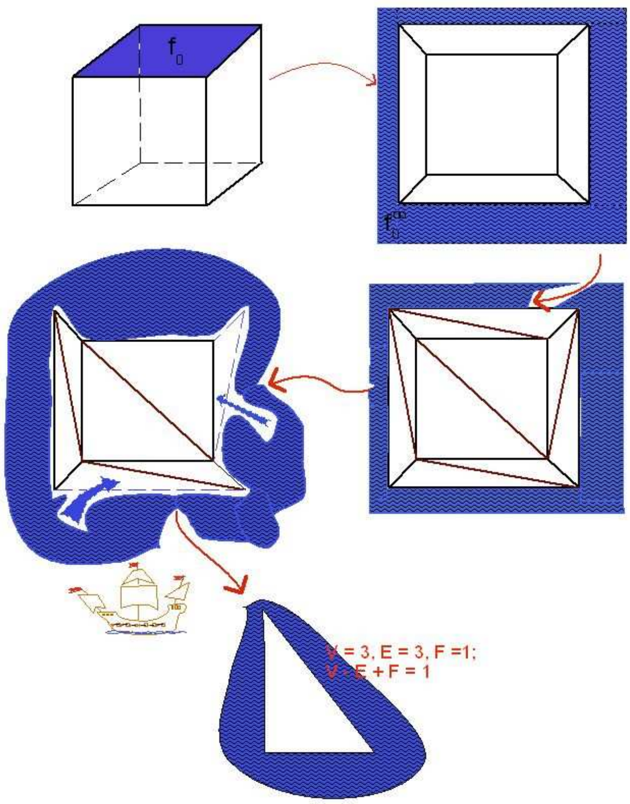

FIGURE 1. The First Proof

based upon spanning trees. Paradoxically, the benefit of this proof is that it does **not** extend to surfaces of positive genus – thus allowing yet a different insight into the topology of surfaces. Applications of Euler's Theorem, such as Kuratowski's Theorem are now natural – through the inequality $E < 3V-6$ ("The little inequality that could"). Its dual inequality $E < 3F - 6$ provides us with an easy proof of the Six Color Theorem (and with an excuse to wander into a discussion of the Four Color Theorem).

Planarity and maps conduct us immediately to consider other surfaces than the plane or the sphere, such as the torus and compact, orientable surfaces of higher genus, and also non-orientable surfaces, in particular the Möbius strip, the Klein bottle and the projective plane (see Figure 2). Also, as an immediate – yet somewhat collateral – development, maximal planar graphs and Fundamental Architectural Arrangements (see [Baglivo and Graver, 1983], pp. 115-119) ensue.

Yet another Proof is needed if one considers further generalizations, such as Star Polyhedra and the even more general Uniform Polyhedra[11] (see Figure 3).

For these one has to appeal to the Spherical Area proof (or Legendre's Proof) (see [Coxeter, 1963]).

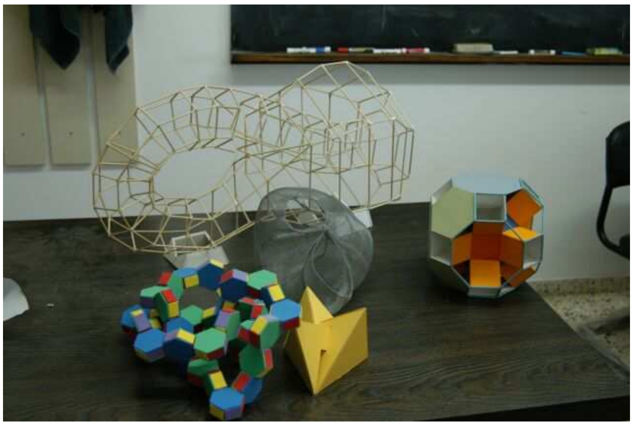

FIGURE 2. Topological Models

FIGURE 3. Star and Uniform Polyhedra

While a formal treatment of surface coverings is, of course, out of the courses scope, its rudiments are highly intuitive. Moreover, a much more technical instrument becomes now tangible: a numerical solution to compute the elements (sides, dihedral angles, radii, etc.) of a Uniform Polyhedron. The method above – based upon a many variable adaptation of Newton's Tangent Method for solving equations – is the one developed in [Har'el, Zvi, 1993]. Even if in the beginning the students tend to oppose (at an instinctive, self-preserving level) the meanders of its computation, in the end, the concreteness and clarity of the numerical results obtained rewards them with a better insight and understanding of the objects they studied until this point only on a descriptive, visual way. This is an intrinsic result sprouting from the very depths of mathematical thought and understanding (but we shall resume this discussion again in the last Section).[13]

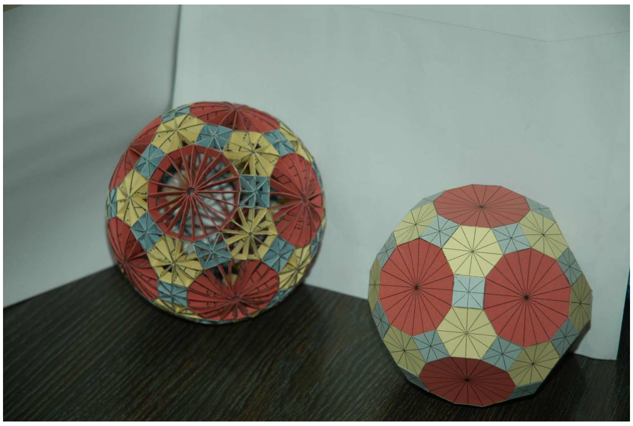

FIGURE 4. An Archimedian Solid and its Spherical Counterpart

3. Didactical Aspects

At this stage is only natural that one should try and estimate the Didactical/Methodolgical benefits of this Course structure. We do believe this approach unifies the different aspects/parts of the course, giving a coherent perception that expunges false compartmentalization (sadly so often permeating Mathematics Curricula and, indeed, Education). Following the natural road opened by Euler's Theorem allows the student to recognize Geometry as the "*Art of posing questions*" ([Gromov, 1998]) – as opposed to the mere ability to solve standard, technical exercises, the revered torturer of our high-school days – and to view it not like a perpetual unpalatable compromise between dual, mutually exclusive entities: Figurative Expression and Computational Technique, but rather to perceive its dialectic nature, as the interplay between two aspects, two functions of the mind, since "*Our brain has two halves: one is responsible for the multiplication of polynomials and languages, and the other half is responsible for the orientation of figures in space and all the things important in real life. Mathematics is Geometry when you have to use both halves.*" [Arnold, 1997].
Moreover, the focus on Polyhedra facilitates the use of Constructive Projects as Marking Tool, thus addressing the demiurgical skills of the students, and adding yet another unifying, summarizing tool at the Course end. Indeed, Architecture students have not only the ability and the habit to express themselves in a constructive manner, they do posses propensity, the proclivity for this Renaissance type of expression: material and spiritual, combined (see Figure 4). Also, it is this author's firm belief that "*People are much smarter when they can use their full intellect*" ([Thuston, 1990]). Even more: creative freedom and trust are stronger moving forces and better guaranties of novel, original ideas (for the final projects) then some constrictive exam frame. Trigonometric equations may be tedious and boring, but the become **your** equations when they help finalize a project. (See, for example the two Origami polyhedra of Figure 5).

FIGURE 5. A Cornucopia of Models

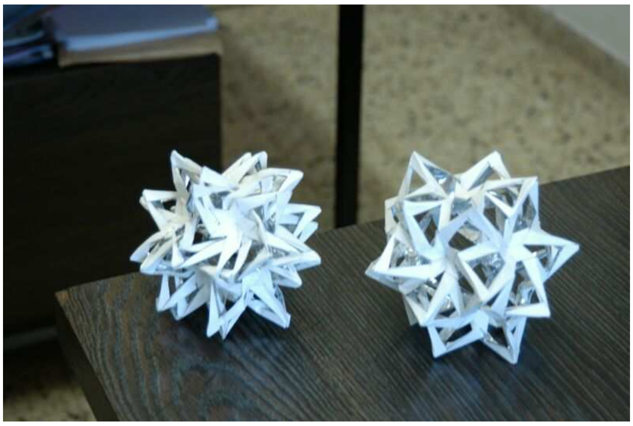

FIGURE 6. Two Origami polyhedra

4. Beyond Euler's Theorem

When the matter is so vast and so generous and the audience is so artistically inclined yet technically able, like in our context, one should not restrict himself to the mere Mathematical and Didactical considerations, but should rather ask himself what is the deepest possible impact of his Course, and what is the intellectual message it should convey.

It is stated in [Consiglieri and Consiglieri, 2003] that "*...mathematics does not lead to emotional forms but abstract ones; that responsibility belongs to aesthetics*". This statement may conform to the common feeling of practicing artists. Nevertheless, it contradicts the very cultural tradition that resides at the inner core of the choice of **any** Course in Mathematics for Architects, the fact that "*Classical mathematics is a quest for structural harmony.*" ([Gromov, 1998]) As yet another of the Titans of Contemporary Mathematics confesses: "*Mathematics has a remarkable beauty, power and coherence, more than we could have ever expected*" ([Thurston],

1990) and "*Mathematics is like a flight of fancy* " ([Thurston], 1990). And again, in Freudenthal's words: "*... mathematics* is an interplay of *content* and *form* "[13]. Yet couldn't this serve as a concise, functional definition of Art?

And even if we restrict ourselves to a more mundane, practical level: if a Course offers his listeners a plethora of examples, a gallery of fantastic forms, to fertilize, to help germinate and to serve as nutrient for growth Pure Art, wouldn't it served his purpose? With this question that contains a hope, a belief in positive answer, we conclude our essay.

5. Acknowledgments

The author would like to express his gratitude to Prof. Tzvi Har'el for introducing him in the fascinating domain of Architectural Mathematics.

Many thanks are also due to Eli Appelboim for his friendly and sustained help and to the anonymous referee for his/hers kind comments and thoughtful suggestions.

6. Notes

[1] From "Ananke" – Greek goddess, personification of unalterable necessity, compulsion, or of the force of destiny. Also "anankastic syndrome" – perfectionistic traits expressed as meticulous conscientiousness, preoccupation with rules and procedures, rigidity of behaviour.

[2] In an even earlier incarnation it also comprised elements of Differential Geometry of Surfaces – see [Har'el, 1985].

[3] This notion requires some mathematical formalism: Let X be a topological space. The k-th Betti number β_k of X is defined as the rank of the k-th homology group of X: $\beta_k = \mathrm{rank} H_k$.

[4] Let S be a compact surface. Then $\iint_S K dA = 2\pi\chi(S)$, where K represents the Gauss curvature and $\chi(S)$ is the Euler characteristic of S.

[5] See [Arnold, 1997] and Section 3 below.

[6] Of nested cosmological regular polyhedra.

[7] The geometric, angle based proof, is also presented, but more like an afterthought, an instructive variation.

[8] Thus reducing the problem to proving that, for the new map $V - E + F = 1$.

[9] The basic idea of Steiner (and Lhuilier's) proof is to project the polyhedron orthogonally on a plane, obtaining a polygon covered twice by a set of polygons, then express the sum of the angles of these polygons in terms of V, E, F.

[10] The main steps of Von Staudt's proof are as follows: Build the spanning tree of the vertex set – the number of its edges will be $E_1 = V - 1$. Construct also

the dual tree (of the face set) – the number of edges will be $E_2 = F - 1$. Since the trees are disjoint we have: $E = E_1 + E_2 = V + F - 2$, i.e. $V - E + F = 2$.

[11] A proof that made this author to chose Mathematics as his profession – it, and Cantor's Diagonal Proof.

[12] And, of course, Regular and Uniform Polyhedra direct us to the study of Symmetry Groups.

[13] Indeed, mathematicians refer to their object of study (or should we say "passion"?!) in esthetic terms: "What a Beautiful Theorem!", a "lovely idea", a "nice", "beautiful" or even "elegant proof".

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