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The Recovery of Distorted Band-Limited Stochastic Processes

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***Abstract*—This paper deals with the problem of recovering a band-limited process after it has been distorted by an instantaneous non-linearity and subsequently band-limited. Several uniqueness theorems for the input–output relationships are derived. In contrast with the deterministic case, no requirement that the nonlinearity be monotonic is made here. An iterative procedure for the recovery of the input is presented. Applications to two-level quantizers are considered, and a new result on the determination of a band-limited Gaussian process from its zero crossings is obtained.**

I. INTRODUCTION

A PRACTICAL problem in communication systems is that of the recovery of a band-limited signal $f(t)$ after it has been transformed by an instantaneous nonlinearity, or compander, and subsequently band-limited to the original bandwidth [1], [2]. As was noted in [1], companding is used to improve the performance of transmission systems which do not generally respond well to very high or very low signal amplitudes. Thus a typical compander has a slope which is very high near the origin but tapers off rapidly at infinity.

If the original signal $f(t)$ is band-limited to $(-W,W)$, say, the output of the compander, $A[f(t)]$, is not band-limited in general. Therefore, subsequent band-limiting of $A[f(t)]$—transmission through a band-limited channel—results in an apparent loss of information. A beautiful theorem by Beurling [1] shows that if the companding function A is monotonic, then knowledge of the spectrum of $A[f(t)]$ in the band $(-W,W)$ is sufficient to determine $f(t)$ uniquely. More precisely, if $f_1(t)$ and $f_2(t)$ are two functions band-limited to $(-W,W)$, and if $A[f_1(t)]$ and $A[f_2(t)]$ have spectra which are equal in the band $(-W,W)$, then $f_1(t) = f_2(t)$ for almost all t. Beurling's theorem is not constructive, however. An explicit procedure for the recovery of $f(t)$ was derived by Landau [1], [2] for a class of companding functions $A(x)$.

It has long been conjectured that a result similar to Beurling's theorem should hold for band-limited stochastic processes. Beurling's theorem, valid for band-limited functions of finite energy, does not apply of course to band-limited stochastic processes, since stationary processes do not have sample paths of finite energy.

In this paper we show that a result similar to Beurling's theorem does indeed hold for a class of stationary band-limited stochastic processes. Moreover, while in Beurling's theorem the monotonicity of the companding function A is essential, we do not require here that A be monotonic. This result is somewhat surprising and has some far-reaching implications. For example, we show that, within the class of jointly stationary and jointly Gaussian band-limited processes, a Gaussian process is uniquely determined by its zero crossings. This is a new result which has no counterpart for deterministic band-limited functions.

In Section II, the problem is formulated and the basic assumptions are stated. In Section III, uniqueness theorems are derived for some broad classes of band-limited processes where the companding function is arbitrary. The implications of these theorems are discussed. In Section IV, we derive a uniqueness theorem for arbitrary band-limited processes where the companding function A is now restricted to be monotonic. We also derive an iterative procedure, based in part on Landau's scheme, for the recovery of the input process.

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II. Preliminaries

Let us first introduce some notation. Let R be the real line, $\mathcal{M}$ the σ-algebra of Lebesgue measurable sets in R, and m the Lebesgue measure. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space.

Consider the configuration depicted in Fig. 1. We shall assume that the input process $\{X(t,\omega),\ t \in R\}$ is a real second-order mean-square continuous stationary band-limited stochastic process over the probability space $(\Omega,\mathcal{F},\mathcal{P})$. Let $S(\lambda)$ be the spectral distribution of the process $X(t)$ and $C(\tau)$ its covariance function

$$C(\tau) = \int_{-W}^{W} e^{i\tau\lambda}\, dS(\lambda) \tag{1}$$

where $(-W,W)$ is the bandwidth of the process and where we have assumed that the process $X(t)$ has been normalized to have zero mean and unit variance. Given two such processes, $X_1(t)$ and $X_2(t)$, we shall assume that they are jointly stationary so that $X = (X_1,X_2)$ constitutes a "2-dimensional stationary process" [3]. Let $\mathcal{B}$ denote the class of processes with the aforementioned properties. Note that if $\mathrm{S} = \{S_{X_iX_j}\}$, $i,j = 1,2$, is the spectral distribution matrix of $X = (X_1,X_2)$, $X_i \in \mathcal{B}$, $i = 1,2$, then [3]

$$|S_{X_iX_j}(B)|^2 \le S_{X_iX_i}(B) S_{X_jX_j}(B), \qquad i,j = 1,2 \tag{2}$$

for any Borel set B in the real line. It then follows by (2) that the processes $X_1(t)$ and $X_2(t)$ are jointly band-limited to $(-W,W)$.

Let A be a real Borel-measurable function on the real line. Then the process $\{Y(t,\omega),\ t \in R\}$ defined by

$$Y(t,\omega) = A[X(t,\omega)], \qquad t \in R \tag{3}$$

is stationary. Since we are dealing with second-order stochastic processes, we shall assume that $E[A^2(X(t))] < \infty$ for all t, where E is the expectation operation. Furthermore, since the spectral representation of stationary processes plays an important role in our derivations, we shall require that the process $Y(t)$ be mean-square continuous. We shall further assume that $E[X(0)A(X(0))] \neq 0$. Since in practice $A(x)$ is monotonic or odd, this last assumption is not restrictive from the point of view of applications, as it is satisfied for a broad class of input processes, e.g., Gaussian processes. Thus let $\mathcal{A}$ denote the class of admissible companding functions $A(x)$ satisfying

i) $E[A^2(X(t))] < \infty, \qquad$ for all t (4a)

ii) $\{A[X(t)],\ t \in R\}$

is a mean-square continuous process (4b)

iii) $E[X(0)A[X(0)]] \neq 0.$ (4c)

Let us note that a simple sufficient condition for (4b) to be valid is that A satisfy the Lipschitz condition

$$|A(y) - A(x)| < M|y - x|, \qquad x,y \in R. \tag{5}$$

Note however that (5) is not necessary for (4b) to be valid. In particular, if the companding function A represents a "hard-limiter"

$$A(x) = \operatorname{sgn} x \tag{6}$$

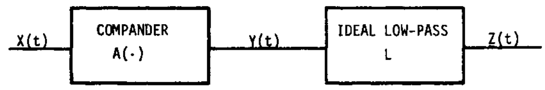

Fig. 1. The transformation $Z(t) = [T(X)](t)$.

and $x \in \mathcal{B}$ is Gaussian, then

$$R_{YY}(\tau) = \left(\frac{2}{\pi}\right) \sin^{-1} C_{XX}(\tau)$$

so that $Y(t) = \operatorname{sgn}[X(t)]$ is a stationary second-order mean-square continuous process. This fact will be used in the next section.

The linear system L is an ideal low-pass filter with transfer function

$$H(i\lambda) = \begin{cases} 1, & |\lambda| \le W \\ 0, & |\lambda| > W. \end{cases} \tag{7}$$

It then follows that its output $Z(t)$ is a second-order mean-square continuous stationary band-limiting process. We shall write

$$Z(t) = [L(A(X))](t) \tag{8}$$

as well as

$$Z(t) = [T(X)](t). \tag{9}$$

III. The Correspondence Between Input and Output for a Class of Band-Limited Processes

In this section, we prove the uniqueness of the input–output relationship $Z(t) = [T(X)](t)$ for a broad class of band-limited stochastic processes. We shall first consider Gaussian processes, and later extend the results to non-Gaussian processes.

A. Input–Output Relationships for Gaussian Processes

Theorem 1: Let $X_1(t)$ and $X_2(t)$ be two jointly Gaussian processes in $\mathcal{B}$. Let

$$Z_i(t) = [L(A(X_i))](t), \qquad i = 1,2 \tag{10}$$

where $A \in \mathcal{A}$. Then

$Z_1(t) = Z_2(t)$ a.s. for *some* $t = t_0$

implies $X_1(t) = X_2(t)$ a.s. for *all* t. (11)

Proof: Consider the functional J

$$J = E\{[X_1(t_0) - X_2(t_0)][Z_1(t_0) - Z_2(t_0)]\} = 0 \tag{12}$$

which vanishes by hypothesis. Now J can be evaluated as

$$J = R_{X_1Z_1}(0) - R_{X_1Z_2}(0) - R_{X_2Z_1}(0) + R_{X_2Z_2}(0)$$

and by the spectral representation

$$\begin{aligned} J &= \int_{-\infty}^{\infty} H(i\lambda)\, d[S_{X_1Y_1}(\lambda) - S_{X_1Y_2}(\lambda) \\ &\qquad - S_{X_2Y_1}(\lambda) + S_{X_2Y_2}(\lambda)] \\ &= \int_{-W}^{W} d[S_{X_1Y_1}(\lambda) - S_{X_1Y_2}(\lambda) \\ &\qquad - S_{X_2Y_1}(\lambda) + S_{X_2Y_2}(\lambda)] \end{aligned} \tag{13}$$

where $S_{X_iY_j}(\lambda)$ is the cross spectral distribution between the processes $X_i(t)$ and $Y_j(t), Y_j(t) = A[X_j(t)]$. Next we note that

$$R_{X_iY_j}(\tau) = C_{X_iY_j}(\tau), \qquad i,j = 1,2 \tag{14}$$

since the input processes have zero means. Moreover, since the Gaussian processes have the "cross-covariance" property [4], we have

$$C_{X_iY_j}(\tau) = a_{i,j}C_{X_iX_j}(\tau), \qquad i,j = 1,2 \tag{15}$$

where the constant $a_{i,j}$ is given by

$$a_{i,j} = E[X_j(t)A(X_j(t))], \qquad i,j = 1,2 \tag{16}$$

and is independent of t. Since the processes $X_1(t)$ and $X_2(t)$ have the same univariate distribution, we have from (16) that (15) can be written as

$$C_{X_iY_j}(\tau) = \alpha C_{X_iX_j}(\tau), \qquad i,j = 1,2 \tag{17}$$

where α is given by

$$\alpha = E[X(t)A[X(t)]]. \tag{18}$$

It then follows from (14) and (17) that (13) may be written as

$$J = \alpha \int_{-W}^{W} d[S_{X_1X_1}(\lambda) - S_{X_1X_2}(\lambda) - S_{X_2X_1}(\lambda) + S_{X_2X_2}(\lambda)]$$

which is equivalent to

$$J = \alpha E|X_1(t) - X_2(t)|^2, \qquad \text{for all } t. \tag{19}$$

Since by (12) $J = 0$, and $\alpha \neq 0$ by (18) and (4c), we have from (19) that

$$X_1(t) = X_2(t) \text{ a.s. for all } t \tag{20}$$

so that the two processes $X_1(t)$ and $X_2(t)$ are equivalent. Q.E.D.

Remark 1: Theorem 1 implies that for each fixed t the random variables $X_1(t)$ and $X_2(t)$ are equal almost surely. If we assume that input processes $X_1(t)$ and $X_2(t)$ are measurable as well (because of mean-square continuity, measurable versions always exist), then Theorem 1 implies equality almost everywhere of the *sample paths* of the two processes because by (20)

$$E|X_1(t) - X_2(t)|^2 = 0, \qquad t \in R$$

so that

$$\int_{-\infty}^{\infty} E|X_1(t) - X_2(t)|^2 m(dt) = 0 \tag{21}$$

and by Tonelli's theorem (21) implies

$$E\left[\int_{-\infty}^{\infty} |X_1(t) - X_2(t)|^2 m(dt)\right] = 0 \tag{22}$$

which in turn implies

$$\int_{-\infty}^{\infty} |X_1(t) - X_2(t)|^2 m(dt) = 0 \text{ a.s.} \tag{23}$$

Finally, (23) implies that for almost all sample functions we have

$$X_1(t) = X_2(t) \text{ a.e. } [m].$$

Before turning to the implications of Theorem 1, let us consider the following question. Assume that $X \in \mathscr{B}$ is Gaussian and the companding function $A \in \mathscr{A}$ as in Theorem 1. Does uniqueness of the input–output relation continue to hold if the low-pass filter is *not* ideal? The answer to this question is affirmative, as stated below.

Theorem 2: Let $X_1(t)$ and $X_2(t)$ be two jointly Gaussian processes in $\mathscr{B}$. Let

$$Z_i(t) = [L(A(X_i))](t), \qquad i = 1,2$$

where $A \in \mathscr{A}$ and the low-pass filter L satisfies

$H(i\lambda)$ is continuous over $(-W,W)$

$$\text{such that } \operatorname{Re} H(i\lambda) > 0 \text{ (or } <0) \text{ over } (-W,W). \tag{24}$$

Then

$Z_1(t) = Z_2(t)$ a.s. for *some* $t = t_0$

implies $X_1(t) = X_2(t)$ a.s. for *all* t.

Proof: Proceeding as in the proof of Theorem 1 we find that the functional J defined by (12) can be written as

$$J = \alpha \int_{-W}^{W} H(i\lambda)\, dS_{ee}(\lambda) \tag{25}$$

where $S_{ee}(\lambda)$ is the spectral distribution of the process $e(t)$

$$e(t) = X_1(t) - X_2(t), \qquad t \in R$$

and the constant α is given by (18). Since $J = 0$ by hypothesis, we have in particular

$$\operatorname{Re} \int_{-W}^{W} H(i\lambda)\, dS_{ee}(\lambda) = 0. \tag{26}$$

Now assumption (24) on $H(i\lambda)$ and (26) imply that $S_{ee}(\lambda)$ has no points of increase in $(-W,W)$. Since the process $e(t)$ is band-limited, we have $R_{ee}(0) = 0$, which implies

$$X_1(t) = X_2(t) \text{ a.s. for all } t. \qquad \text{Q.E.D.}$$

Remark: Let us note that if the input processes have spectral density functions, then condition (24) on $H(i\lambda)$ can be replaced by the weaker condition

$H(i\lambda)$ is measurable over $(-W,W)$ such that

$$\operatorname{Re} H(i\lambda) > 0 \text{ (or } <0) \text{ a.e. } [m] \text{ over } (-W,W). \tag{24'}$$

B. Applications

Let us note that the most remarkable feature of theorems 1 and 2 is the fact that they do not require monotonicity of the companding function A. This is a surprising result, since no counterpart for deterministic band-limited functions is known. In fact, the assumption in Beurling's theorem that A is monotonic is crucial and essential for the validity of that theorem.

An important application of Theorems 1 and 2 is obtained in the case of a two-level quantizer,

$$A(x) = \operatorname{sgn} x. \tag{27}$$

It was shown in Section II that the compander (27) is an admissible transformation for Gaussian processes, i.e., $A \in \mathscr{A}$. Clearly, the output $Y(t) = \operatorname{sgn}[X(t)]$ is a stationary binary process associated with the zero crossings of the Gaussian process $X(t)$. Let us note that the zero crossings of a Gaussian process $X(t) \in \mathscr{B}$ are well defined, since $X(t)$ has analytical sample paths, so that all level crossings are "genuine" in the sense that they consist of upcrossings and downcrossings with no tangencies [5]. The binary process $Y(t)$ is clearly not band-limited. Let us call the process $Z(t)$ obtained by band-limiting $Y(t)$ the "band-limited version" of $Y(t)$. The band-limiting low-pass filter L may be ideal (7) or nonideal (24). We then have the following important result which follows from Theorems 1 and 2.

Theorem 3: Let $\mathscr{G}$ be the class of real zero-mean jointly stationary and jointly Gaussian band-limited processes. Every process $X(t) \in \mathscr{G}$ is uniquely determined within $\mathscr{G}$ by the "band-limited version" of the binary process $Y(t)$ associated with its zero crossings. The uniqueness is up to a multiplicative constant.

Theorem 3 clearly implies that if

$$X_1(t), X_2(t) \in \mathscr{G} \text{ then } \operatorname{sgn}[X_1(t)] = \operatorname{sgn}[X_2(t)] \text{ a.s. for all } t \Rightarrow X_1(t) = bX_2(t) \text{ a.s. for all } t, \quad b \text{ is a real constant} \tag{28}$$

Now since $\operatorname{sgn}[X(t)]$ is uniquely determined by the zero crossings of $X(t) \in \mathscr{G}$ (up to a multiplicative $\pm$ sign), it then follows from this and (28) that no two processes in $\mathscr{G}$ can have the same zero crossings unless they are a constant multiple of each other. We thus have the following corollary.

Corollary: Every process $X(t) \in \mathscr{G}$ is uniquely determined, within $\mathscr{G}$, by its zero crossings. The uniqueness is up to a multiplicative constant.

It should be noted that the uniqueness in Theorem 3 and its corollary is within the class of *jointly* stationary and Gaussian processes. It would be of some significance to be able to drop the assumption on joint stationarity and Gaussianness and retain individual stationarity and Gaussianness. So far, we have not succeeded in accomplishing this. However, the Gaussian assumption in Theorem 3 can be dropped if one can extend Theorem 1, and thus Theorem 3 and its corollary, to non-Gaussian processes. This is done in Theorem 4.

In the meantime, we note that even within the class $\mathscr{G}$ the results of theorem 3 and its corollary are surprising since they have no counterpart in the deterministic case. Let us discuss briefly what is known in that case. Let $f(t)$ be a real square-integrable function which is band-limited to $(-W, W)$, such that for all t

$$f(t) = \frac{1}{2\pi}\int_{-W}^{W} F(i\lambda)e^{i\lambda t}\, d\lambda$$

where $F(i\lambda)$ is the Fourier transform of $f(t)$. Define the entire function $f(z)$ by

$$f(z) = \frac{1}{2\pi}\int_{-W}^{W} F(i\lambda)e^{i\lambda z}\, d\lambda, \qquad z = t + iu$$

and let $\{z_n\}_{n=1}^{\infty}$ be the zeros of $f(z)$. A classical theorem by Titchmarsh [6] states that $f(z)$ is uniquely determined by its zeros; specifically,

$$f(z) = f(0)e^{iWz}\prod_{n=1}^{\infty} 1 - \frac{iz}{z_n}$$

so that

$$f(t) = f(0)e^{iWt}\prod_{n=1}^{\infty} 1 - \frac{it}{z_n}. \tag{29}$$

Let us note, however, that the zeros $\{z_n\}$ are complex in general. Clearly, the complex zeros of $f(t)$ are not observable, so that $f(t)$ is not determined by its (real) zero crossings. Real-zero interpolation can produce spectacular amplitude fluctuations in the reconstructed signal [7]. Manipulation of complex zeros and real-zero interpolation are discussed in a recent work by Voelcker [8]. The basic fact remains, however, that an arbitrary real band-limited function is not uniquely determined by its (real) zero crossings.

Thus we see that the uniqueness relationship, within the class $\mathscr{G}$, between a real band-limited Gaussian process and its zero crossings that is given by Theorem 3 and its corollary is quite surprising. Let us note that the configuration of Fig. 1 with a "hard-limited" compander can represent a model for many common systems, i.e., sequential binary data, black-and-white facsimile or television, clipped speech and so on. We remark finally that we have found no procedure for the *perfect recovery* of the process $X(t)$ from its zero crossings.

C. Extension to Non-Gaussian Processes

In Sections III-A and III-B, the assumption of a Gaussian input is made. The question arises whether the results of Theorems 1 and 2 are valid for other classes of input processes. The answer is affirmative. To this end let us note that the proof of Theorems 1 and 2 is based on the "cross-covariance" property of Gaussian processes (15). Now there is a broad class of stochastic processes for which the cross-covariance property is valid. (See [9] for the most comprehensive discussion on this topic.) For example, the envelope of the output of a bandpass filter with a Gaussian input is a process having the "cross-covariance" property. The extension of Theorem 1 and 2 to these processes is straightforward. We state only one result.

Theorem 4: Let $X_1(t)$ and $X_2(t)$ be two real zero-mean jointly stationary band-limited processes. Let

$$Z_i(t) = [L(A(X_i))](t), \qquad i = 1,2$$

where $A \in \mathscr{A}$ and L is an ideal low-pass filter. Then, under the assumptions

i) the processes $X_1(t)$ and $X_2(t)$ have individually and jointly the cross-covariance property

ii) the variances σ_1 and σ_2 are equal
iii) $E[X_1(t)A(X_1(t))] = E[X_2(t)A(X_2(t))] \neq 0$

we have

$$Z_1(t) = Z_2(t) \text{ a.s. for } \textit{some } t = t_0 \text{ implies } X_1(t) = X_2(t) \text{ a.s. for } \textit{all } t.$$

As a final remark, let us note that the uniqueness theorems of this section are nonconstructive (as is Beurling's theorem). In the next section we shall impose further constraints on the companding function A and derive a uniqueness theorem *and* a constructive procedure for the recovery of arbitrary band-limited processes.

IV. Uniqueness and Explicit Recovery of Arbitrary Band-Limited Processes

It would be of considerable practical importance to find a general procedure for the recovery of the input process $X(t)$ from its distorted band-limited version $Z(t) = [T(X)](t)$, where T is the nonlinear system depicted in Fig. 1. This is a very complex problem, and we have found no general solution. However, if the companding function is sufficiently smooth, we can derive both uniqueness and an explicit procedure for the recovery of the input process $X(t)$, where $X(t)$ is now an arbitrary band-limited process.

We begin by stating our assumptions. The input process is assumed to be a real zero-mean second-order mean-square continuous stationary band-limited process, with spectral distribution $S(\lambda)$ over $(-W,W)$. Let $\mathscr{B}'$ denote the class of these processes. The companding function $A(x)$ is assumed to be monotonic (say increasing) and continuous such that

$$u(x_1 - x_2) \le A(x_1) - A(x_2) \le U(x_1 - x_2), \qquad \text{for all } x_1,x_2 \in R \quad (30)$$

where u and U are positive constants. In particular, if $A(x)$ is absolutely continuous with derivative $A'(x)$ satisfying $u \le A'(x) \le U$, then (30) is satisfied. The low-pass filter L is assumed to be ideal as in (7). Under the assumption (30), it is easily seen that the process $Y(t) = A[X(t)]$ is mean-square continuous. We use the notation $\|X\|^2 = E|X|^2$ to denote the second moment of a random variable X.

The uniqueness theorem and the constructive recovery procedure are based on a contraction mapping in the space $\mathscr{B}'$. Specifically, define the operator $K: \mathscr{B}' \to \mathscr{B}'$ by

$$K[X](t) = X(t) - c[T(X)](t), \qquad X(t) \in \mathscr{B}'. \quad (31)$$

Then K is a contraction mapping for some values of the parameter c as stated in the following proposition. Its proof is given in the Appendix.

Proposition: Let $X_1(t)$ and $X_2(t)$ be two jointly stationary processes in $\mathscr{B}'$. Then

$$\|K[X_1](t) - K[X_2](t)\| \le \theta\|X_1(t) - X_2(t)\|, \qquad \theta < 1 \quad (32)$$

for all $0 < c < 2/U$.

We next show the uniqueness of the input–output relationship. The proof is given in the Appendix.

Theorem 5 (Uniqueness): Let $X_1(t)$ and $X_2(t)$ be two jointly stationary processes in $\mathscr{B}'$. Let

$$Z_i(t) = [T(X_i)](t) = [L(A(X_i))](t), \qquad i = 1,2$$

where $A(x)$ is monotonic and satisfies (30) and L is an ideal low-pass filter. Then

$$Z_1(t) = Z_2(t) \text{ a.s. for all } t \text{ implies } X_1(t) = X_2(t) \text{ a.s. for all } t. \quad (33)$$

Next we turn to the recovery of the input. The iterative procedure given below is based on the fixed point theorem [10] and Landau's scheme [1]. The proof is given in the Appendix.

Theorem 6 (Reconstruction): Let $X(t) \in \mathscr{B}'$ and

$$Z(t) = [T(X)](t) = [L(A(X))](t) \quad (34)$$

where $A(x)$ is monotonic and satisfies (30) and L is an ideal low-pass filter. Then the iterative procedure

$$X_{n+1}(t) = X_n(t) + c\{Z(t) - [T(X_n)](t)\}, \quad n = 1,2,\cdots$$
$$X_1(t) \equiv 0 \quad (35)$$

converges in the mean-square sense to the input process $X(t)$ for all $0 < c < 2/U$.

If the companding function A is monotonically decreasing such that

$$-U(x_1 - x_2) \le A(x_1) - A(x_2) \le -u(x_1 - x_2), \qquad x_1, x_2 \in R \quad (30')$$

where u and U are positive constants, then the proposition, Theorem 5, and Theorem 6 remain valid.

V. Conclusion

In this paper we have considered the problem of recovering a band-limited stochastic process after it has been distorted by an instantaneous nonlinearity and subsequently band-limited. We derived several uniqueness theorems for the input-output relationship which were shown to be valid even if the nonlinearity is not monotonic. Applications to two-level quantizers were considered, and a new result on the determination of a band-limited Gaussian process by its zero crossings was obtained. In the case where the nonlinearity is smooth, we also derived an iterative procedure for the recovery of the input process.

The question of the recovery of the input when the companding function is not smooth is being investigated at the present time. Some further results and extensions of this work will be reported in the future.

Appendix

The proofs of the proposition and Theorems 5 and 6 of Section IV are given in the following.

A. Proof of the Proposition

Define the process $r(t)$ by

$$r(t) = X_1(t) - X_2(t) - c\{A[X_1(t)] - A[X_2(t)]\}, \quad t \in R \quad \text{(A-1)}$$

which is clearly not band-limited. Moreover, the process $[L(r)](t)$ is given by

$$\begin{aligned}[L(r)](t) &= [L(X_1 - X_2)](t) - c\{[T(X_1)](t) - [T(X_2)](t)\} \\ &= X_1(t) - X_2(t) - c\{[T(X_1)](t) - [T(X_2)](t)\}\end{aligned} \quad \text{(A-2)}$$

where the second equality follows from the fact that $X_1(t) - X_2(t)$ is a band-limited process and L is an ideal low-pass filter. If $S_{rr}(\lambda)$ is the spectral distribution of the process $r(t)$, then by the spectral representation theory we have

$$\begin{aligned}\|[L(r)](t)\|^2 &= \int_{-\infty}^{\infty} |H(i\lambda)|^2 \, dS_{rr}(\lambda) \\ &= \int_{-W}^{W} dS_{rr}(\lambda) \quad \text{(A-3)} \\ &\leq \int_{-\infty}^{\infty} dS_{rr}(\lambda) = \|r(t)\|^2 \\ &= \|X_1(t) - X_2(t) - cA[X_1(t)] + cA[X_2(t)]\|^2 \\ &= \left\| [X_1(t) - X_2(t)] \left[1 - c\frac{A(X_1(t)) - A(X_2(t))}{X_1(t) - X_2(t)}\right] \right\|^2 \quad \text{(A-4)}\end{aligned}$$

and, since $A(x)$ satisfies (30), we have for $0 < c < 2/U$ that

$$\theta = \sup_{x_1, x_2} \left| 1 - c\frac{A(x_1) - A(x_2)}{x_1 - x_2} \right| < 1. \quad \text{(A-5)}$$

It then follows from (A-4) and (A-5) that

$$\|[L(r)](t)\|^2 \leq \theta^2 \|X_1(t) - X_2(t)\|^2, \qquad \theta < 1. \quad \text{(A-6)}$$

Equation (A-6) is the required result since

$$[L(r)](t) = K[X_1](t) - K[X_2](t). \qquad \text{Q.E.D.}$$

B. Proof of Theorem 5

By hypothesis $Z_1(t) = Z_2(t)$ a.s., so that $[T(X_1)](t) = [T(X_2)](t)$. By the proposition we then have

$$\|X_1(t) - X_2(t)\| < \theta \|X_1(t) - X_2(t)\|, \qquad \theta < 1. \quad \text{(B-1)}$$

Equation (B-1) can hold only if

$$\|X_1(t) - X_2(t)\| = 0$$

which implies

$$X_1(t) = X_2(t) \text{ a.s. for all } t. \quad \text{(B-2)}$$

Q.E.D.

C. Proof of Theorem 6

Let us first note that all iterations given by (35) are band-limited processes since each term on the right-hand side of (35) is a band-limited process. Consider, for each fixed t, the distance $\|X(t) - X_{n+1}(t)\|$, which by (35) can be written as

$$\begin{aligned}\|X(t) - X_{n+1}(t)\| &= \|X(t) - X_n(t) - c\{Z(t) - [T(X_n)](t)\}\| \\ &= \|X(t) - X_n(t) - c\{[T(X)](t) \\ &\quad - [T(X_n)](t)\}\|.\end{aligned} \quad \text{(C-1)}$$

Applying the proposition to the right-hand side of (C-1), we immediately have

$$\|X(t) - X_{n+1}(t)\| \leq \theta \|X(t) - X_n(t)\|, \qquad \theta < 1, \quad n = 1,2,\cdots. \quad \text{(C-2)}$$

Hence,

$$\|X(t) - X_{n+1}(t)\| \leq \theta^n \|X(t) - X_1(t)\|, \qquad n = 1,2,\cdots. \quad \text{(C-3)}$$

Since $\theta < 1$, we have by (C-3) that

$$\|X(t) - X_{n+1}(t)\| \to 0 \text{ as } n \to \infty. \quad \text{(C-4)}$$

Q.E.D.

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