Supplementary materials for:
Power-efficient Cameras Using Natural Image Statistics

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1. Proofs

1.1. Optimal bit allocation and relevant proofs

To prove Lemmas 1 and 2, we start with solving the optimization problem in (7) in the general setting.

Consider the following model for energy consumption of a scalar quantizer

$$E(\mathbf{R}) = \sum_{i=1}^{n} 2^{2\gamma R_i}, \tag{1}$$

where $\gamma > 0$ is some parameter corresponding to the particular quantization system.

Similarly, we model the distortion of such a quantizer as

$$D(\mathbf{R}) = \sum_{i=1}^{n} D_i(R_i) = \sum_{i=1}^{n} \sigma_i 2^{-2\alpha R_i}, \tag{2}$$

where $\alpha > 0$ is also a parameter of the system. We will also assume that $\alpha > \gamma$.

For a given energy budget E, the optimal bit allocation is the solution to the following convex optimization problem:

$$\begin{aligned} \min_{R_i} \quad & D(\mathbf{R}) \\ \text{subject to} \quad & E(\mathbf{R}) \leqslant E \\ & R_i \geqslant 0, \forall i. \end{aligned} \tag{3}$$

Lemma 1. *The optimal bit allocation is*

$$R_i = \begin{cases} \frac{1}{2(\alpha+\gamma)} \log_2 \left(\frac{\sigma_i^2}{\eta^*} \frac{\alpha}{\gamma} \right) & \text{if } \eta^* < \sigma_i^2 \frac{\alpha}{\gamma} \\ 0 & \text{if } \eta^* \geqslant \sigma_i^2 \frac{\alpha}{\gamma} \end{cases}, \tag{4}$$

where η^* is chosen such that

$$\sum_{i=1}^{n} \max\left(1, \left(\frac{\sigma_i^2}{\eta^*}\frac{\alpha}{\gamma}\right)^{\frac{\gamma}{\alpha+\gamma}}\right) = E. \tag{5}$$

Proof. The solution of the optimization problem is given using the common "reverse water-filling" approach. Consider the Lagrangian

$$L(\mathbf{R}, \boldsymbol{\lambda}, \eta) = D(\mathbf{R}) - \boldsymbol{\lambda}^T \mathbf{R} + \eta(E(\mathbf{R}) - E). \tag{6}$$

From the KKT conditions we get

$$R_i \geqslant 0, \quad E(\mathbf{R}) = E, \quad \lambda_i^* \geqslant 0, \quad \lambda_i^* R_i = 0, \tag{7}$$
$$-\sigma_i^2 2\alpha 2^{-2\alpha R_i} - \lambda_i^* + \eta^* 2\gamma 2^{2\gamma R_i} = 0. \tag{8}$$

We can get rid of λ_i^* since they are slack variables

$$R_i \geqslant 0, \quad E(\mathbf{R}) = E, \quad \lambda_i^* \geqslant 0, \tag{9}$$
$$R_i \left(\eta^* \gamma 2^{2\gamma R_i} - \sigma_i^2 \alpha 2^{-2\alpha R_i}\right) = 0, \tag{10}$$
$$\eta^* \geqslant \sigma_i^2 \frac{\alpha}{\gamma} 2^{-2R_i(\alpha+\gamma)}. \tag{11}$$

Thus, if $\eta^* < \sigma_i^2 \frac{\alpha}{\gamma}$ then $R_i > 0$ from (11), and from (10) we have that (11) is an equality. If $\eta^* \geqslant \sigma_i^2 \frac{\alpha}{\gamma}$ we get $R_i = 0$ and so we have (4). (5) follows directly from the constraint $E(\mathbf{R}) = E$. □

Note that when $\eta^* < \frac{1}{n^2} \frac{\alpha}{\gamma}$ we get that $R_i > 0$ for all i. The following is a simple corollary.

Lemma 2. *If $E > \frac{\alpha+\gamma}{\alpha-\gamma}(n+1)$ then the distortion of the i-th element is*

$$D_i = \sigma_i^{\frac{2\gamma}{\alpha+\gamma}} \left(\sum_{j=1}^{n} \sigma_j^{\frac{2\gamma}{\alpha+\gamma}}\right)^{\frac{\alpha}{\gamma}} E^{-\frac{\alpha}{\gamma}}, \tag{12}$$

and the total distortion of the quantizer is

$$D_{\text{direct}}(E) = \left(\sum_{i=1}^{n} \sigma_i^{\frac{2\gamma}{\alpha+\gamma}}\right)^{\frac{\alpha+\gamma}{\gamma}} E^{-\frac{\alpha}{\gamma}}. \tag{13}$$

1.2. Proof of Lemmas 1 and 2

Lemma 1 follows directly from (4) when all σ_i are equal and $\gamma = 0.5$.

Lemma 2 follows directly from Lemma 2 by choosing $\gamma = 0.5$.

1.3. Proof of Corollary 1

Using Theorem 1, it is enough to show

$$k\left(\frac{E}{k}\right)^{-2\alpha} < \frac{km}{m-k-1}\frac{k}{m}\left(\frac{E}{m}\right)^{-2\alpha} \iff \tag{14}$$

$$k^{2\alpha} < \frac{k}{m-k-1}m^{2\alpha}, \tag{15}$$

which is true for $\alpha > 0.5$ and $m > k+1$.

1.4. Proof of Corollary 2

In order to show

$$k\left(\frac{E-2n}{k}\right)^{-2\alpha} < \frac{k^2}{m-k-1}\left(\frac{E}{m}\right)^{-2\alpha}, \tag{16}$$

we first note that when $E > 10n$, $\left(\frac{E-2n}{k}\right)^{-2\alpha} < \left(0.8\frac{E}{k}\right)^{-2\alpha}$, so it's enough to show

$$\left(0.8\frac{E}{k}\right)^{-2\alpha} < \frac{k}{m}\left(\frac{E}{m}\right)^{-2\alpha} < \frac{k}{m-k-1}\left(\frac{E}{m}\right)^{-2\alpha} \tag{17}$$

$$1 < \left(\frac{m}{k}\right)^{2\alpha-1} 0.8^{2\alpha}, \tag{18}$$

which is true when $m > 2k$ and $\alpha \geqslant 0.85$.

1.5. Proof of Corollary 3

We start with providing a lower bound for $\mathsf{mmse}_{\mathsf{random}}(E)$. As stated, we're interested in the function

$$G(x,z) = 1 - \frac{F(x,z)}{4xz}, \tag{19}$$

where function $F(x,z)$ is defined in [4] as

$$F(x,z) = \left(\sqrt{x(1+\sqrt{z})^2+1} - \sqrt{x(1-\sqrt{z})^2+1}\right)^2, \tag{20}$$

with $z = \beta^{-1} = n/m$ and

$$x = (n\sigma_{\mathbf{w}})^{-2} = \frac{1}{m}\frac{1}{\sum\sigma_i^2}\left(\frac{E}{m}\right)^{2\alpha} \tag{21}$$

in our case. We first note that for $z > 0$, $\frac{\partial}{\partial z}G(x,z) > 0$, hence we can assume that $z = 1$ and so

$$G(x,z) = 1 - \frac{4x+2-2\sqrt{4x+1}}{4x}. \tag{22}$$

On the other hand we note that under the conditions of Lemma 2 we can use the integral bound to get

$$D_{\mathsf{direct}}(E) < c_1(n+1)^{\frac{\alpha-\gamma}{\gamma}}E^{-\frac{\alpha}{\gamma}}, \tag{23}$$

where $c_1 = \left(\frac{\alpha+\gamma}{\alpha-\gamma}\right)^{\frac{\alpha+\gamma}{\gamma}}$. Thus, it is enough to show

$$c_1(n+1)^{\frac{\alpha-\gamma}{\gamma}}E^{-\frac{\alpha}{\gamma}} < \frac{1}{n}\left(1 - \frac{4x+2-2\sqrt{4x+1}}{4x}\right). \tag{24}$$

Since in our model $\gamma = 0.5$ we plug it into the above and get

$$\frac{4c_1}{\sum\sigma_i^2}\frac{(n+1)^{2\alpha-1}}{m^{2\alpha}} + 2 < 2\sqrt{4x+1} \tag{25}$$

$$\left(\frac{4c_1}{\sum\sigma_i^2}\frac{(n+1)^{2\alpha-1}}{m^{2\alpha}}\right)^2 + 4\left(\frac{4c_1}{\sum\sigma_i^2}\frac{(n+1)^{2\alpha-1}}{m^{2\alpha}}\right) + 4 < 16x+4 \tag{26}$$

$$\left(\frac{c_1}{\sum\sigma_i^2}\frac{(n+1)^{2\alpha-1}}{m^{2\alpha}}\right)^2 + \left(\frac{c_1}{\sum\sigma_i^2}\frac{(n+1)^{2\alpha-1}}{m^{2\alpha}}\right) < \frac{1}{m}\frac{1}{\sum\sigma_i^2}\left(\frac{E}{m}\right)^{2\alpha} \tag{27}$$

$$\frac{c_1^2}{\sum\sigma_i^2}m^{-2\alpha+1}(n+1)^{4\alpha-2} + c_1 m(n+1)^{2\alpha-1} < E^{2\alpha} \tag{28}$$

$$\frac{c_1^2\beta^{-2\alpha+1}}{\sum\sigma_i^2}n^{-2\alpha+1}(n+1)^{4\alpha-2} + c_1\beta n(n+1)^{2\alpha-1} < E^{2\alpha}. \tag{29}$$

Note that for $n > 10$ and $\alpha \geqslant 0.85$ there's a constant $c_2 < 1.01$ such that the above holds when

$$E > \left(\frac{c_2c_1^2\beta^{-2\alpha+1}}{(n+1)\sum\sigma_i^2} + c_1\beta\right)^{\frac{1}{2\alpha}}(n+1), \tag{30}$$

and the result follows immediately.

2. ADC distortion

In this section we show simulation results to motivate our ADC distortion model

$$D(R) = c\sigma^2 2^{-2\alpha R}. \tag{31}$$

We show empirically that for $\alpha \approx 0.85$, this function approximates optimal (according to [2]) uniform quantization of Gaussian and Laplacian signals in the bit rate range $0-15$, which covers most ADC implementations [3].

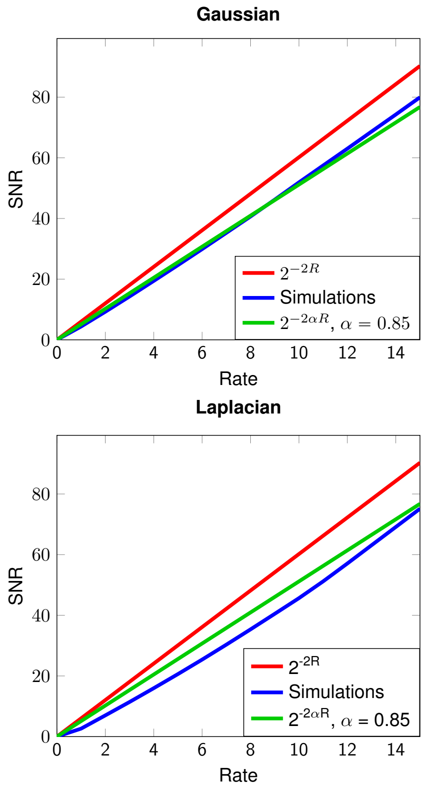

Figure 1. SNR of fixed rate uniform quantization of a Gaussian with unit variance, where the support of the quantizer was chosen according to [2], compared with our distortion model ($2^{-2\alpha R}$) and the distortion model used in [1].

In Fig. 1 we plot our distortion model, the model $c2^{-2R}$ used by Goyal *et al*. in [1], and simulation results were the samples were quantized uniformly with supports (*i.e*. left and right quantizer bounds) calculated according to [2].

Note that no choice of constant c in the distortion model $c2^{-2R}$ would correctly model the simulation results, since the slopes of the model and the simulation SNR are different. We believe our model approximates the simulations closely enough in the bit rate range of interest.

References

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