TIME DELAY ESTIMATION: COMPRESSED SENSING OVER AN INFINITE UNION OF SUBSPACES

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ABSTRACT

Sampling theorems for signals that lie in a union of subspaces have been receiving growing interest. A recent model that describes analog signals over a union is that of a union of shift-invariant (SI) subspaces. Until now, sampling and recovery algorithms have been developed only for a finite union of SI subspaces. Here we extend this paradigm to a special case of an infinite union, in which the SI subspaces are generated by pulses with unknown delays, taken from a continuous interval. We develop a unified approach to time delay recovery of the pulses, from low rate samples of the signal taken at the lowest possible rate. In particular, we derive sufficient conditions on the pulses and the sampling filters in order to ensure perfect recovery of the signal. We then show that by properly manipulating the low-rate samples, the time delays can be recovered using the well-known ESPRIT algorithm.

***Index Terms*—** sampling, union of subspaces, time delay estimation.

1. INTRODUCTION

One of the traditional assumptions underlying analog-to-digital conversion is that in order to perfectly reconstruct an analog signal from its samples, it must be sampled at the Nyquist rate, i.e. twice its highest frequency. This assumption is required when the only prior knowledge on the signal, is that it is bandlimited. Other priors on the signal structure can lead to more efficient sampling schemes [1].

Recently, there has been growing interest in sampling theorems for signals that lie in a union of subspaces. In [2, 3] necessary and sufficient conditions are derived for a sampling operator to be invertible in such a signal model. However, no concrete sampling methods and recovery algorithms were developed, that allow perfect recovery of the signal from its samples. Various papers treated this signal model on finite-dimensional subspaces. The work in [4, 5] considered the case in which the signal lies in a finite union of finite dimensional spaces. Conditions for unique and stable recovery of the signal from its samples were derived, and efficient recovery algorithms were proposed. Another example is the recent work on signals with finite rate of innovation (FRI) [6, 7]. In that context, efficient schemes were derived for sampling streams of weighted pulses, in which the time delays and amplitudes of each pulse are unknown.

To treat analog signals with infinitely many degrees of freedom, a signal model comprising a union of shift-invariant (SI) subspaces was proposed in [8]. Under this model, each signal lies in a SI subspace spanned by K generating functions with shifts of T, chosen out of a finite set of possible generators. The sampling scheme proposed in [8] is based on passing the signal though a bank of filters whose outputs are sampled at a rate of $1/T$. Using compressed sensing (CS) [9] tools, it is shown that by proper design of the filters, the proposed sampling scheme can achieve a sampling rate of $2K/T$ without knowing the active generators.

In this paper we extend the signal model proposed in [8] to the case of an infinite union of SI subspaces. Similarly, we assume that each signal lies in a SI subspace spanned by K generating functions. However, in our setup, each generating function is defined using a parametric function which depends on a parameter, taken from a continuous interval. Therefore, there are an infinite number of possible generating functions to choose from.

We focus on the case in which the unknown parameters are a set of delays, i.e each generating function is a pulse with unknown delay. For this case we develop an efficient sampling scheme that can perfectly recover such a signal from its samples at the minimal possible rate. To this end we use a sampling scheme similar to [8], based on parallel sampling channels. We derive sufficient conditions on the generating functions and the choice of sampling filters which guarantee unique recovery of the signal's parameters. In particular, at least $2K$ sampling channels are required in order to ensure unique recovery. By appropriate manipulation of the sampling sequences, we formulate our problem within the framework of direction of arrival (DOA) [10] and rely on the ESPRIT algorithm [10], developed in that context, in order to recover the unknown delays.

Although conventional CS tools are not used here, we still consider this work as a part of the CS framework, in which a sparsity prior is exploited in order to compress the signal in the sampling stage, i.e reduce the sampling rate. The sparsity in our model is expressed by the fact that only K generators are active, out of the possible infinite number of generators.

This paper is organized as follows. In Section 2, we present the union of SI subspace model and the special case of unknown delays. A general sampling scheme is proposed in Section 3. Section 4 provides sufficient conditions ensuring a unique recovery. The relation with FRI sampling is discussed in Section 5. Numerical experiments are presented in Section 6.

2. PROBLEM FORMULATION

A signal class that plays an important role in sampling theory are signals in SI spaces. In [8] the standard SI model is extended to a union of SI subspaces. A signal in such a union can be written as

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$$x(t)=\sum_{|k|=K}\sum_{n\in\mathbb{Z}}a_k[n]\,g_k(t-nT), \qquad (1)$$

where $a_k[n]$ are arbitrary sequences in ℓ_2 and $g_k(t)\in L_2$ are known functions. The notation $|k|=K$ denotes a sum over at most K elements, where the assumption in [8] is that there are N possible generators $g_k(t), 1\le k\le N$. Therefore, each signal $x(t)$ lies in a SI subspace spanned by K generators $g_k(t)$ selected from the set of N possibilities. Since we do not know in advance which K are chosen, the class of signals of the form (1) constitute a union of $\binom{N}{K}$ SI subspaces.

Our goal is to extend the results of [8] to an infinite union of SI subspaces. To this end, we consider signals of the form

$$x(t)=\sum_{k=1}^{K}\sum_{n\in\mathbb{Z}}a_k[n]\,g(t-nT,\theta_k), \qquad (2)$$

where $g(t,\theta)\in L_2$ is a parametric function and $\{\theta_k\}_{k=1}^{K}$ are a set of parameters taken from some continuous interval Θ. In the model (2) there are an infinite number of possible generators as the parameter θ varies in the continuous interval Θ. Therefore, (2) describes an infinite union of SI subspaces.

In this paper we consider a special case of (2) where each generating function is a pulse with unknown time delay. More precisely we deal with signals of the form

$$x(t)=\sum_{k=1}^{K}\sum_{n\in\mathbb{Z}}a_k[n]\,g(t-t_k-nT), \qquad (3)$$

where $\tau=\{t_k\}_{k=1}^{K}$ is a set of K distinct unknown time delays in the continuous interval $[0,T)$ and $g(t)\in L_2$ is a known function.

This signal model can describe, for example, a transmission of a pulse $g(t)$ at a constant rate of $1/T$ through a time-variant multipath channel, where t_k and $a_k[n]$ represent the time delay and time-variant gain coefficient of the kth path respectively.

Our goal is to develop an efficient sampling scheme for signals of the form (3), allowing perfect reconstruction of the signal from its samples, when sampling at the lowest possible rate. Since $x(t)$ is defined by the set of delays τ and sequences $a_k[n]$, our problem is equivalent to recovering these parameters from the samples.

3. SAMPLING SCHEME

To sample $x(t)$ we propose a sampling scheme comprised of p parallel channels. In each channel $x(t)$ is pre-filtered using the filter $s_\ell^*(-t)$ and sampled uniformly at times $t=nT$ to produce the sampling sequence $c_\ell[n]$, as depicted in the left-hand side of Fig. 1. We assume that $p\ge K$; exact conditions on the number of sampling channels p will be given in the next section.

It was shown in [11] that the discrete-time Fourier transform (DTFT) of the ℓth sampling sequence can be expressed as

$$\begin{aligned}C_\ell\left(e^{j\omega T}\right) &= \sum_{k=1}^{K}A_k\left(e^{j\omega T}\right)e^{-j\omega t_k}\frac{1}{T}\sum_{m\in\mathbb{Z}}S_\ell^*\left(\omega-\frac{2\pi}{T}m\right)\cdot\\ &\qquad G\left(\omega-\frac{2\pi}{T}m\right)e^{j\frac{2\pi}{T}mt_k},\end{aligned} \qquad (4)$$

where $A_k\left(e^{j\omega T}\right)$ denotes the DTFT of the sequence $a_k[n]$, $G(\omega)$ and $S_\ell(\omega)$ denotes the Fourier transforms of $g(t)$ and $s_\ell(t)$ respectively.

Denoting by $\mathbf{c}\left(e^{j\omega T}\right)$ the length-p column vector whose ℓth element is $C_\ell\left(e^{j\omega T}\right)$ and by $\mathbf{a}\left(e^{j\omega T}\right)$ the length-K column vector whose kth element is $A_k\left(e^{j\omega T}\right)$, we can write (4) in matrix form as

$$\mathbf{c}\left(e^{j\omega T}\right)=\mathbf{M}\left(e^{j\omega T},\tau\right)\mathbf{b}\left(e^{j\omega T}\right), \qquad (5)$$

where $\mathbf{M}\left(e^{j\omega T},\tau\right)$ is a $p\times K$ matrix with ℓkth element

$$\begin{aligned}&\mathbf{M}_{\ell k}\left(e^{j\omega T},\tau\right)=\\ &\qquad \frac{1}{T}\sum_{m\in\mathbb{Z}}S_\ell^*\left(\omega-\frac{2\pi}{T}m\right)G\left(\omega-\frac{2\pi}{T}m\right)e^{j\frac{2\pi}{T}mt_k},\end{aligned} \qquad (6)$$

and

$$\mathbf{b}\left(e^{j\omega T}\right)=\mathbf{D}\left(e^{j\omega T},\tau\right)\mathbf{a}\left(e^{j\omega T}\right), \qquad (7)$$

with $\mathbf{D}\left(e^{j\omega T},\tau\right)$ denoting a diagonal matrix with kth diagonal element $e^{-j\omega t_k}$.

To proceed, we focus our attention on sampling filters $S_\ell(\omega)$ with finite support in the frequency domain, contained in the range

$$\mathcal{F}=\left[\frac{2\pi}{T}\gamma,\frac{2\pi}{T}(p+\gamma)\right], \qquad (8)$$

where $\gamma\in\mathbb{Z}$ is an index which determines the working frequency band $\mathcal{F}$. This choice should be such that it matches the frequency occupation of $g(t)$. Under this choice of filters, the matrix $\mathbf{M}\left(e^{j\omega T},\tau\right)$ can be expressed as

$$\mathbf{M}\left(e^{j\omega T},\tau\right)=\mathbf{W}\left(e^{j\omega T}\right)\mathbf{N}(\tau) \qquad (9)$$

where $\mathbf{W}\left(e^{j\omega T}\right)$ is a $p\times p$ matrix whose ℓmth element is given by

$$\begin{aligned}\mathbf{W}_{\ell m}\left(e^{j\omega T}\right) &= \frac{1}{T}S_\ell^*\left(\omega+\frac{2\pi}{T}(m-1+\gamma)\right)\cdot\\ &\qquad G\left(\omega+\frac{2\pi}{T}(m-1+\gamma)\right)\end{aligned} \qquad (10)$$

and $\mathbf{N}(\tau)$ is a $p\times K$ with mkth element

$$\mathbf{N}_{mk}(\tau)=e^{-j\frac{2\pi}{T}(m-1+\gamma)t_k}. \qquad (11)$$

If $\mathbf{W}\left(e^{j\omega T}\right)$ is stably invertible, then we can define the modified measurement vector $\mathbf{d}\left(e^{j\omega T}\right)$ as

$$\mathbf{d}\left(e^{j\omega T}\right)=\mathbf{W}^{-1}\left(e^{j\omega T}\right)\mathbf{c}\left(e^{j\omega T}\right). \qquad (12)$$

From (5) and (9), this vector satisfies

$$\mathbf{d}\left(e^{j\omega T}\right)=\mathbf{N}(\tau)\,\mathbf{b}\left(e^{j\omega T}\right). \qquad (13)$$

Since $\mathbf{N}(\tau)$ is independent of ω, using the linearity of the DTFT, we can express (13) in the time domain as

$$\mathbf{d}[n]=\mathbf{N}(\tau)\,\mathbf{b}[n], \quad n\in\mathbb{Z}. \qquad (14)$$

The elements of the vectors $\mathbf{d}[n]$ and $\mathbf{b}[n]$ are the discrete time sequences, obtained from the inverse DTFT of the elements of the vectors $\mathbf{b}\left(e^{j\omega T}\right)$ and $\mathbf{d}\left(e^{j\omega T}\right)$ respectively.

Equation (14) describes an infinite set of measurement vectors, each obtained by the same measurement matrix $\mathbf{N}(\tau)$, which depends on the unknown delays τ. This problem is reminiscent of the

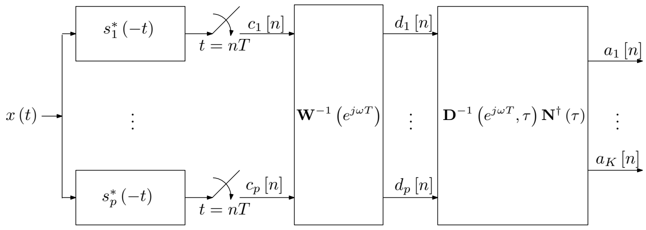


Fig. 1. Sampling and reconstruction scheme

type of problems that arise in DOA estimation. One efficient algorithm for parameter estimation, which was originally developed in that context, is the ESPRIT [10] method. This technique can be used in our setting to recover the unknown delays τ. Therefore, our approach is to first recover τ from the measurements using ESPRIT. After τ is known, the vector $\mathbf{a}\left(e^{j\omega T}\right)$ can be found using the following linear filtering relation

$$\mathbf{a}\left(e^{j\omega T}\right) = \mathbf{D}^{-1}\left(e^{j\omega T}, \tau\right)\mathbf{N}^{\dagger}(\tau)\,\mathbf{d}\left(e^{j\omega T}\right). \tag{15}$$

The resulting sampling scheme is depicted in Fig. 1.

Our last step, therefore, is to derive conditions on the filters $s_\ell^*(-t)$ and the function $g(t)$ in order that the matrix $\mathbf{W}\left(e^{j\omega T}\right)$ is stably invertible. To this end, we can decompose $\mathbf{W}\left(e^{j\omega T}\right)$ as

$$\mathbf{W}\left(e^{j\omega T}\right) = \mathbf{S}\left(e^{j\omega T}\right)\mathbf{G}\left(e^{j\omega T}\right) \tag{16}$$

where $\mathbf{S}\left(e^{j\omega T}\right)$ is a $p \times p$ matrix with ℓmth element

$$\mathbf{S}_{\ell m}\left(e^{j\omega T}\right) = \frac{1}{T}S_\ell^*\left(\omega + \frac{2\pi}{T}(m-1+\gamma)\right) \tag{17}$$

and $\mathbf{G}\left(e^{j\omega T}\right)$ is a $p \times p$ diagonal matrix with mth diagonal element

$$\mathbf{G}_{mm}\left(e^{j\omega T}\right) = G\left(\omega + \frac{2\pi}{T}(m-1+\gamma)\right). \tag{18}$$

Each of these matrices should be stably invertible, leading to the following conditions:

Condition 1 *the function $g(t)$ needs to satisfy*

$$0 < a \le |G(\omega)| \le b < \infty \ a.e\ \omega \in \mathcal{F}. \tag{19}$$

Condition 2 *The filters $s_\ell^*(-t)$ should be chosen in such a way that they form a stably invertible matrix $\mathbf{S}\left(e^{j\omega T}\right)$.*

Examples for choices of filters that satisfy condition (2) are given in [11]. These examples include a bank of complex bandpass filters and sampling channels with different time delays (interleaved sampling).

4. SUFFICIENT CONDITIONS FOR PERFECT RECOVERY

We now derive sufficient conditions for a unique solution to the set of infinite equations (14).

We begin by introducing some notation. Let $\mathbf{d}[\Lambda]$ be the measurement set containing all measurement vectors $\mathbf{d}[\Lambda] = \{\mathbf{d}[n], n \in \mathbb{Z}\}$ and let $\mathbf{b}[\Lambda] = \{\mathbf{b}[n], n \in \mathbb{Z}\}$ be the unknown vector set. We can then rewrite (14) as

$$\mathbf{d}[\Lambda] = \mathbf{N}(\tau)\,\mathbf{b}[\Lambda]. \tag{20}$$

The following proposition provides sufficient conditions for a unique solution to (20). For a proof see [11].

Proposition 1 *If $\left(\bar{\tau}, \bar{\mathbf{b}}[\Lambda] \neq \mathbf{0}\right)$ is a solution to* (20) *and*

$$p > 2K - dim\left(span\left(\bar{\mathbf{b}}[\Lambda]\right)\right) \tag{21}$$

then $\left(\bar{\tau}, \bar{\mathbf{b}}[\Lambda]\right)$ is the unique solution of (20).

The notation span $\left(\bar{\mathbf{b}}[\Lambda]\right)$ is used for the subspace of minimal dimension containing $\bar{\mathbf{b}}[\Lambda]$.

Proposition 1 suggests that a unique solution to (14) is guaranteed, under proper selection of the number of sampling channels p. This parameter, in turn, determines the average sampling rate, given by p/T. Condition (21) depends on dim (span ($\mathbf{b}[\Lambda]$)), which is generally not known in advance. In order to satisfy the uniqueness condition (21) for every signal of the form (3), we must have $p > 2K - 1$ or a minimal sampling rate of $2K/T$. Using the results of [2] it can be shown that this is the theoretical minimum sampling rate required for signals of the form (3).

Our method can achieve the minimum sampling rate suggested by Proposition 1. When $p \ge 2K$ the unknown delays are recovered from the measurement vectors using the ESPRIT method. When dim (span ($\mathbf{b}[\Lambda]$)) $< K$ an additional stage, based on the smoothing technique proposed in [12], has to be performed first. For further details see [11].

5. RELATION TO FRI SAMPLING

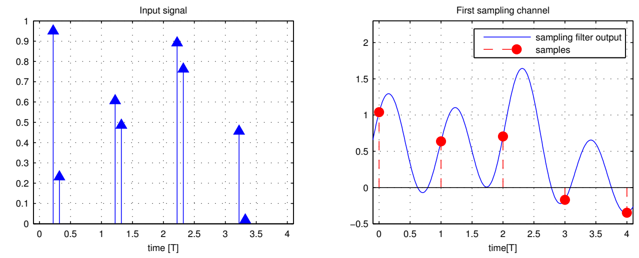
An interesting class of signals that has been treated recently in the sampling literature are FRI signals [6, 7]. Such signals have a finite number of degrees of freedom per unit time, referred to as the rate of innovation. A general form of an FRI signal is given by [6]

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \phi(t - t_n), \tag{22}$$

where $\phi(t)$ is a known function, t_n are unknown time shifts and c_n are unknown weights. Our signal model (3) can be seen as a special case of (22), where additional shift invariant structure is imposed, so that in each period T the time delays are constant.

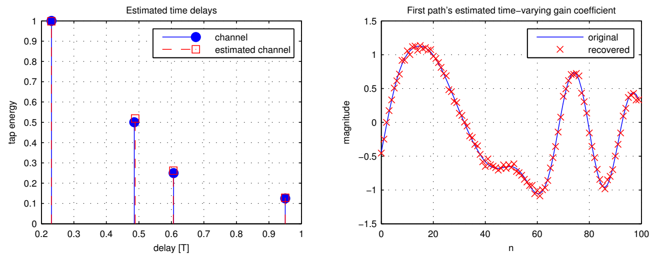
Sampling and reconstruction of infinite length FRI signals was treated in [7]. The method in [7] is based on the use of specific sampling kernels and the function $\phi(t)$ is limited to diracs, differentiated diracs, or compact support pulses. The reconstruction algorithm proposed in [7] is local, namely it recovers the signal's parameters in each time interval separately. Naive use of this approach in our context has two main disadvantages. First, in our method the unknown delays are recovered from all the samples of the signal $x(t)$. A local algorithm is less robust to noise and does not take the shared information into account. In addition, in terms of computational complexity, in our method all the samples are collected to form a finite size correlation matrix, and then the ESPRIT algorithm is applied once. Using the local algorithm requires applying the annihilating filter method, used for FRI recovery, on each time interval over again.

A final disadvantage of the FRI approach is the higher sampling rate required. The theorem for unique recovery of streams of diracs in [7] requires that in each interval of size $2KLT_s$ there are at most K diracs, where L is the support of the sampling kernel and T_s is the sampling period. Since in each interval of size T we have K diracs, the minimal sampling rate is $2KL/T$, which is a factor of L larger than the rate achieved by our scheme. For example, when B-spline kernel is used, it requires $L \ge 2K$.



(a) (b)

Fig. 2. Stream of Diracs. (a) $K = 2$ Diracs per period. (b) First sampling channel output.



(a) (b)

Fig. 3. Channel estimation with $p = 5$ and SNR=20dB. (a) delays recovery. (b) Estimated first path time-varying gain coefficient.

6. NUMERICAL EXPERIMENTS

In the setup of our simulation we chose $g(t) = \delta(t)$. The sampling scheme is composed of a bank of complex bandpass filters, each one of them covering a different frequency band.

We first consider the case where there are $K = 2$ Diracs per period of $T = 1$, as illustrated in Fig. 2(a). In Fig. 2(b), we show the output of the first sampling channel. This example demonstrates the need for the sampling filters when sampling short-length pulses at a low sampling rate. The sampling kernels have the effect of smoothing the short pulses. Consequently, even when the sampling rate is low, the samples contain information about the signal. If we were to sample the signal in Fig. 2(a) directly at a low rate, then we would often obtain only zero samples. In contrast, if we were to sample the signal in Fig. 2(a) directly at a low rate, then we would often obtain only zero samples which contain no information about the signal.

In the next simulation we consider the example described in Section 2 of a time-varying multipath channel. We chose a channel with $K = 4$ paths. The channel's time-varying gain coefficients $a_k[n]$ are modeled as colored random processes with decreasing energies. For the estimation 100 pulses were used and the samples were corrupted by complex-valued Gaussian white noise with an SNR of 20dB. The number of sampling channels is $p = 5$, corresponding to $K + 1$. Although we have seen that $2K$ sampling channels are required for perfect recovery of every signal of the form (3), for some signals, satisfying $\dim(\mathrm{span}(\mathbf{b}[\Lambda])) = K$, lowering the number of channels is possible.

In Fig. 3(a) the original and estimated time delays and averaged energy of the gain coefficients are shown. In Fig. 3(b), we plot the magnitude of the original and estimated gains of the first path versus time. From Figs. 3(a) and (b) it is evident that our method can provide a good estimate of the channel's parameters, even when the samples are noisy, when sampling at the lowest possible rate.

7. CONCLUSION

In this paper we proposed a model for analog signals that lie in a infinite union of SI subspaces. We focused on a time delay estimation problem that can be seen as a special case of this model. We showed that a sampling rate of $2K/T$ is sufficient to guarantee perfect recovery of a signal composed of K delayed pulses per period T, and proposed a recovery algorithm.

While previous works on unions of subspaces have focused mainly on finite unions or finite dimensional subspaces, the problem we treated here can be seen as a first example of a systematic sampling theory for analog signals defined over an infinite union of SI subspaces.

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