Generalized Sampling Methods
Problem Set 1 - Review of Linear Algebra

1. Let A be a Hermitian operator on a complex Hilbert space $\mathcal{H}$. Prove that if $\langle x, Ax\rangle = 0$ for all $x \in \mathcal{H}$, then $A = 0$.

2. Define the inner product $\langle x, y\rangle = x^T W y$ for all $x, y \in \mathbb{R}^m$, where W is a symmetric positive definite matrix. Let A be any $m \times m$ matrix. Determine the adjoint A^* with respect to the given inner product.

3. Let T be a linear transformation. Prove that
   (a) $\mathcal{N}(T^*T) = \mathcal{N}(T)$.
   (b) $\mathcal{R}(T^*T) = \mathcal{R}(T^*)$.

4. Let $\{f_n(t)\}_{n=-\infty}^{\infty}$ be a set of functions defined over $[-\pi, \pi]$ as follows:
$$f_n(t) = \begin{cases} t & n = 0 \\ e^{jnt} & n \neq 0. \end{cases}$$
   Let $X : \ell_2 \to L_2[-\pi, \pi]$ be the set transformation corresponding to $\{f_n(t)\}$.
   (a) Find the range space $\mathcal{R}(X)$.
   (b) Find the null space $\mathcal{N}(X)$.

5. Let $\hat{x} = \arg\min_{v \in \mathcal{V}} \|x - v\|^2$ where $\mathcal{V}$ is a closed subspace of a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Show that $\hat{x} = P_{\mathcal{V}} x$ is the orthogonal projection of x onto $\mathcal{V}$.

6. Let $T : \mathcal{H} \to \mathcal{S}$ be a continuous linear transformation with closed range. Prove that the pseudoinverse $T^\dagger$ of T satisfies
   (a) $TT^\dagger = P_{\mathcal{R}(T)}$.
   (b) $T^\dagger T = P_{\mathcal{N}(T)^\perp}$.
   (c) $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$.
   (d) $\mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp$.