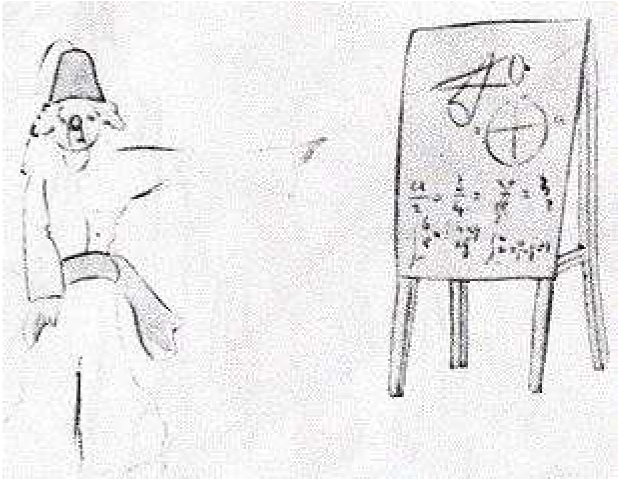


What is Curvature?

Emil Saucan

25.01.2006.

Our main goal: to try to answer the following basic question:

What is the curvature of a surface?

A few apologetic words are in order before we start:

- This is far from being an exhaustive presentation (indeed, this would be impossible!...)

- In consequence a fair amount of personal view (and taste) are involved but one still hopes the most relevant ideas are presented.

- The accent is placed upon the main geometric insight and historical development.

- No formal proofs are given and no full formula development is provided...

- ... so no complicated differentials are considered, so no discretization is described,...

- ... rather the interplay continuous-discrete is emphasized, as the natural way of describing and understanding mathematical phenomena.

We can now return to our question and begin answering by remarking that:

For *Parametrized* surfaces:

$$S = f(u, v); f : U = int(U) \subseteq \mathbf{R}^2 \to \mathbf{R}^3$$

the answer is easy:

$$K = -\frac{1}{4(EG - F^2)^2} \begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} -$$

$$-\frac{1}{\sqrt{EG - F^2}} \left(\frac{\partial}{\partial v} \frac{E_v - F_u}{\sqrt{EG - F^2}} - \frac{\partial}{\partial u} \frac{F_v - G_u}{\sqrt{EG - F^2}} \right)$$

where:

$$E = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial u},\ F = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v},\ G = \frac{\partial f}{\partial v} \cdot \frac{\partial f}{\partial v}$$

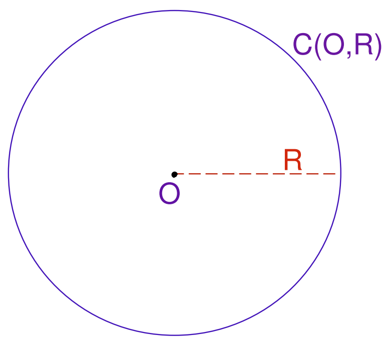
(Frobenius)

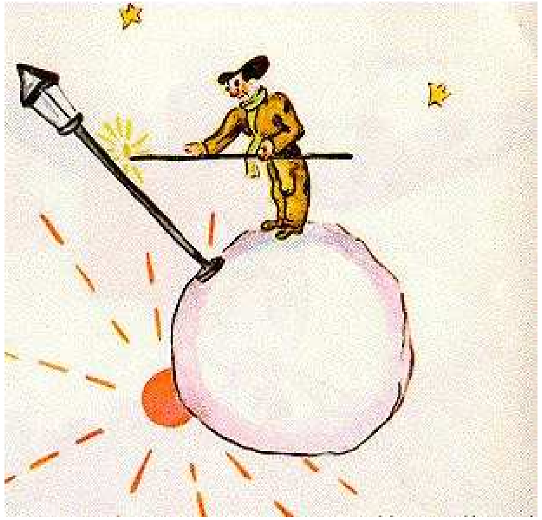
Oooops!... Let's start from the beginning!

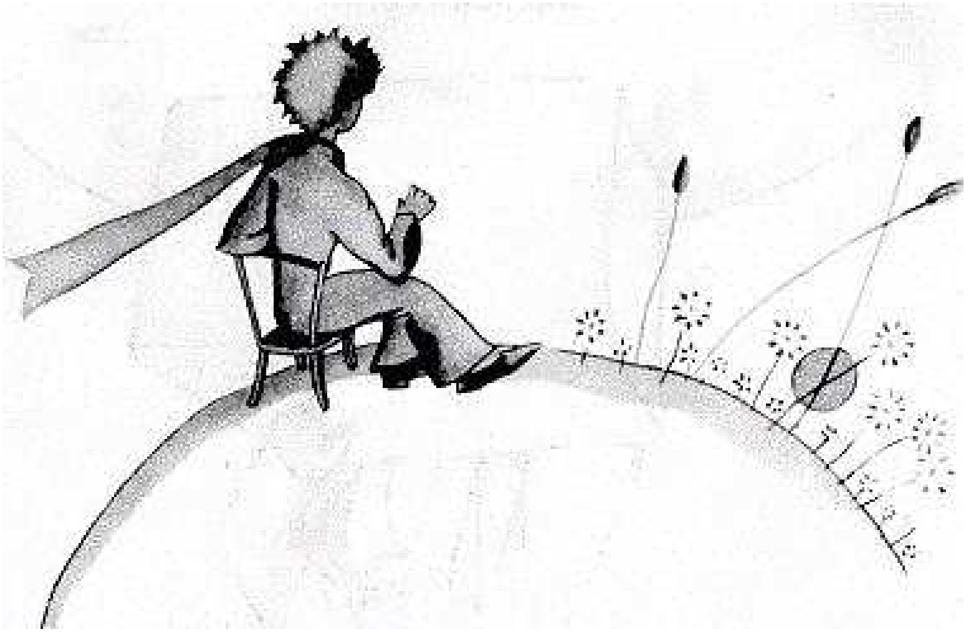
First define curvature for the simpler case of *planar curves*

A line "has no curvature" so define $K \equiv 0$.

The next simplest curve is the circle and here too an answer was known since the Ancient Greeks: $K \equiv \frac{1}{R}$







But, how to “fit” this to more general curves?

The answer is very simple (if you happen to be Newton!*...):

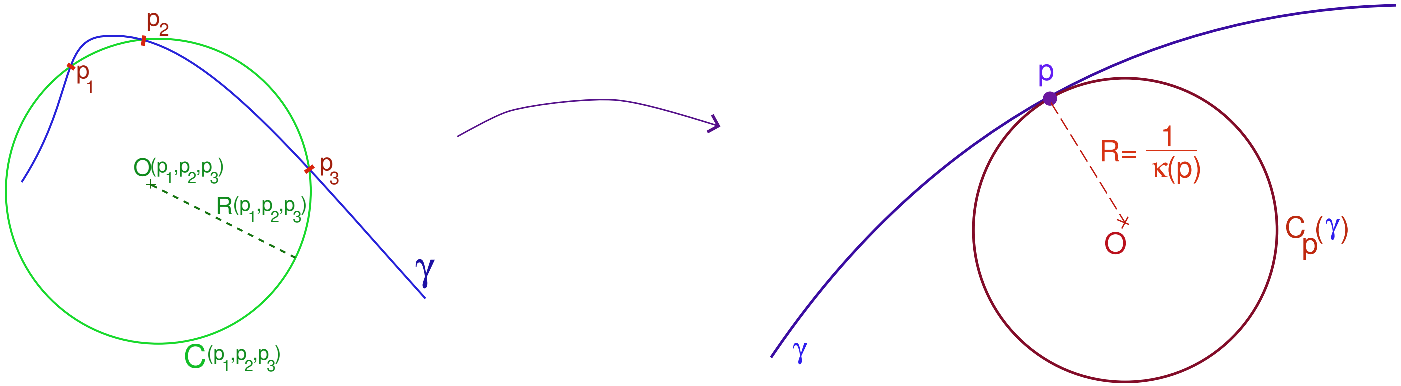
The curvature of the curve at the point p is the curvature of the best “fitting” circle to c at p.

But ... how to find it?

* and the year is 1665

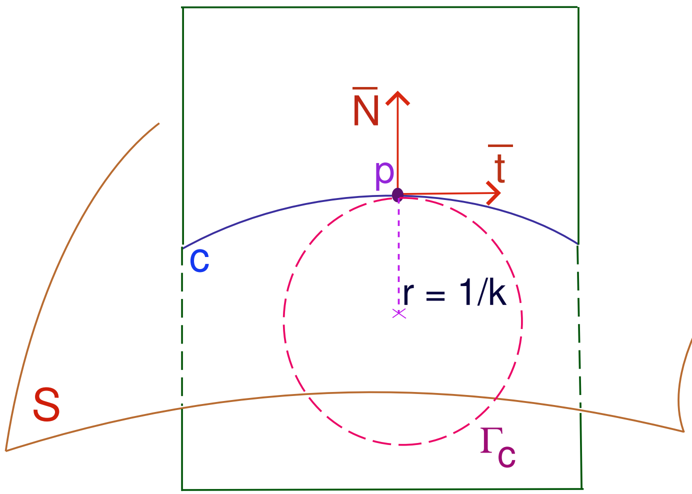
Simple: use the trick employed in defining the tangent to a general curve (i.e. define the tangent as the limit of secants).

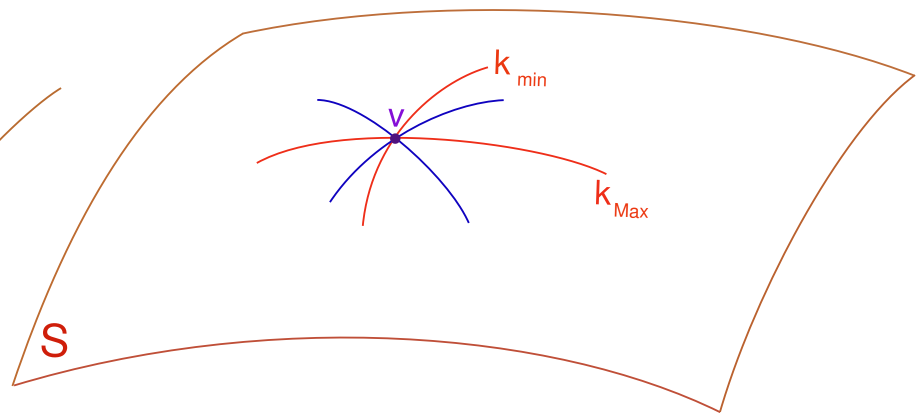
In this case: Define the **Osculatory Circle** as the limit of circles that have 3 common points with the curve:



But this idea, however nice, won't work for surfaces*, because there are far to many curves in too many directions to choose from!...

However, a few things can be told by analyzing curves through a given point on a surface. First, you start by restricting yourself to **Normal Sections**.





* and, anyhow, the first time the notion of **Osculatory sphere** was mentioned was in 1820 (by N. Fuss)

However, the results above are far from satisfactory, because:

- There are two many directions and even more curves.

- Even if you compute all curvatures, what should one choose?

- Indeed, do these **curves** curvatures represent in any way **the curvature of the surface**?

- And, more important, what **is** the curvature of a surface?!

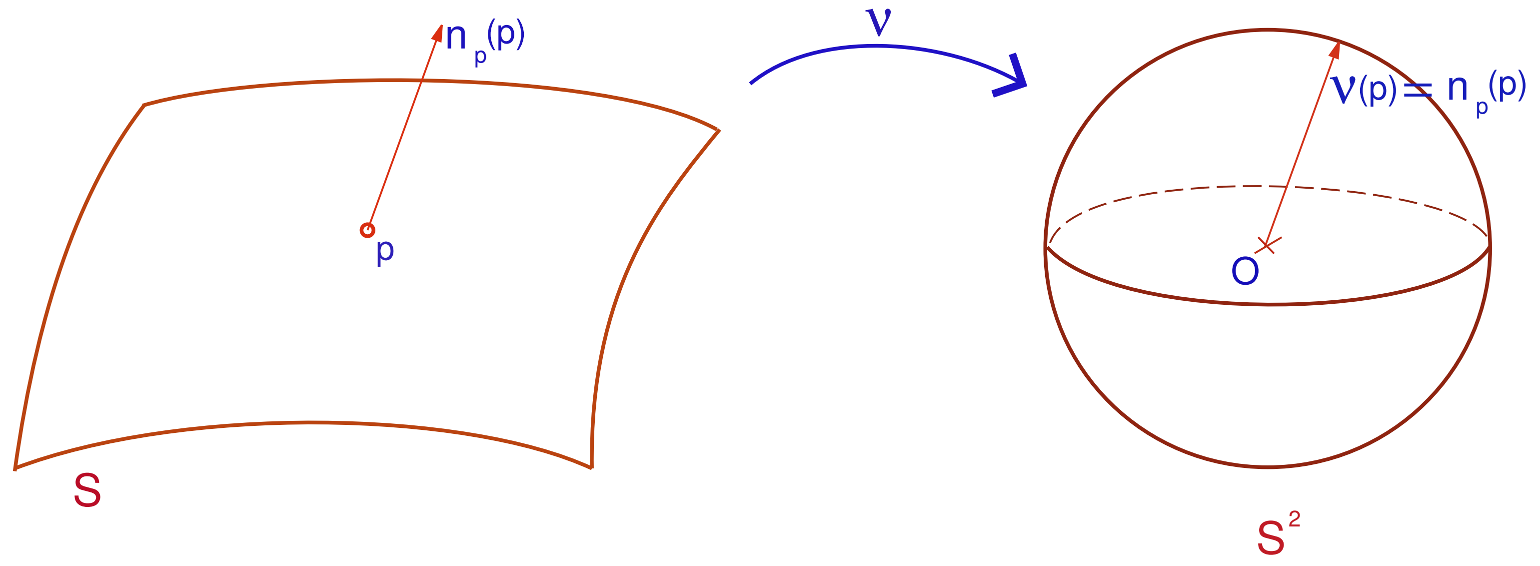
The answer – or rather *answers to all the important questions* – was given by Gauss, in 1827 (in a paper entitled: "Disquisitiones generales circa superficies curvas")

We shall not try and outsmart Gauss (it would the most meaningless, hubris-laden exercise in futility, anyhow!), so we shall step in his steps and start from the more basic, interesting (and fun!) question:

"What is the curvature of a surface (at a point)?!"

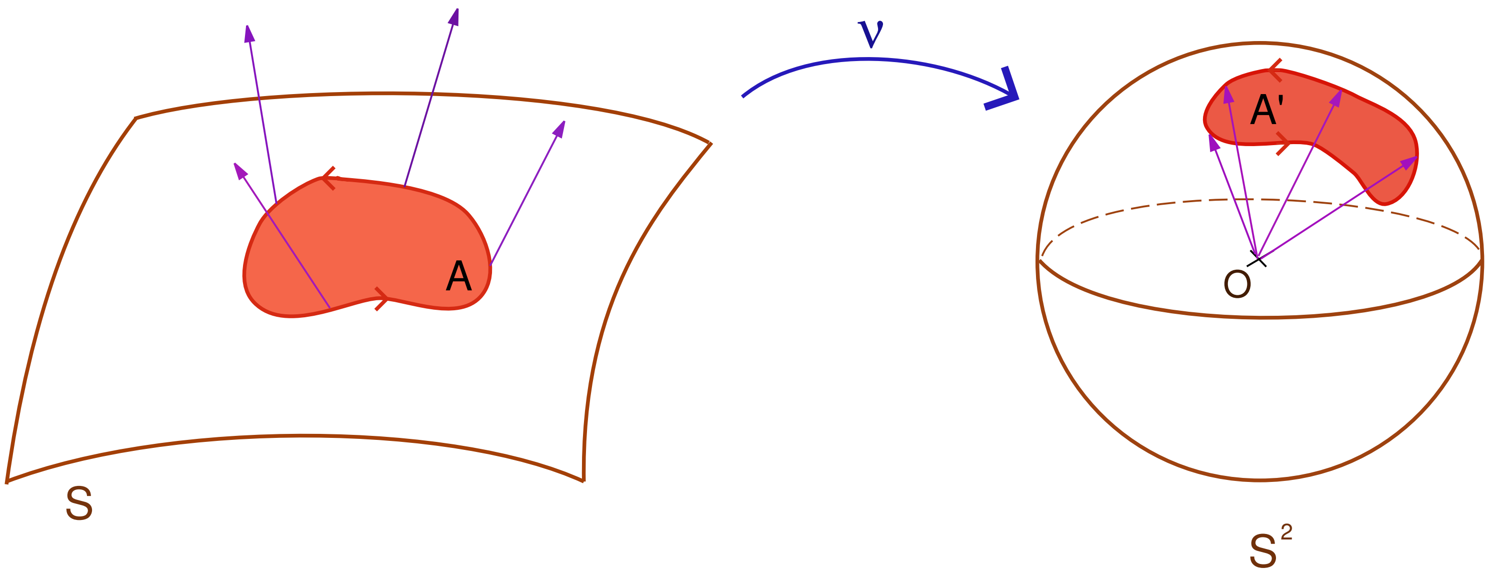
The idea is to define curvature as a measure of a surface from "being straight", or equivalently, a measure of how much a surface has to be bent in order to obtain a certain standard surface, i.e. the *unit sphere* $\mathbf{S}^2$.

Gauss achieved this by considering the *normal mapping* $\nu : S \to \mathbf{S}^2$.

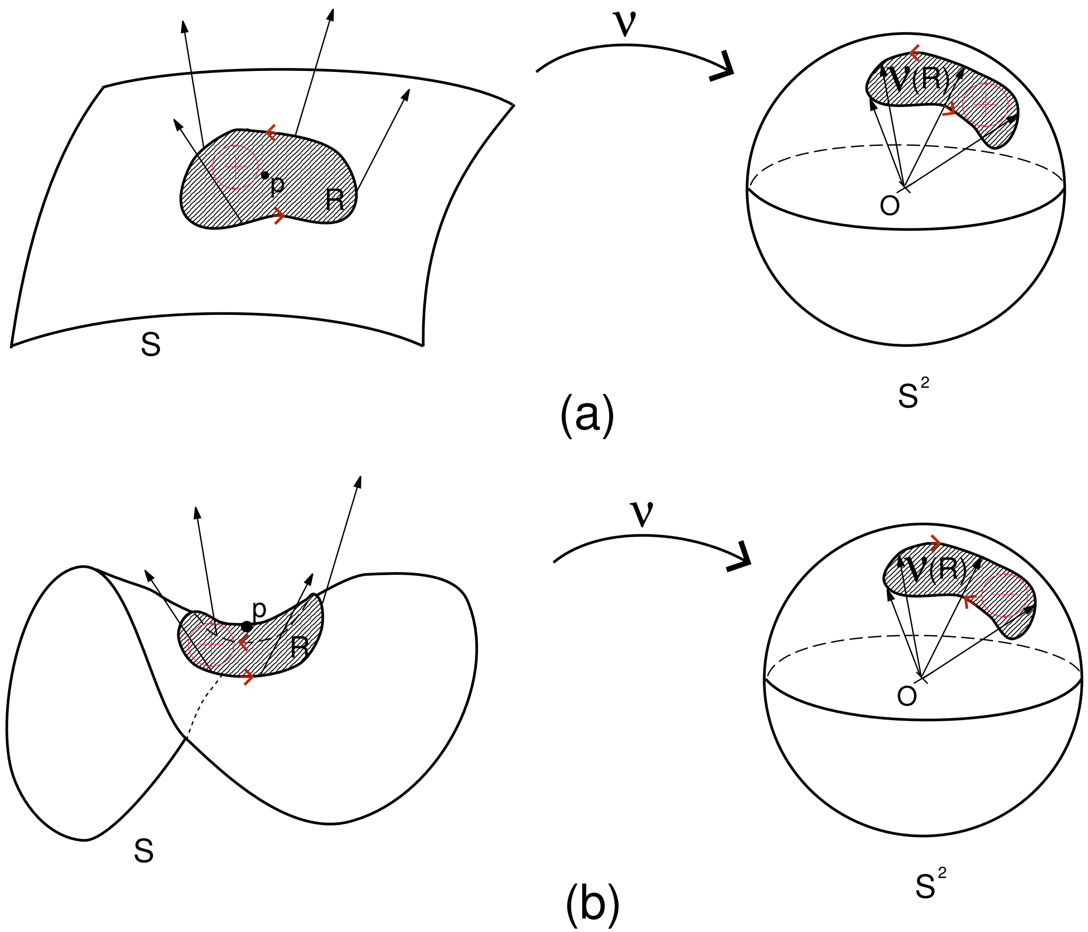


Then the *Gaussian Curvature* of S at p is defined as:

$$K_S(p) = \lim_{diam(R)\to 0} \frac{Area(\nu(R))}{Area(R)}$$



A sign is attached to $K(p)$ in a natural way (for a notion defined by a integral!...):



While not trivial to show that this limit exists, this formula provides us with a sound definition for surfaces' curvature, without employing **anything exterior** to the surface (s.a. normal planes), i.e this definition is *intrinsic* to the surface (no need to embed the surface in $\mathbb{R}^3$).

And indeed, it can be shown that this definition is really independent of the way the surface is embedded in $\mathbb{R}^3$: it invariant under *bendings* i.e. transformations that preserve *lengths* and *angles*. This fact represents

Gauss' Theorema Egregium ("Excellent Theorem")

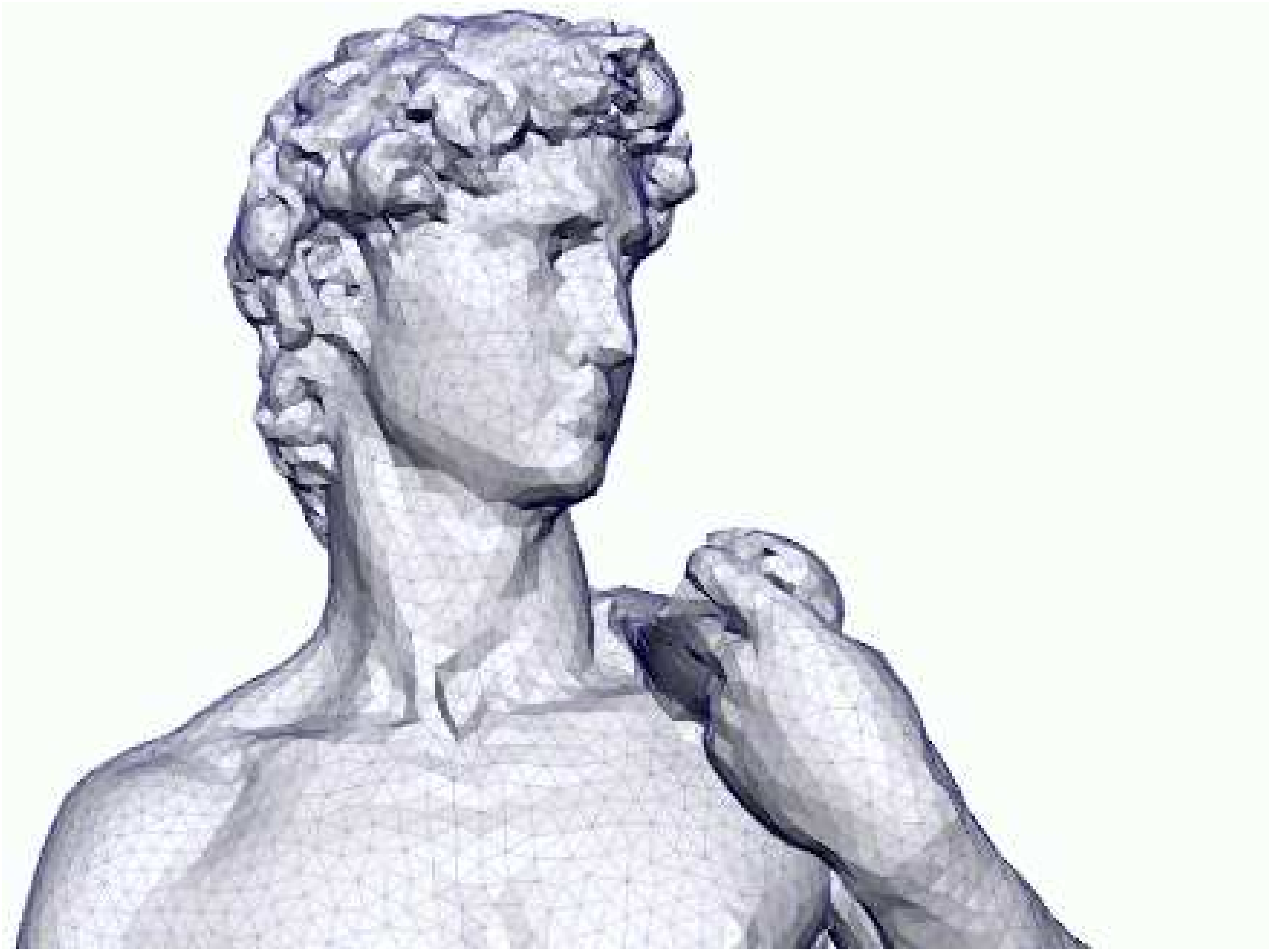
Gaussian curvature is invariant under bendings.

Moreover, by considering a special coordinate system in which $p = (0,0,0)$ and $T_p(S) = xOy$, and S as a graph $S = S(x,y) = \Big(x, y, f(x,y)\Big)$ Gauss also proved that:

$$\boxed{K(p) = k_{min} k_{Max}}$$

and this formula is unfortunately – since it is neither immediate nor natural – employed as the definition of the Gaussian curvature.

But what for the Geometry relevant to such fields as Computer Graphics, Image Processing, Computer Aided Geometric Design, Computerized Tomography, Bio-Geometric Modelling?



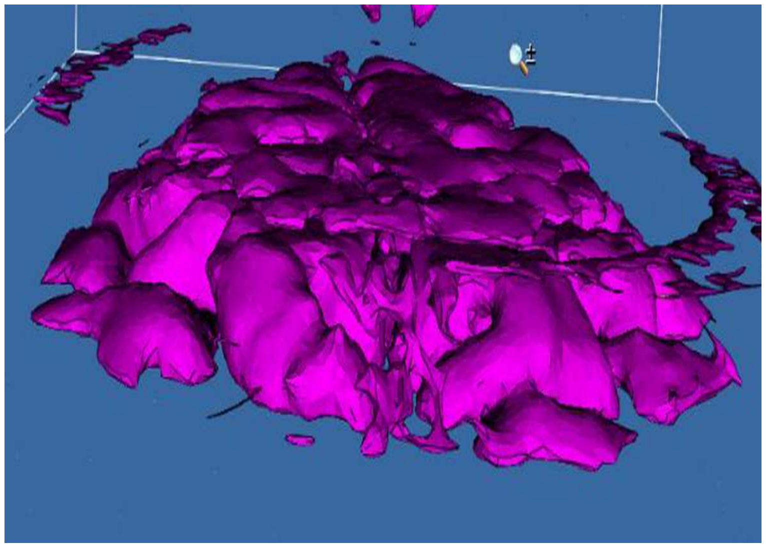
For triangulated (PL) surfaces, one could (following **Descartes** – and in more recent times **Hilber–Cohn-Vossen, Pólya, Banchoff,...**) define K at every vertex as the *defect* of the sum of angles surrounding it:

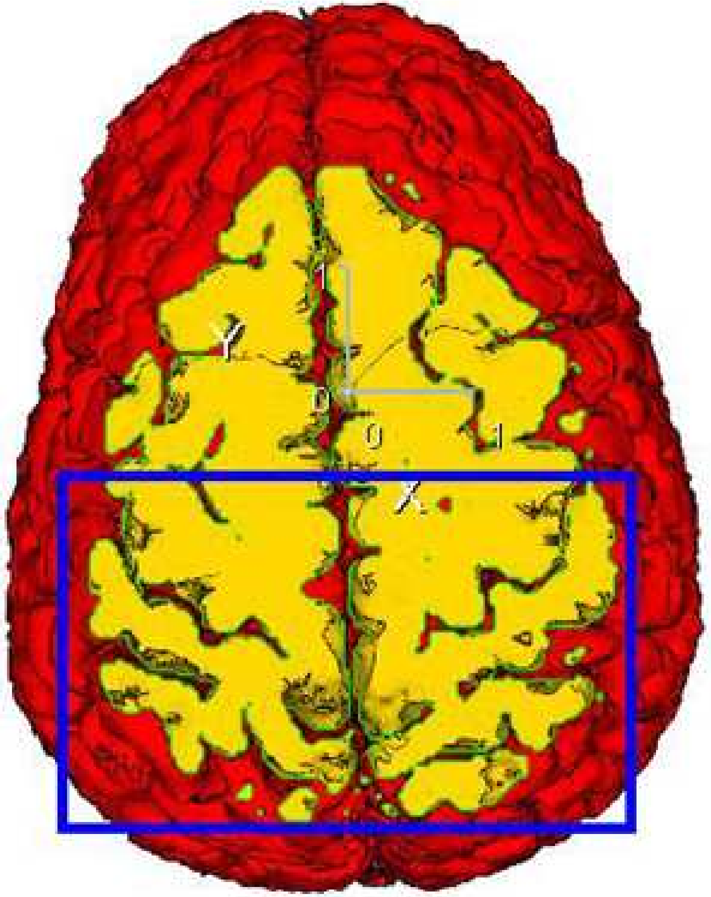
$$K(v) = \delta(v) = 2\pi - \sum_i \alpha_i \,.$$

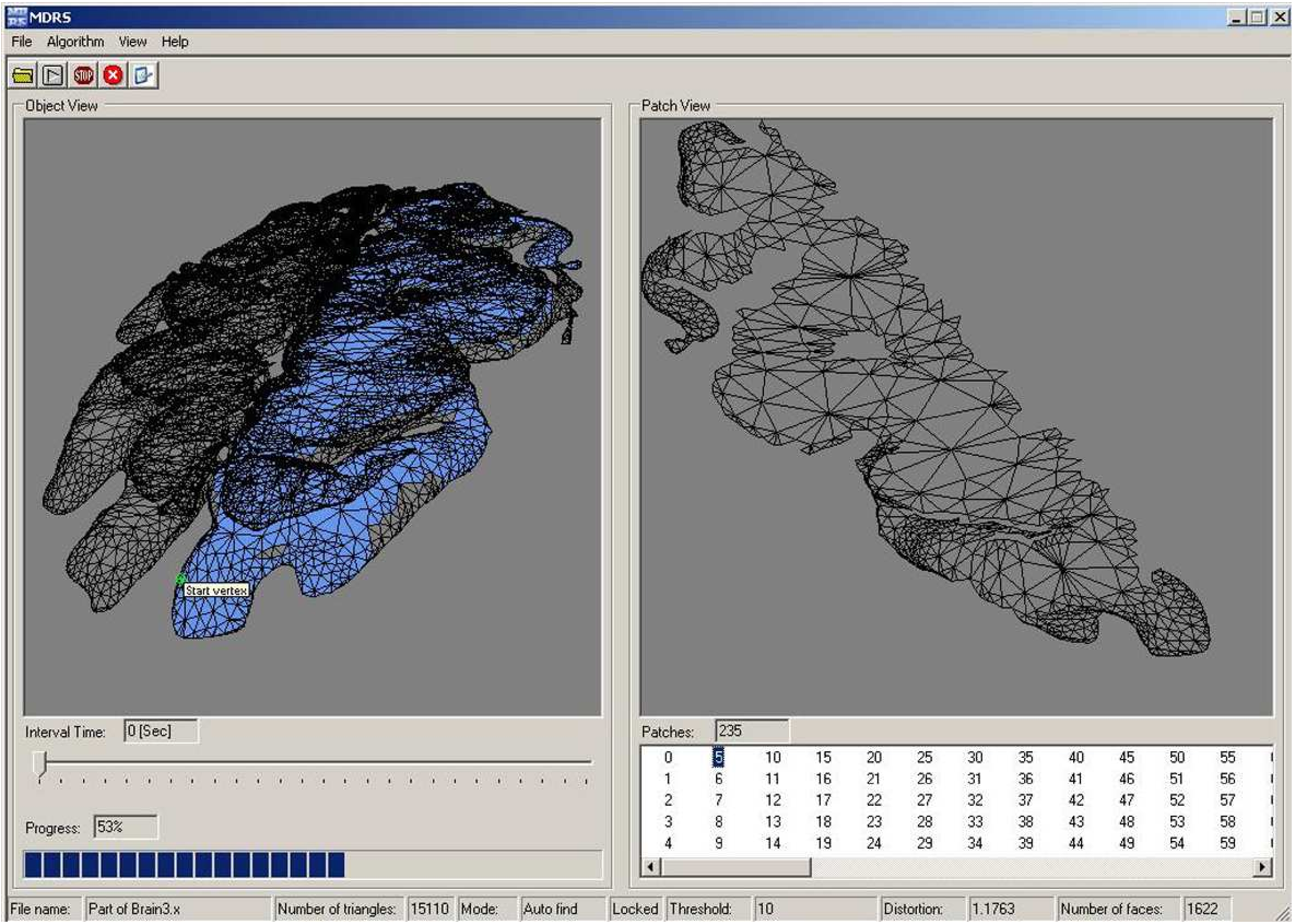
One (non mathematical) application* of this definition...

* "Not only can we mathematicians be useful, but we can create works of art at the same time, partly inspired by the ouside world" (Sir Michael Atiyah)

... is in **Medical Imaging**:



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Object View
Patch View
Start vertex
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4 9 14 19 24 29 34 39 44 49 54 59
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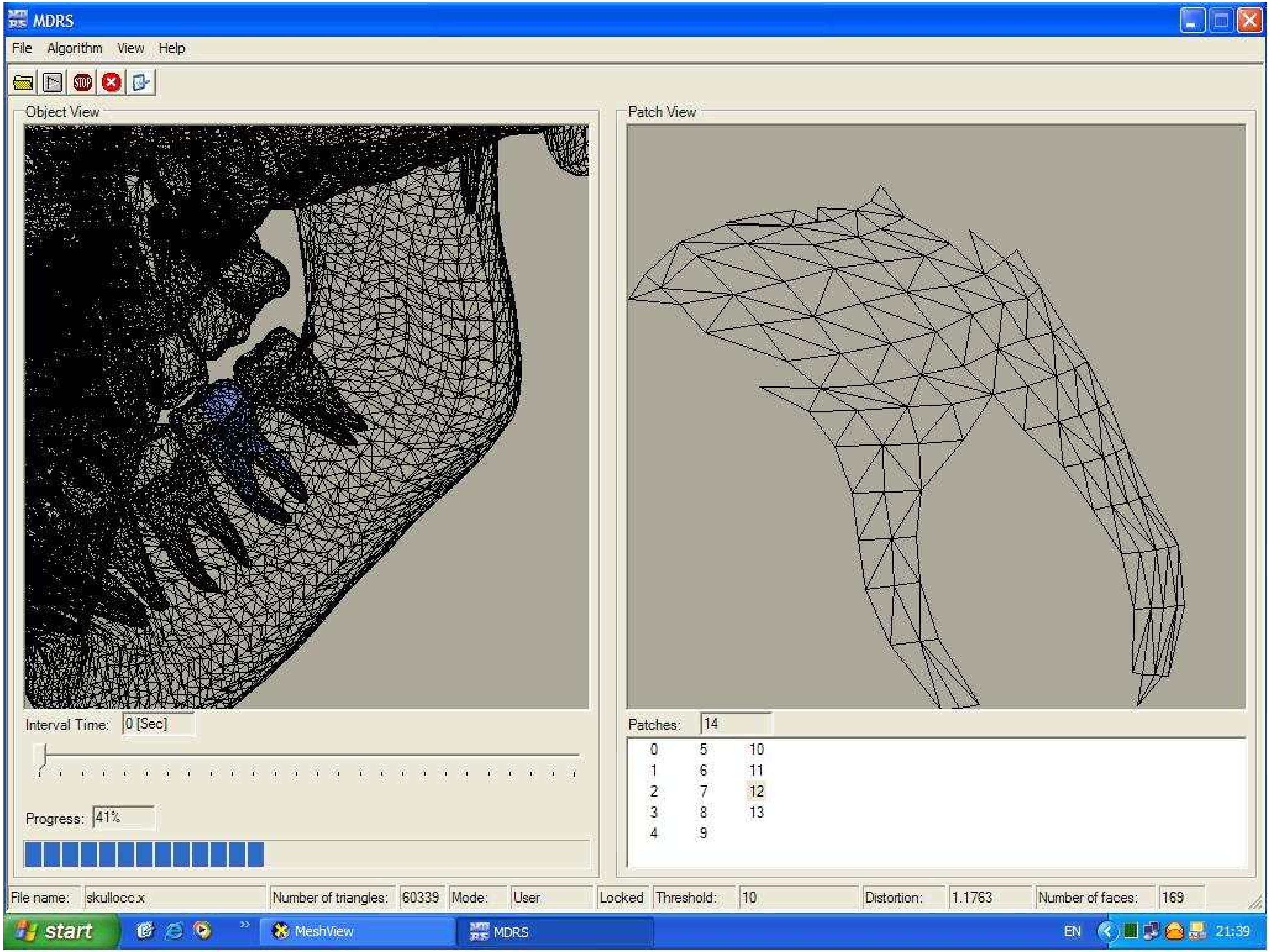
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2 7 12
3 8 13
4 9
File name: skullocc.x
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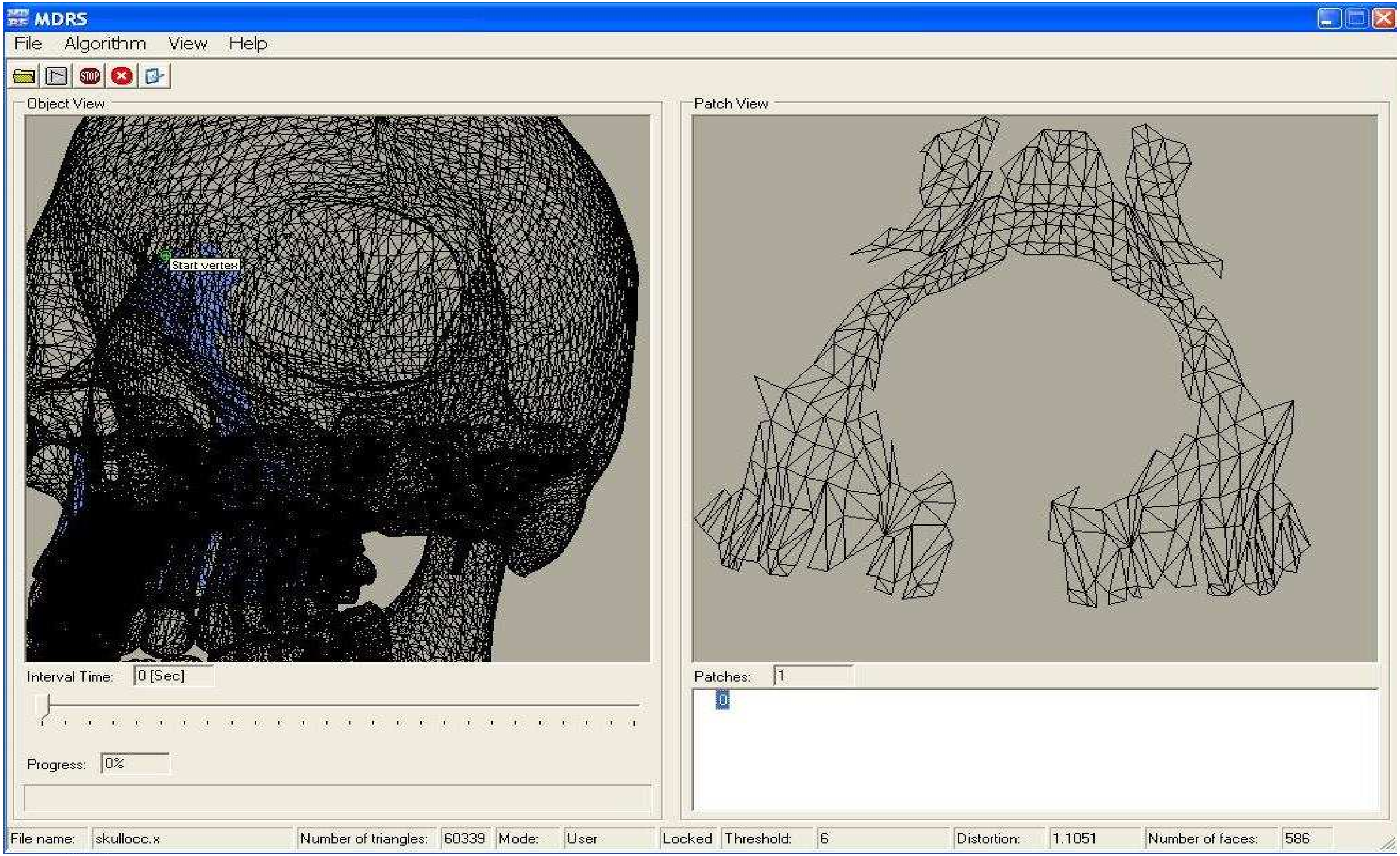
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User
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586

The following question arises naturally:

Can we define curvature without angles?

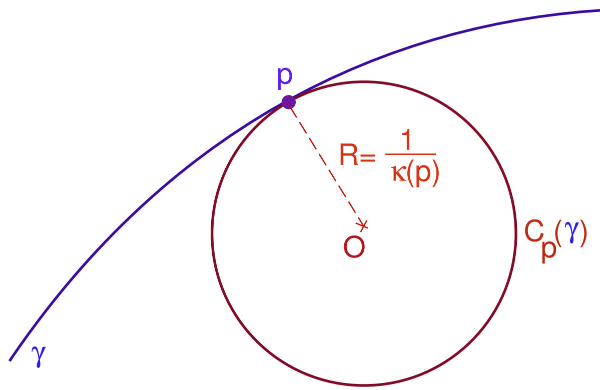
Even more, can we do this without embedding our space in $\mathbb{R}^n$?

That is, can we define a notion of curvature considering only distances, i.e. in (general) *metric spaces?*

First let's try and define *metric curvature* for *curves*:

The **first approach** is the most direct:

We shall mimic the *osculatory circle* in the metric context.



Remember that we have to do this **without** referring to tangency – since we should the be forced to define **it**, too – employing distances, in exclusivity.

This approach is based upon two most familiar high school formulas for the aria of the triangle of sides a, b, c:

[1] **Heron's Formula**

$$\boxed{S = \sqrt{p(p-a)(p-b)(p-c)}}$$

and

[2]

$$\boxed{S = \frac{abc}{4R}}$$

where $p = (a+b+c)/2$ and R denotes the radius of the circumscribed circle (*circumradius*)

Now the following definition seems easy and natural:*

Definition 1 (The Menger Curvature) *Let (M, d) be a metric space, and let $p, q, r \in M$ be three distinct points. Then:*

$$\boxed{K_M(p, q, r) = \frac{\sqrt{(pq+qr+rp)(pq+qr-rp)(pq-qr+rp)(-pq+qr+rp)}}{pq \cdot qr \cdot rp}};$$

where $pq = d(p, q)$, etc. , is called the Menger Curvature *of the points p, q, r.*

* almost trivial!...

We can now define the *Menger Curvature* at a given point by passing to the limit:

Definition 2 *Let (M,d) be a metric space and let $p \in M$ be an accumulation point. Then M has at p* Menger Curvature *$\kappa_M(p)$ iff for any $\varepsilon > 0$ there exists $\delta > 0$ s.t.*

$$\boxed{d(p, p_i) < \delta\,;\ i = 1, 2, 3 \implies |K_M(p_1, p_2, p_3) - \kappa_M(p)| < \varepsilon}$$

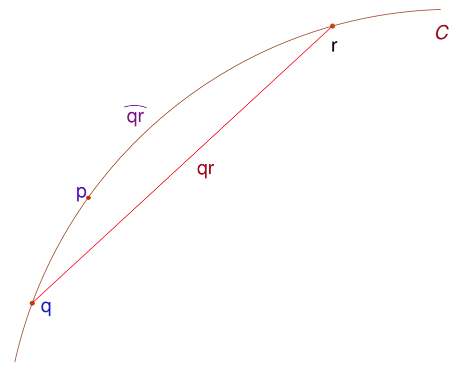
.

However, the very simplicity of the definition above its its own undoing: the Menger curvature is defined in an *intrinsically Euclidian* manner, so it may impose an Euclidian structure upon general spaces, with possibly paradoxical results.*

However, the next definition doesn't mimic closely curves in $\mathbb{R}^2$ so it better fitted for generalizations:

* But not everything is lost since the Menger curvature controls (Melnikov, 1971) – via the Cauchy integral – of all things! – the smoothness (regularity) of fractals and flatness of sets in the plane!...

Definition Let (M,d) be a metric space, let $c : I = [0, 1] \xrightarrow{\simeq} M$ be a homeomorphism, and let $p, q, r \in c(I)$, $q, r \neq p$. Denote by $\widehat{qr}$ the arc of $c(I)$ between q and r, and by qr segment from q to r.



Then c has *Haantjes Curvature** $\kappa_H(p)$ at the point p iff:

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\left(l(\widehat{qr})\right)^3};$$

where "$l(\widehat{qr})$" denotes the length†

*1947

† given by the intrinsic metric induced by d of $\widehat{qr}$.

Apparently, the Haantjes Curvature is a much more restricted notion than the Menger Curvature, since it applies only to rectifiable curves. However the two definitions coincide, whenever then are both applicable, as the following theorem shows:

Theorem 3 (Haantjes) *Let γ be a rectifiable arc in (M, d) and let $p \in \gamma$. If κ_M and κ_H exist, then they are equal.*

Of course, all this would be a nice exercise in esoteric pass-time, where it not for the following result:

Theorem 4 *Let* $\gamma \in \mathcal{C}^1$ *be smooth curve in* $\mathbb{R}^3$ *and let* $p \in \gamma$ *be a regular point. Then the metric curvatures* $\kappa_M(p)$ *and* $\kappa_H(p)$ *exist and they both equal the classical curvature of* γ *at* p*.*

And now for some (possible) applications (at last!...)

In the view of the Theorem above, it is clear that one can use both $\kappa_M(p)$ and $\kappa_H(p)$ as approximations of sectional curvatures for triangulated surfaces.* But one can expect obvious errors (since you use an approximation of classical on an approximation of a smooth curve.) But these curvatures are ideally fitted for the intelligence of PL-curves (and surfaces).

Also, an application of Menger curvature to the problem of reconstruction of curves by a Travelling Salesman Method is due to Giesen.†

*to say nothing about objects of a fractal nature

†1999

But the main flaw of applying Menger or Haantjes curvatures to the study of (metric!) surfaces, resides in the fact that they – as analogues of sectional curvatures – do not convey an intrinsic measure of surface curvature.*

Therefore a proper notion is to be searched for...

And, once again, Gauss' idea of comparing the given surface to a model one provides the answer!...

*as we have already remarked in the classical case

However, we can't restrict ourselves to the unit sphere $\mathbb{S}^2$ as a *gauge* surface, but we shall compare the given surface S to any of the *the complete, simply connected surface of constant curvature* κ, i.e.

$$\boxed{S_\kappa \equiv \mathbb{R}^2, \text{ if } \kappa = 0}$$

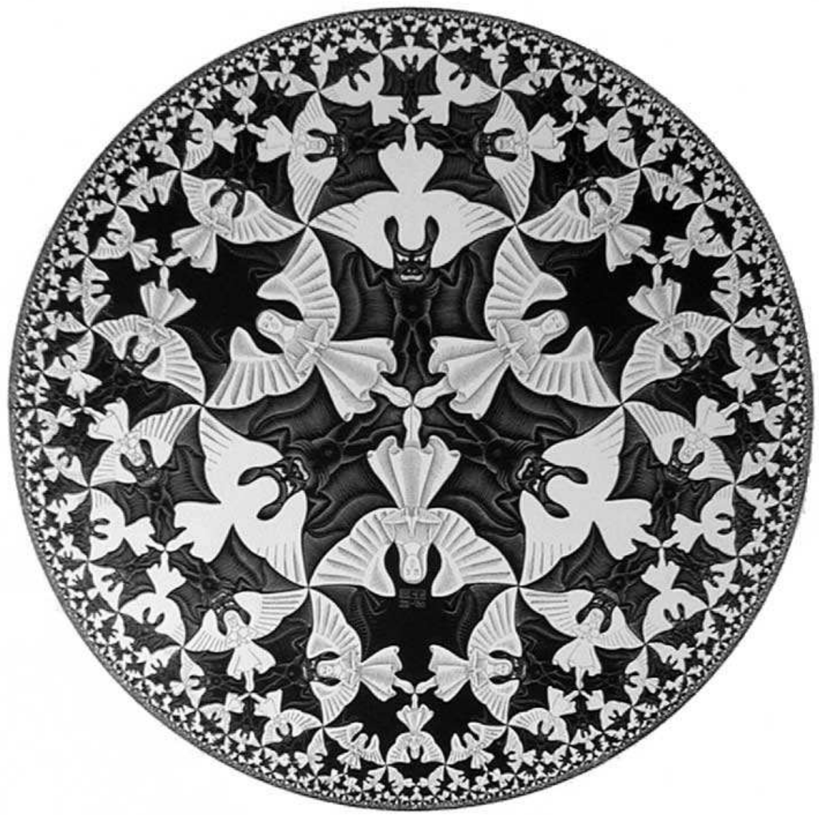
$$\boxed{S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}, \text{ if } \kappa > 0}$$

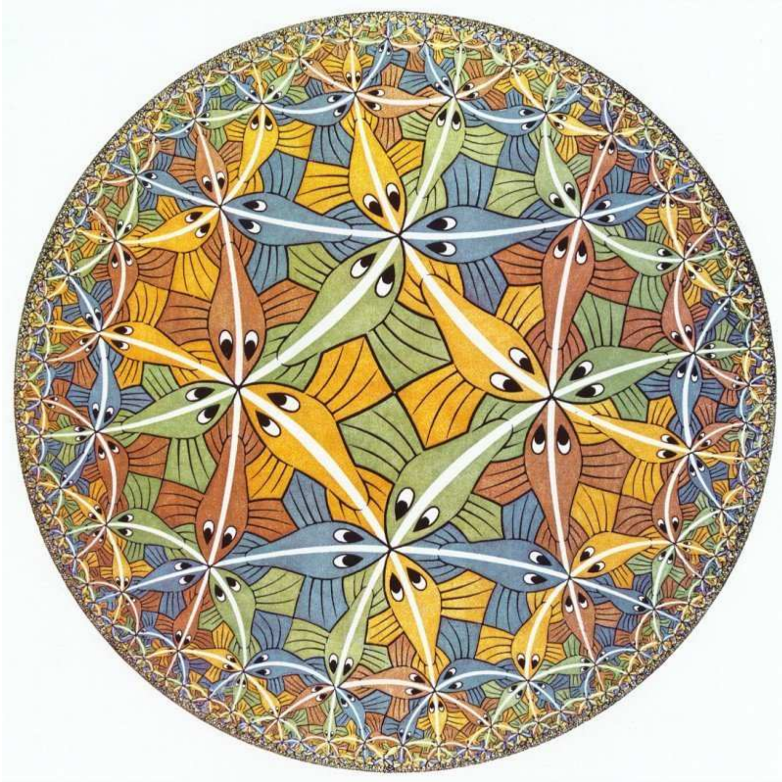
$$\boxed{S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}, \text{ if } \kappa < 0}$$

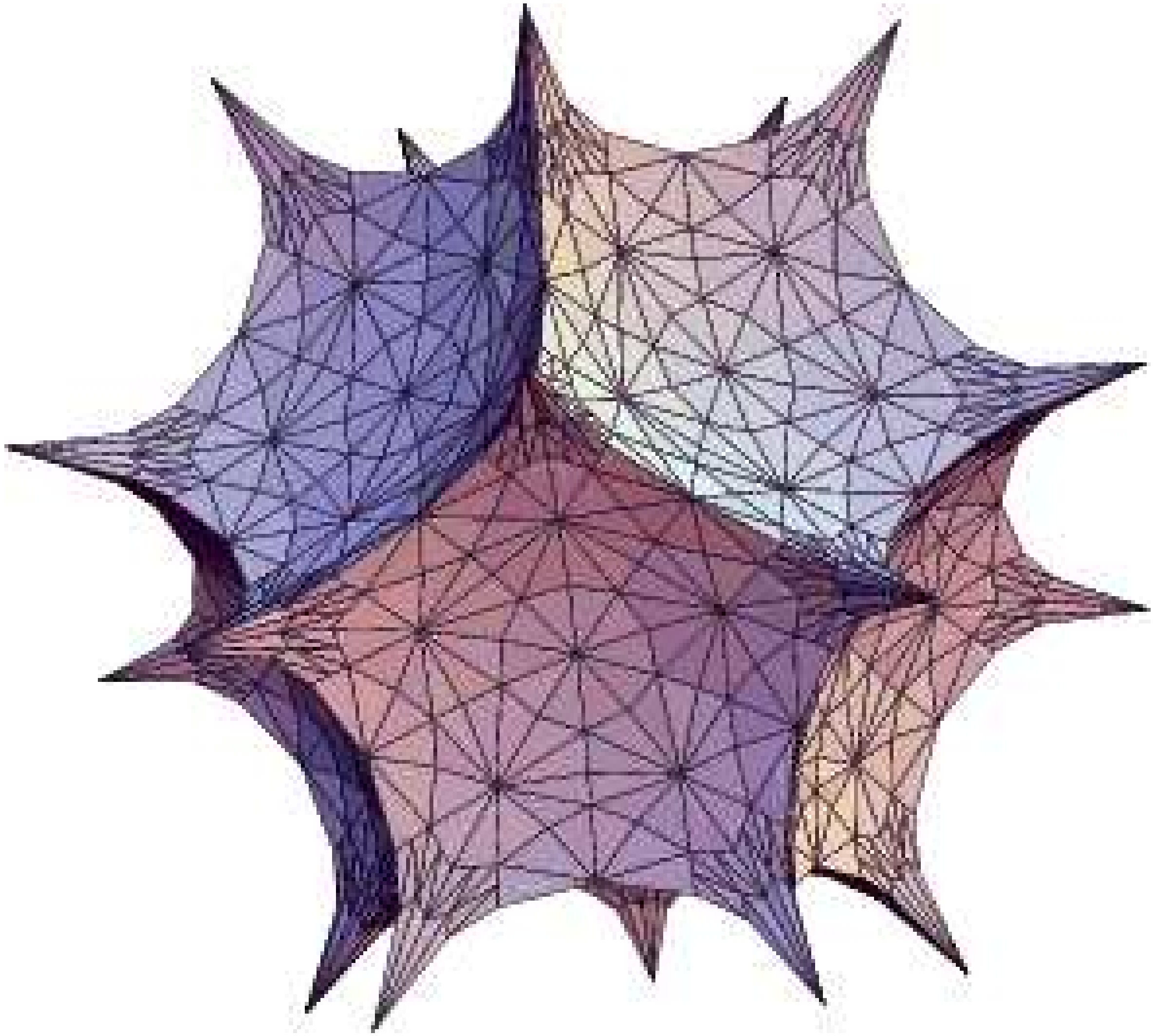
Here $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$ denotes the Sphere of radius $R = 1/\sqrt{\kappa}$,

and

$S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$ stands for the *Hyperbolic Plane* of curvature $\sqrt{-\kappa}$, as represented by the *Poincare Model* of the plane disk of radius $R = 1/\sqrt{-\kappa}$.







We can now start towards our goal of defining an intrinsic metric curvature for surfaces.

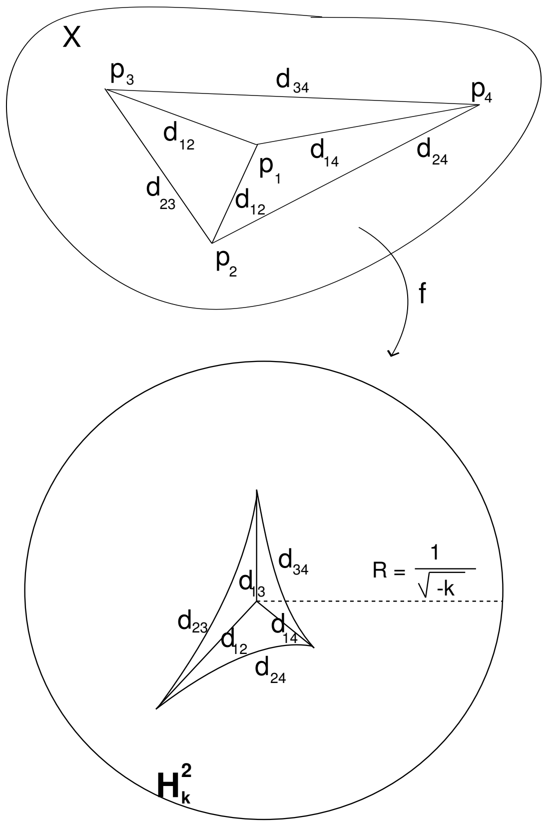
We do this by comparing quadruples on the given metric space, to those in a gauge surface. It is, in fact, a natural idea, since quadruples are classically the "minimal" geometric figures that allow the differentiation between metric spaces.

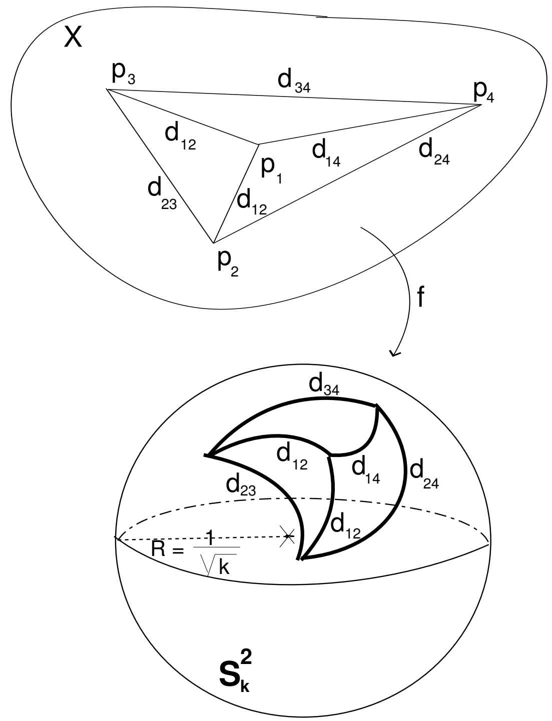
Definition 5 *Let* (M, d) *be a metric space, and let* $Q = \{p_1, ..., p_4\} \subset M$*, together with the mutual distances:* $d_{ij} = d_{ji} = d(p_i, p_j)$*;* $1 \leq i, j \leq 4$*. The set* Q *together with the set of distances* $\{d_{ij}\}_{1 \leq i,j \leq 4}$ *is called a* metric quadruple*.*

Remark 6 *One can define metric quadruples in slightly more abstract manner, without the aid of the ambient space: a metric quadruple being a* 4 *point metric space; i.e.* $Q = \left(\{p_1, ..., p_4\}, \{d_{ij}\}\right)$*, where the distances* d_{ij} *verify the axioms for a metric.*

The following definition is almost obvious:

Definition 7 *The* embedding curvature $\kappa(Q)$ *of the metric quadruple* Q *is defined be the curvature* κ *of* S_κ *into which* Q *can be isometrically embedded.*





We can now define the embedding curvature at a point in a natural way by passing to the limit (but without neglecting the existence conditions), more precisely:

Definition 8 *Let* (M, d) *be a metric space, and let* $p \in M$ *be an accumulation point. Then* p *is said to have* Wald curvature $\kappa_W(p)$ *iff*
(i) $\nexists\, N \in \mathcal{N}(p), N$ *linear** ;
(ii) $\forall \varepsilon > 0,\ \exists \delta > 0$ *s.t.* $Q = \{p_1, ..., p_4\} \subset M$, *and s.t.* $d(p, p_i) < \delta\, (i = 1, ..., 4) \Longrightarrow |\kappa(Q) - \kappa_W(p)| < \varepsilon$.

*The neighborhood N of p is called linear iff N is contained in a geodesic.

So the notion of *Embedding Curvature*, however interesting, may prove to be either ambiguous or even – in some cases – empty!...

However, for "good" metric spaces* the embedding curvature exists and it is unique. And, what is even more relevant for us, this embedding curvature coincides with the classical Gaussian curvature.

Indeed, the discussion above would be nothing more than a nice intellectual exercise where it not for the fact that the **metric** **(Wald)** and the **classical** **(Gauss)** curvatures coincide whenever the second notion makes sense, that is for smoothsurfaces in $\mathbb{R}^3$.

*i.e. spaces that are locally "plane like"

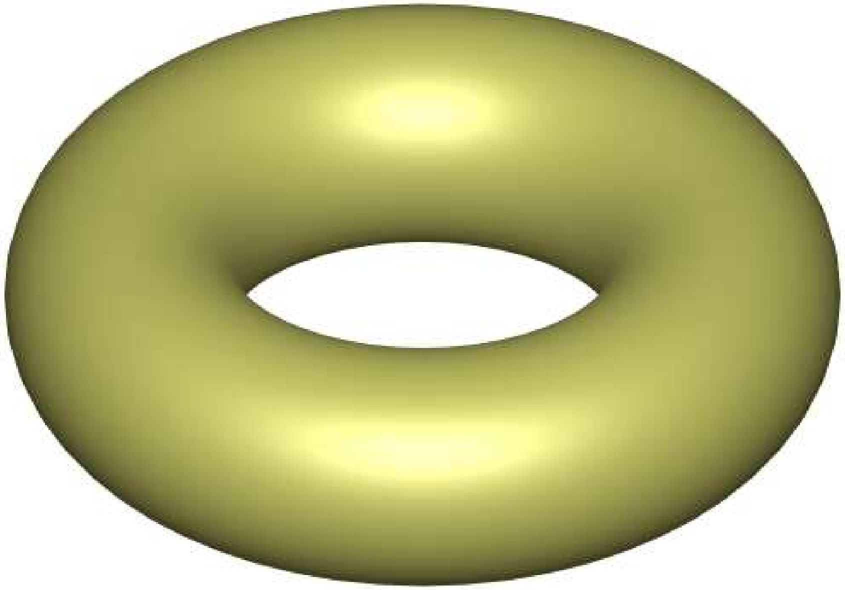
More precisely the following Theorem holds:

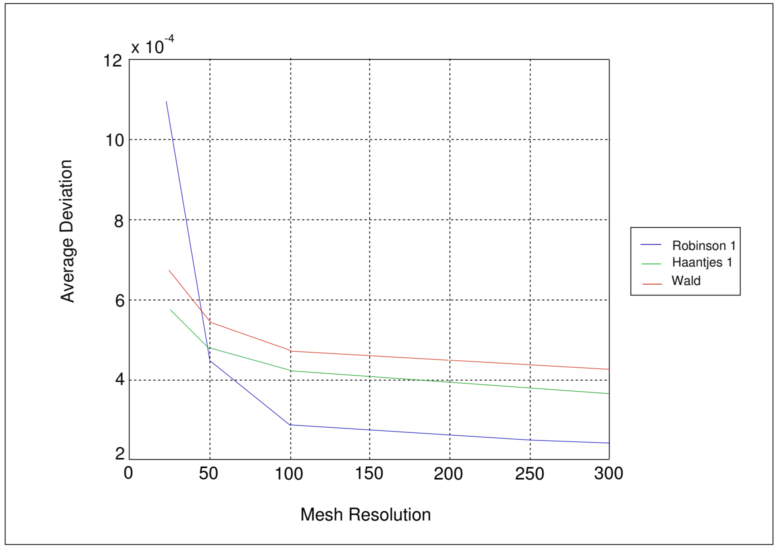
Theorem 9 (Wald) *Let $S \subset \mathbb{R}^3$, be a smooth surface.*. Then $\kappa_W(p)$ exists, for all $p \in S$, and $\kappa_W(p) = \kappa_G(p), \forall\, p \in M$.*

Moreover, Wald also proved that a partial reciprocal theorem holds, more precisely he proved the following:

Theorem 10 (Wald) *Let M be a compact and convex metric space. If $\kappa_W(p)$ exists, for all $p \in M$, then M is a smooth surface and $\kappa_W(p) = \kappa_G(p), \forall\, p \in M$.*

*i.e. $S \in \mathcal{C}^m$, $m \geq 2$




x 10^-4
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Average Deviation
Mesh Resolution
Robinson 1
Haantjes 1
Wald

And now, to an application (in a non-graphic context): for **DNA Microarraay Data Analysis**

We start by adapting Haantjes curvature to *vertex weighted graphs*:

Definition 11 *Let (G, E, μ) be a connected vertex weighted graph. Define (for all $v \sim u$):*

$$d(v,w) = \begin{cases} \frac{|\mu(v)|+|\mu(w)|}{|\mu(v)\mu(w)|} & v \neq w\,,\ \mu(v), \mu(w) \neq 0; \\ 1 & v \neq w\,,\ \mu(v) = 0 \text{ or } \mu(w) = 0; \\ 0 & v = w\,. \end{cases}$$

Remark 12 *In our context is natural to choose positive, integer weights.*

Remark 13 *The metric just defined may appear arbitrary but in fact it is rather general, because of the following reasons:*

- *One can easily "jiggle" the given metric to obtain an equivalent one by applying a function with certain properties (s.a. $\sqrt{d}$, $|\ln d|$)*

- *Any family of (bounded) metric spaces $\left\{(M_i, d_i)_i\right\}$ admits an isometric embedding in some (bounded) metric space (M, d).*

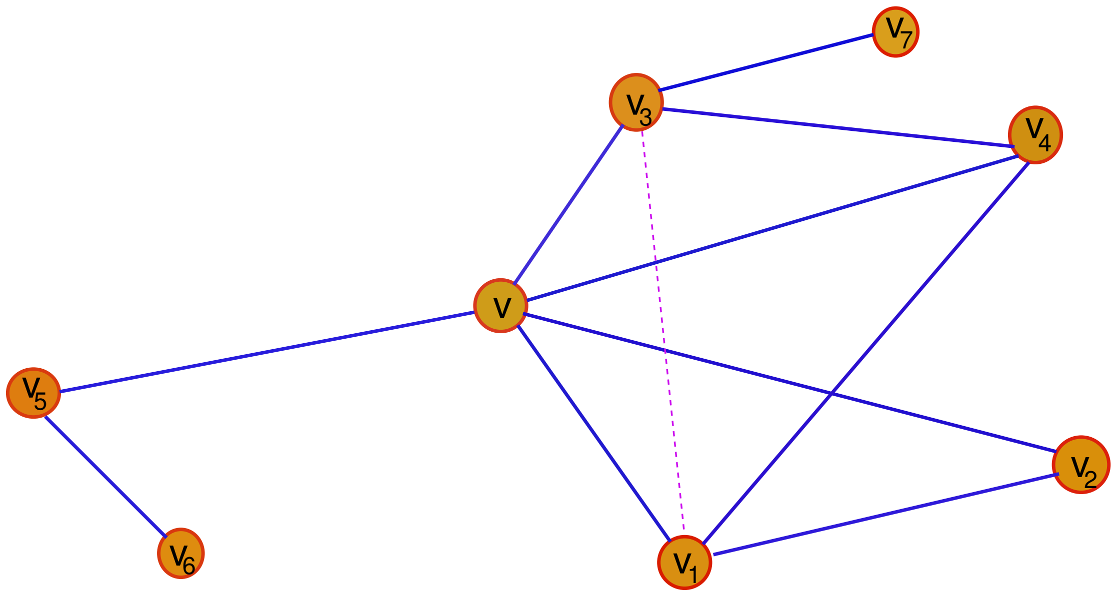
- *The metrics of any finite family of metric spaces are Lipschitz equivalent.*

Definition 14 *Let $G = (V, E, \mu)$ be as before, let d be the metric on G defined above, and let $v \in V$. Let $\pi = v_1 v v_2$ be a path through v. First we define the* curvature of triangles *with vertex v as being:*

$$\kappa'_H(\triangle v_1 v v_2) = \begin{cases} \sqrt{24\frac{|d(v_1,v)+d(v,v_2)-d(v_1,v_2)|}{\left(d(v_1,v)+d(v,v_2)\right)^3}} & e = (v_1, v_2) \in E; \\ 0 & e = (v_1, v_2) \notin E. \end{cases}$$

Then the *modified* Haantjes curvature $\kappa'_{H,\pi} = \kappa'_H(v)$ of π at v is defined to be the arithmetic mean of the curvatures of all the triangles with apex v:

$$\kappa'_H(p) = \frac{\sum_{v_i \sim v,\, v_{ij} \sim v,\, v_{ij} \neq v_i} \kappa'_H(\triangle v_i v v_{ij})}{|\{\triangle v_i v v_{ij} \mid v_i \sim v, v_{ij} \sim v, v_{ij} \neq v_i\}|}$$



In this variation of the definition the curvature at v is computed as the *mean of the curvatures* off all the triangles with apex at v, so in a sense the curvature at each point depends on the curvatures at the points in $v_i \sim v$.

We compare the clustering performance of our (metric) curvature to that of the combinatorial curvature:

Definition Let G be a (connected) graph and let v be a vertex of G, s.t. $\rho(v) \geq 2$, where $\rho(v)$ denotes the degree of v i.e. $\rho(v) = \left|\{u \,|\, u \sim v\}\right|$. The combinatorial curvature of G at v is defined as:

$$\boxed{curv(v) \triangleq \frac{|\{\triangle v v_i v_{ij} \,|\, v_i \sim v, v_{ij} \sim v_i, v_{ij} \neq v\}|}{\rho(v)(\rho(v)-1)/2}}\;;$$

that is, it represents the ratio between the actual number of triangles and the maximum number of possible triangles with apex at v.

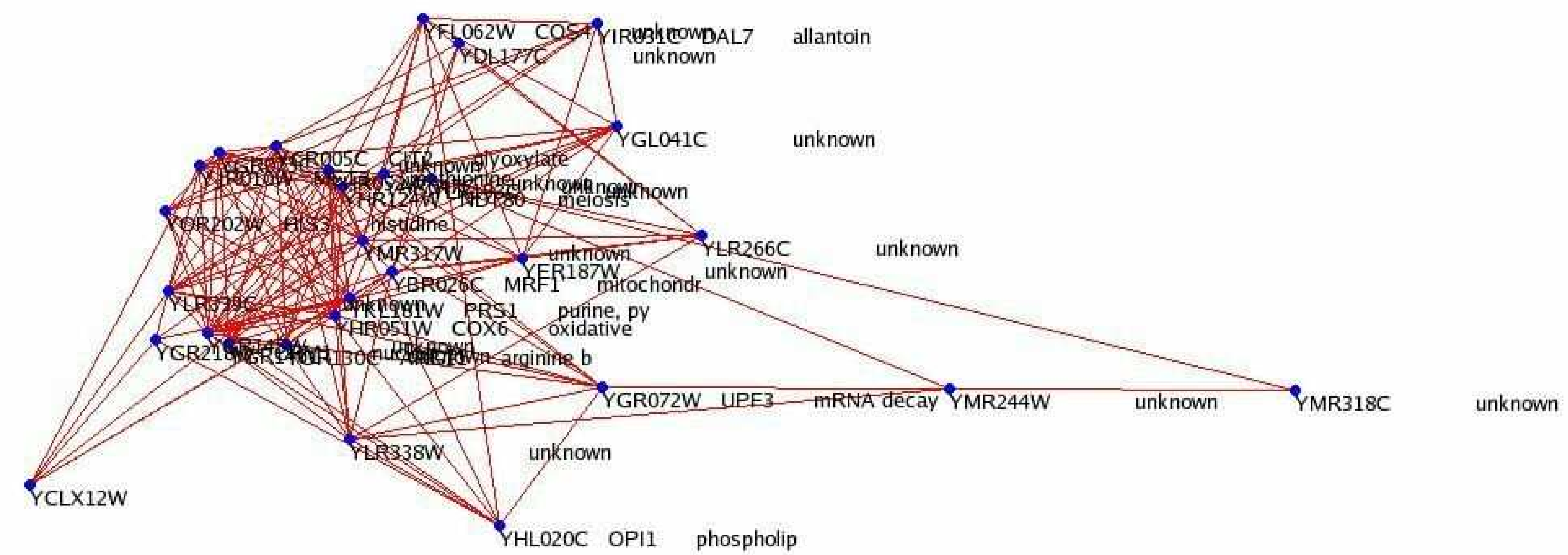
Remark $0 \leq curv(v) \leq 1$.

Remark One can show that $curv(v) = 2 - < d_v >$, where $< d_v >$ represents the average combinatorial distance between pairs of neighbours of v.

To perform clustering, one selects a curvature threshold $T_{curv} \in [0,1]$* and selects a subgraph $H_{T_{curv}} \subseteq G$ by removing all nodes of curvature $< T_{curv}$ together with their adjacent edges.

DNA microarray data taken from http://rana.lbl.gov/EisenData.html is made into a graph by a method of correlation based "edging". Namely, one computes the correlation between different DNA microarrays and sets an edge between them according to a (correlation) threshold. For that we used the open source Trixy (J. Rougemont and P. Hingamp).

Afterwards the obtained graph undergoes clustering according to curvature. For the metric we used gene length as vertices weights for they were shown to be relevant for the functioning of genes.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YIR031C DAL7 allantoin
YDL177C unknown
YGL041C unknown
YGR005C CIT2 glyoxylate
YHR124W NDT80 meiosis
YOR202W HIS3 histidine
YMR317W
YLR266C unknown
YER187W unknown
YBR026C MRF1 mitochondr
YKL181W PRS1 purine, py
YHR051W COX6 oxidative
arginine b
YGR072W UPF3 mRNA decay
YMR244W unknown
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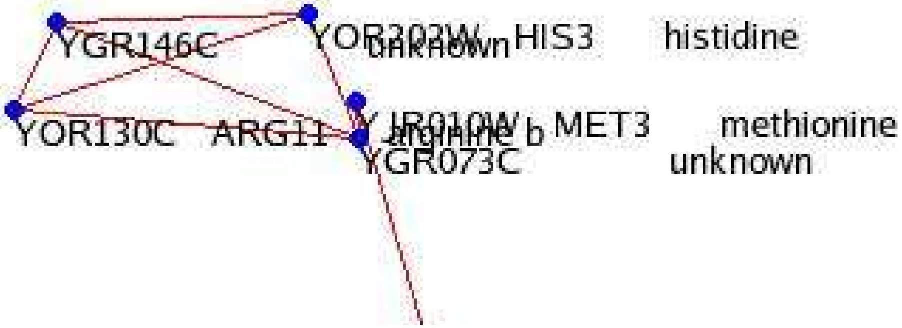
YLR338W unknown
YCLX12W
YHL020C OPI1 phospholip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MET3
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ARG11
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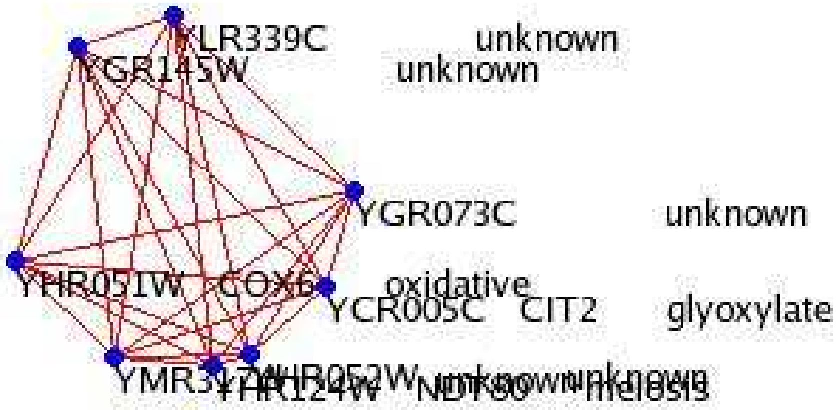
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YGR145W
unknown
YGR073C
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COX6
oxidative
YCR005C
CIT2
glyoxylate


YOR130C
ARG11
arginine b
YGR073C
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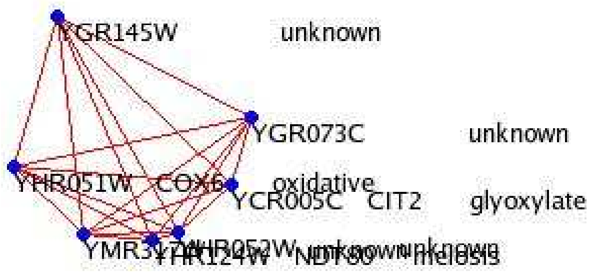
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YGR073C
unknown
COX6
oxidative
YCR005C
CIT2
glyoxylate

We conclude with the words of the great Dutch Geometer **N.H.Kuiper**:

Real understanding in mathematics means an intuitive grasp of a fact. Therefore the urge to understand will seek satisfaction in simplicity of stated theorems, simplicity of methods and proofs, and simplicity of tools. It is simplicity which can give rise to a sensation of beauty that goes with real understanding...
Thus the specific interest of a geometrically-minded mathematician, who deals with figures like curves, surfaces, with structures like metric, ..., is influenced by this simplicity as well as by the success of the methods and tools.

In the hope that we have added a bit to your – and our – intuition of curvature, we conclude this presentation.