Compressive Sensing with Unknown Parameters

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***Abstract*—This work addresses target detection from a set of compressive sensing radar measurements corrupted by additive white Gaussian noise. In previous work, we studied target localization using compressive sensing in the spatial domain, i.e., the use of an undersampled MIMO radar array, and proposed the Multi-Branch Matching Pursuit (MBMP) algorithm, which requires knowledge of the number of targets. Generalizing the MBMP algorithm, we propose a framework for target detection, which has several important advantages over previous methods: (*i*) it is fully adaptive; (*ii*) it addresses the general multiple measurement vector (MMV) setting; (*iii*) it provides a finite data records analysis of false alarm and detection probabilities, which holds for any measurement matrix. Using numerical simulations, we show that the proposed algorithm is competitive with respect to state-of-the-art compressive sensing algorithms for target detection.**

I. Introduction

In general, target localization with radar consists of two stages: detection and estimation [1]. The detection process establishes the presence of a target in a prescribed resolution cell. This process is characterized by two parameters [2]: probability of false alarm (P_{FA}) and probability of detection (P_D). The goal is to maximize the probability of detection for a fixed level of false alarms. Classical detection is a process that inherently relies on a single target point of view. Detection performance is usually represented by receiver operating curves (ROC). Estimation builds on detection by seeking to improve the accuracy of localization for detected targets. In principle, estimation adopts a multi-target viewpoint. For example, maximum likelihood estimation accounts for the interaction between closely spaced targets, hence localization performance is generally improved compared to detection.

Undersampling, inherent to compressive sensing [3], causes ambiguities, i.e., interaction between targets may give rise to false peaks. Classical detection, in which resolution bins are tested one-by-one for the presence of a target, is then not suitable for compressive sensing scenarios. In contrast, classical estimation algorithms can handle compressive sensing scenarios, but are not equipped to handle unknown number of targets. In this work, we seek compressive sensing methods that bridge between detection and estimation in the sense of supporting detection of multiple targets, while accounting for mutual effects between the targets. In particular, we focus on the application of multiple-input multiple-output (MIMO) radar [4] to the detection and estimation of targets from direction-of-arrival (DOA) measurements. In other words, we seek to recover information about the scene by compressive sensing methods without a priori information about the number of targets.

The sparsity assumption, which permeates compressive sensing, blurs the distinctions between detection and estimation. In compressive sensing, the main unknown is the signal support, which must be recovered. The recovery of the support is essentially an estimation problem, but it requires decisions on zero and non-zero elements, which is a detection problem. In the MMV setting, we aim to recover a sparse signal matrix $\mathbf{X}$ from noisy compressive measurements $\mathbf{Y} = \mathbf{AX} + \mathbf{E}$, where $\mathbf{A}$ is a measurement matrix and $\mathbf{E}$ represents the noise. It has been shown in [5] that, under certain conditions on the matrix $\mathbf{A}$ and the sparsity K, the optimal method to recover $\mathbf{X}$ is by solving the nonconvex noisy l_0-norm problem, i.e., $\min_{\mathbf{X}} \|\mathbf{Y} - \mathbf{AX}\|_F^2 + \nu \|\mathbf{X}\|_0$, where $\|\mathbf{X}\|_0$ counts the number of non-zero norm *rows* of $\mathbf{X}$. The *regularization parameter* ν governs the trade-off between fitting the data ($\|\mathbf{Y} - \mathbf{AX}\|_F^2$) and reducing the solution cardinality ($\|\mathbf{X}\|_0$), hence it can be set based on prior information, for example, the number of targets K or the noise level σ^2. In a radar detection problem, where there is no a priori information on the number of targets or on the noise level, setting the parameter ν is non-trivial.

In [6], the authors addressed target detection in the so-called single measurement vector (SMV) setting (when a single snapshot is available, i.e., the matrix $\mathbf{Y}$ reduces to a column vector) using the Complex Approximate Message Passing (CAMP) algorithm, which aims to solve the l_1-norm convex approximation of the l_0-norm problem. Building on an asymptotic analysis of CAMP, the authors proposed a detection framework, where K and σ^2 are unknown. In addition to [6], several other authors proposed compressed sensing algorithms to address this type of problem, but either assumed the noise level and/or the number of targets to be known, or required as input the regularization parameter ν.

In this work, extending our previously proposed Multi-Branch Matching Pursuit (MBMP) algorithm [7], we present a framework for target detection which has several important advantages over previous methods: (*i*) it is fully adaptive, i.e., it does not require prior knowledge of the number of targets K or noise level σ^2; (*ii*) it addresses the general MMV setting (rather than the SMV setting addressed in [6]); (*iii*) it provides an analysis of false alarm and detection probabilities that holds for finite data records and any measurement matrix $\mathbf{A}$ (rather than an asymptotic analysis based on a Gaussian measurement matrix $\mathbf{A}$ as in [6]). The proposed algorithm is

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tested in a spatial compressive sensing-radar setting, where undersampling in space enables considerable savings in the number of array elements, while still enabling high resolution localization. However, we point out that the work in this paper does not rely on the particular structure of the spatial compressive sensing matrix $\mathbf{A}$, therefore our findings are relevant to other radar compressive sensing scenarios, for instance, radar compressive sensing in the time domain [8] or in the frequency domain [6].

The following notation is used: boldface denotes matrices (uppercase) and vectors (lowercase); for a matrix $\mathbf{X}$, $[\mathbf{X}]_{i,j}$ denotes the element at i-th row and j-th column, $\mathbf{X}(i,:)$ denotes the i-th row, and $\mathrm{vec}(\mathbf{X})$ produces a column vector by stacking the columns of $\mathbf{X}$; $(\cdot)^*$ denotes the complex conjugate operator; $(\cdot)^T$ denotes the transpose operator; $(\cdot)^H$ is the complex conjugate-transpose operator, $(\cdot)^\dagger$ is the pseudo-inverse operator, $\|\mathbf{X}\|_F$ is the Frobenius norm of $\mathbf{A}$. The symbol "$\otimes$" denotes the Kronecker product. Given a set S of indices, $|S|$ denotes its cardinality, $\mathbf{A}_S$ is the sub-matrix obtained by considering only the columns indexed in S, and $\Pi^{\perp}_{\mathbf{A}_S} \triangleq \mathbf{I} - \mathbf{A}_S\mathbf{A}_S^\dagger$ is the orthogonal projection matrix onto the null space of $\mathbf{A}_S^H$. Given two set of indexes, S and S', $S \backslash S'$ contains the indexes of S which are not present in S'. We define a K-sparse matrix to have only K non-zero norm rows, $\|\mathbf{X}\|_0$ counts the number of non-zero norm rows of $\mathbf{X}$ and we say that a set of indices S is the support of the matrix $\mathbf{X}$ if the set S collects the indexes of non-zero norm rows of $\mathbf{X}$. For a vector, $\mathbf{x} \sim \mathcal{CN}(\mu, \mathbf{C})$ means that $\mathbf{x}$ has a circular symmetric complex normal distribution with mean μ and covariance matrix $\mathbf{C}$. $F_{a,b}$ denotes an F distribution with a numerator degrees of freedom and b denominator degrees of freedom, while $F'_{a,b}(\eta)$ denotes a non-central F distribution with a numerator degrees of freedom and b denominator degrees of freedom, and non-centrality parameter η. Finally, for a probability density function X, the right-tail probability at γ is denoted by $P = Q_X(\gamma)$, while $\gamma = Q_X^{-1}(P)$ denotes its inverse function.

II. System Model

In our spatial compressive sensing framework for MIMO radar, N sensors collect a finite train of P pulses sent by M transmitters and returned from K stationary targets. We assume that transmitters and receivers form linear arrays of aperture $Z/2$: the m-th transmitter is at position ξ_m, $\xi_m \in [0, Z/2]\ \forall m$, on the x-axis; the n-th receiver is at position ζ_n, $\zeta_n \in [0, Z/2]\ \forall n$. Targets are assumed in the far-field, meaning that a target's aspect angle[1] θ_k is constant across the array. The purpose of the system is to detect the presence of targets and determine their DOA angles. Following [7], the DOA estimation problem can be cast in a sparse localization framework. Neglecting the discretization error, it is assumed that the target possible locations comply with a grid of G points $\phi_{1:G}$ (with $G \gg K$). By defining the $MN \times G$ matrix $\mathbf{A} = [\mathbf{a}(\phi_1), \ldots, \mathbf{a}(\phi_n)]$ where $\mathbf{a}(\theta) \triangleq \mathbf{c}(\theta) \otimes \mathbf{b}(\theta)$ with $\mathbf{b}(\theta) = \left[\exp\left(-j2\pi\frac{Z\theta}{\lambda}\zeta_1\right), \ldots, \exp\left(-j2\pi\frac{Z\theta}{\lambda}\zeta_N\right)\right]^T$ the transmitter steering vector and $\mathbf{c}(\theta) = \left[\exp\left(-j2\pi\frac{Z\theta}{\lambda}\xi_1\right), \ldots, \exp\left(-j2\pi\frac{Z\theta}{\lambda}\xi_M\right)\right]^T$ the receiver steering vector (see [7] for further details), the signal model is expressed:

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E} \tag{1}$$

where $\mathbf{E} \in \mathbb{C}^{MN \times P}$ represents the noise, which is assumed to be independent and identically distributed (i.i.d.) complex Gaussian, i.e., $\mathrm{vec}(\mathbf{E}) \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$, with unknown σ^2. The unknown matrix $\mathbf{X} \in \mathbb{C}^{G \times P}$ contains the targets locations and gains. Zero rows of $\mathbf{X}$ correspond to grid points without a target. The problem (1) is sparse in the sense that $\mathbf{X}$ has only $K \ll G$ non-zero rows.

The properties of the measurement matrix $\mathbf{A}$ are governed by the grid-points $\phi_{1:G}$ and by the sensors' number and positions $\xi_{1:M}$ and $\zeta_{1:N}$. Since the sensors' positions are assumed random (described by the probability density functions (pdf) $p(\xi)$ and $p(\zeta)$), the elements of the measurement matrix $\mathbf{A}$ are also random. In the following, we chose $p(\xi)$ and $p(\zeta)$ to be uniform distributions, and we chose $\phi_{1:G}$ as a uniform grid of $2\lambda/Z$-spaced points in the range $[-1, 1]$.

III. Detection using MBMP

The goal of the detection problem is to identify the non-zero norm rows of $\mathbf{X}$ (i.e., its support) given the measurements $\mathbf{Y}$ in (1). It has been shown [5] that, under certain conditions on the matrix $\mathbf{A}$ and the sparsity K, the matrix $\mathbf{X}$ in (1) can be recovered by solving the nonconvex noisy l_0-norm problem:

$$\min_{\mathbf{X}} \|\mathbf{Y} - \mathbf{A}\mathbf{X}\|_F^2 + \nu \|\mathbf{X}\|_0 \tag{2}$$

where ν is a *regularization parameter* which depends on prior information, e.g. the number of targets K or the noise level σ^2. In the following, we first review the MBMP algorithm [7] to solve (2) assuming the number of targets K is known. Then, we detail the proposed framework for target detection in which we extend the MBMP algorithm to handle an unknown number of targets K.

To present the proposed framework, it is instructive to reformulate problem (2) in terms of the support S of the solution $\mathbf{X}$. Problem (2) is equivalent to

$$\min_{S} \left\|\Pi^{\perp}_{\mathbf{A}_S}\mathbf{Y}\right\|_F^2 + \nu |S| \tag{3}$$

in the sense that, given the optimal solution S^{opt} of (3), the non-zero norm rows of the optimal solution to (2) are $\mathbf{X}_{S^{opt}} = \mathbf{A}^\dagger_{S^{opt}}\mathbf{Y}$, and, vice versa, given the optimal solution $\mathbf{X}^{opt}$ of (2) the optimal solution S^{opt} of (3) is the support of $\mathbf{X}^{opt}$. The reformulation follows by noticing that for a fixed support S, the non-zero rows of the solution to problem (2) are $\mathbf{X}_S = \mathbf{A}_S^\dagger\mathbf{Y}$. Plugging this into the fit term, we have $\mathbf{Y} - \mathbf{A}\mathbf{X} = \mathbf{Y} - \mathbf{A}_S\mathbf{A}_S^\dagger\mathbf{Y} = \Pi^{\perp}_{\mathbf{A}_S}\mathbf{Y}$, where the last step follows from the definition of $\Pi^{\perp}_{\mathbf{A}_S}$. Finally, for a fixed support S, the cardinality term is simply $\|\mathbf{X}\|_0 = |S|$, thus obtaining the reformulation.

[1] θ_k is defined as the sine of the k-th target's DOA angle.

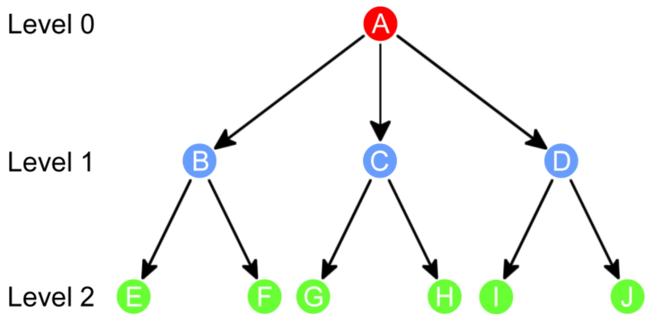

Figure 1. Graph of MBMP algorithm for a branch vector $\mathbf{d} = [3, 2]$.

The regularization parameter ν in (3) affects only the cardinality of the solution's support S. Therefore, for a fixed cardinality, the support can be found from

$$S_j = \arg\min_{S} \left\|\Pi^{\perp}_{\mathbf{A}_S}\mathbf{Y}\right\|_F^2 \;\; \text{s.t.} \;\; |S| = j \tag{4}$$

which does not require knowledge of ν. In general, solving this problem requires combinatorial complexity. In the following we describe how the solution (with and without knowledge of the sparsity K) can be efficiently approximated by using the MBMP algorithm.

A. MBMP with known sparsity K

When the sparsity K is known, we aim to solve (4) with $j = K$. The MBMP algorithm approximates the solution of this problem by generalizing the matching pursuit strategy. It is possible to visualize the MBMP algorithm as a tree of nodes, as shown in Fig. 1, where each node is populated with a provisional support, such that node's level indicates the cardinality of its associated support, e.g., all nodes at level 2 contain supports with cardinality $|S| = 2$. The process stops when all nodes at level K have been populated, and the provisional support, among nodes at level K, achieving the minimum data-fit ($\left\|\Pi^{\perp}_{\mathbf{A}_S}\mathbf{Y}\right\|_F^2$) is elected as the solution. The structure of the tree depends on the number of levels, K, and on the number of allowed branches at each level (assumed constant for nodes within the same level of the tree). The structure can be specified using a vector $\mathbf{d}$ (referred to as *branch vector*) of length K: d_i represents the number of branches at level $i-1$. For instance, the tree in Fig. 1 has $\mathbf{d} = [3, 2]$ (node 0 has $d_1 = 3$ branches, and each node at level 1 possesses $d_2 = 2$ branches). See [7] for more details.

B. MBMP with unknown sparsity K

We now extend the MBMP to handle an unknown number of targets K. While K is unknown, we do assume that an upper limit $\bar{K}$ is available. Note that K has to be lower than the number of rows of $\mathbf{A}$, i.e., $K < MN$, since otherwise the uniqueness of the solution is not guaranteed, even in a noiseless scenario [5].

The MBMP algorithm is extended as follows: (1) MBMP is applied using the upper limit $\bar{K}$, to obtain $\bar{K}$ supports, $S_1, \ldots, S_{\bar{K}}$; (2) by relying on results from detection theory, we choose one among the $\bar{K}$ supports. The idea is to approximate the solution to (4) for $j = 1$ to $\bar{K}$, to obtain $\bar{K}$ supports, $S_1, \ldots, S_{\bar{K}}$, with cardinality ranging from 1 to $\bar{K}$. This entire process can be efficiently approximated by using the MBMP algorithm to solve (4) for $j = \bar{K}$. The provisional support achieving the minimum data-fit ($\left\|\Pi^{\perp}_{\mathbf{A}_S}\mathbf{Y}\right\|_F^2$), among nodes at level j, can be used to approximate S_j. In the following, we analyze the detection process, meaning determining which of the supports $S_1, \ldots, S_{\bar{K}}$ is the true one.

The idea of the detection process is to check whether a test statistic is higher than a threshold. For a given support S, consider the data model $\mathbf{Y} = \mathbf{A}_S\tilde{\mathbf{X}} + \mathbf{E}$, where $\mathbf{A}_S \in \mathbb{C}^{MN\times|S|}$ ($MN > |S|$) is a known measurement matrix of rank $|S|$, $\tilde{\mathbf{X}} \in \mathbb{C}^{|S|\times P}$ is a matrix of unknown parameters, and the noise term $\mathbf{E} \in \mathbb{C}^{MN\times P}$ satisfies $\mathrm{vec}(\mathbf{E}) \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$, where σ^2 is unknown. The goal of the detection process is to decide which of the rows of $\tilde{\mathbf{X}}$ are non-zero. The SMV setting ($P = 1$) for a real measurement matrix was addressed in [2, Pag. 345]. In the following theorem we address the general MMV ($P \geq 1$) complex case:

Theorem 1 ***(GLRT for MMV Model - σ^2 Unknown)*** *The hypothesis testing problem of whether a specific row i of $\tilde{\mathbf{X}}$ is non-zero given that the other rows $l \neq i$ are known to be non-zero, is formulated*

$$\begin{aligned}\mathcal{H}_{i,0} &: \left\|\tilde{\mathbf{X}}(i,:)\right\| = 0, \sigma^2 > 0 \\ \mathcal{H}_{i,1} &: \left\|\tilde{\mathbf{X}}(i,:)\right\| \neq 0, \sigma^2 > 0\end{aligned} \tag{5}$$

Then:
1) The Generalized Log-likelihood Ratio Test (GLRT) for deciding $\mathcal{H}_{i,1}$ is

$$T_i(\mathbf{Y}, S) = \frac{\left\|\hat{\mathbf{X}}(i,:)\right\|_2^2}{\frac{\left\|\Pi^{\perp}_{\mathbf{A}_S}\mathbf{Y}\right\|_F^2}{MN-|S|}\left[\left(\mathbf{A}_S^H\mathbf{A}_S\right)^{-1}\right]_{i,i}} > \gamma \tag{6}$$

where $\hat{\mathbf{X}} \triangleq \mathbf{A}_S^{\dagger}\mathbf{Y}$ is the MLE of $\tilde{\mathbf{X}}$ under $\mathcal{H}_{i,1}$.
2) The exact probability of false alarm for finite data records is given by

$$P_{FA} = Q_{F_{2P,2P(MN-|S|)}}(\gamma). \tag{7}$$

3) If the elements of the i-th row of $\tilde{\mathbf{X}}$ have constant modulo, i.e., $\left|\left[\tilde{\mathbf{X}}\right]_{i,t}\right| = \beta$ for every t, the exact probability of detection for finite data records is given by

$$P_D = Q_{F'_{2P,2P(MN-|S|)}(\eta_i)}(\gamma) \tag{8}$$

where the non-centrality parameter is given by $\eta_i = P\beta^2/\left(\frac{\sigma^2}{2}\left[\left(\mathbf{A}_S^H\mathbf{A}_S\right)^{-1}\right]_{i,i}\right)$.

Proof: See Appendix. ∎

The theorem is applicable when we want to test the support $S \triangleq S_{K-1} \cup i$ for $i = 1, \ldots, G$ and $i \notin S_{K-1}$, where S_{K-1} is a subset of the true support with cardinality $K-1$. The case when i matches the remaining non-zero row index

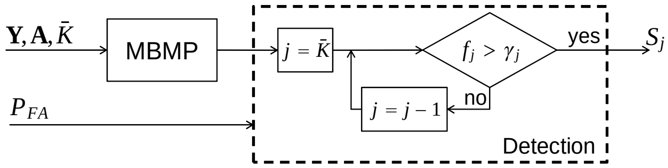

Figure 2. Block diagram of the proposed architecture for CS-radar detection.

(i.e., the true support index not in S_{K-1}) is equivalent to the $\mathcal{H}_{i,1}$ hypothesis. Furthermore, the theorem also applies when we want to test for $S \triangleq S^{true} \cup i$ for $i = 1, \ldots, G$ and $i \notin S^{true}$, where S^{true} is the true support: this case matches the $\mathcal{H}_{i,0}$ hypothesis. The importance of this theorem is that it characterizes the test statistic $T_i\left(\mathbf{Y}, S\right)$ and its distribution under $\mathcal{H}_{i,0}$. In other words, the theorem tells us that, assuming S contains the true support S^{true}, the test statistics for a row's index i with only noise (i.e., $i \in S \backslash S^{true}$) follows the $F_{2P,2P(MN-|S|)}$ distribution.

In MBMP, when the correct support is estimated and we add a new index to the support, the new index is the one that correlates the most with the noise realization $\mathbf{E}$ among the $G-K$ columns with indices outside the support. Therefore the test statistic, at cardinality $K+1$, is $\max_{i \notin S^{true}} T_1\left(\mathbf{Y}, i \cup S^{true}\right)$, and its distributions is the maximum among $G-K$ random variables, each having a $F_{2P,2P(MN-K-1)}$ distribution. The dependency among these random variables is hard to analyze, hence a closed form seams difficult to obtain. In the numerical results, we show that a reasonable assumption is to approximate them as independent. Using this assumption, the test statistic $\max_{i \notin S^{true}} T_1\left(\mathbf{Y}, i \cup S^{true}\right)$ is distributed as the maximum of $G-K$ i.i.d. $F_{2P,2P(MN-K-1)}$ random variables (which for ease of reference we denote as $g\left(K+1\right)$). Therefore, the threshold for the test statistic at cardinality j can be set as

$$\gamma_j \triangleq Q^{-1}_{g(j)}\left(P_{FA}\right) \tag{9}$$

for a given probability of false alarm P_{FA}. Notice that the threshold γ_j depends on the cardinality of the support under test. Focusing at cardinality j, the detection process aims to detect whether all the indices in S_j are non-zero, given that the indices in S_{j-1} are non-zero (i.e., contain targets plus noise). This translates to check whether the minimum test statistic among the indices of S_j not contained in S_{j-1}, i.e.,

$$f_j \triangleq \min_{i \in S_j \backslash S_{j-1}} T_i\left(\mathbf{Y}, S_j\right) \tag{10}$$

is higher then the given threshold γ_j. A high value of the metric f_j indicates that S_j is likely to be the true support, whereas a small value of this metric indicates that S_j is likely to contain at least one index that contains only noise.

Summarizing, in order to detect the support among $S_1, \ldots, S_{\bar{K}}$, obtained from MBMP, we let $j = \bar{K}$, compute γ_j using (9) and f_j using (10), and check whether $f_j > \gamma_j$. In this case the process stops and we decide for $S_{\bar{K}}$ as the support of the detected targets; otherwise we iterate $j = j - 1$ until $f_j > \gamma_j$ or $j = 0$ (which corresponds to the case of

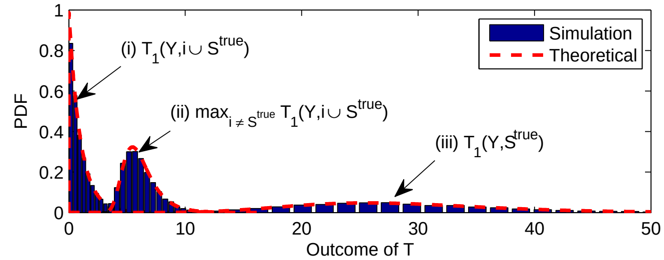

Figure 3. Simulated and theoretical pdf of the test statistic $T_i\left(\mathbf{Y}, S_j\right)$. System settings: $G = 181$, $K = 8$, $P = 1$, $M = N = 7$, SNR $= 15$db.

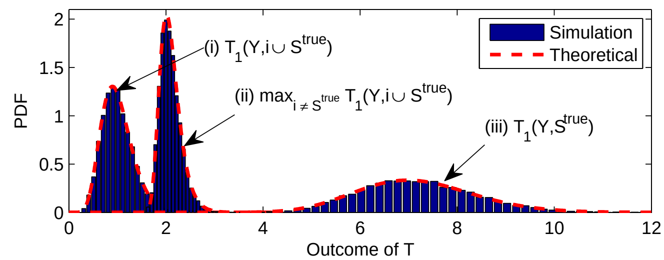

Figure 4. Simulated and theoretical pdf of the test statistic $T_i\left(\mathbf{Y}, S_j\right)$. System settings: $G = 181$, $K = 8$, $P = 10$, $M = N = 7$, SNR $= 8$db.

detecting no targets). Fig. 2 depicts a block diagram of the process.

IV. Numerical Results

In this section, we present numerical results to demonstrate the potential of the MBMP algorithm for detection using the spatial compressive sensing signal model (1). To produce each figure, we first draw a random realization of the array sensors' positions, which is maintained fixed throughout independent Monte-Carlo realizations of the noise ($\mathrm{vec}\left(\mathbf{E}_p\right) \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right)$ $\forall p$) and of the targets' positions and responses ($x_{k,p} = \exp\left(-j\varphi_k\right), \forall p,\ p = 1, ..., P$, and $\varphi_k \sim \mathcal{U}\left(0, 2\pi\right)$ $\forall k,\ k = 1, ..., K$). The signal-to-noise ratio (SNR) is defined as $10 \log_{10} \sigma^2$.

We start by analyzing the simulated and theoretical probability density functions of the test statistic (6), over the ensemble of random targets and noise, in three different cases: (*i*) the test statistic $T_1\left(\mathbf{Y}, i \cup S^{true}\right)$, where i is a *fixed* index outside the support; (*ii*) the maximum test statistic over indices i outside the support, i.e., $\max_{i \notin S^{true}} T_1\left(\mathbf{Y}, i \cup S^{true}\right)$; (*iii*) the test statistic $T_1\left(\mathbf{Y}, S^{true}\right)$ for the first index in the true support. We set $K = 8$ targets and an array aperture of $Z = 180\lambda$ (where λ is the transmitted signal wavelength). The grid size is $G = 181$ grid-points. In Fig. 3, we plot the results for an SMV scenario ($P = 1$), while Fig. 4 shows an MMV scenario ($P = 10$). The theoretical distributions for cases (*i*) and (*iii*) (used in (7) and (8), respectively) are exact. It can be seen that the theoretical distributions in case (*ii*), obtained by assuming independent random variables, closely matches the Monte-Carlo simulations results.

We next investigate the performance of the proposed detection scheme based on MBMP (i.e., Fig. 2). We compare

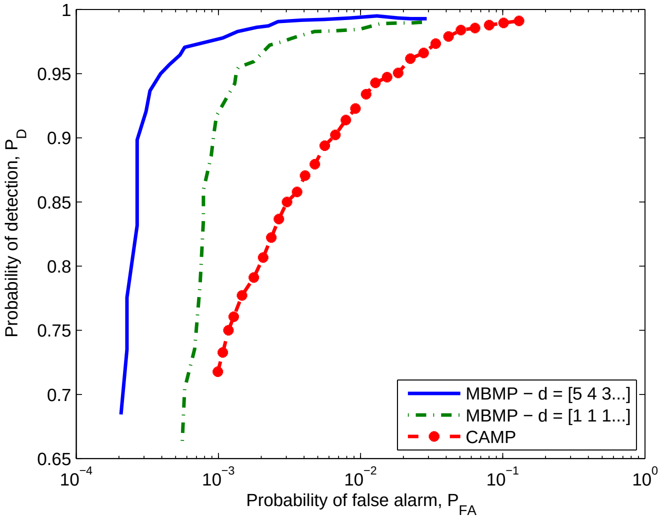

Figure 5. ROC curves for the two MBMP architectures and CAMP. System settings: $G = 251$, $K = 9$, $P = 1$, $M = N = 7$, SNR = 15db.

it with the detection scheme proposed in [6] based on the CAMP algorithm. The MBMP algorithm's complexity can be adjusted by varying the branch vector $\mathbf{d}$. To give an idea of how the complexity of the algorithm affects the performance, the MBMP is run with two different choices of the branch vector $\mathbf{d}$, both of length $\bar{K} = 20$: the first uses $\mathbf{d} = [5, 4, 3, 2, 1, 1, \ldots, 1]$, while the second uses $\mathbf{d} = [1, 1, \ldots, 1]$. We set $K = 9$ targets and an array aperture of $Z = 250\lambda$. The grid size is $G = 181$ grid-points. In order to compare with the architecture proposed in [6], we simulated an SMV scenario. Denoting with S^{true} the true support, and with $\widehat{S}$ the support determined by a detection algorithm, we define $\left(\widehat{S} \backslash S^{true}\right) / (G - K)$ as the empirical probability of false alarm and $\left(\widehat{S} \cap S^{true}\right) / K$ as the empirical probability of detection.

In Fig. 5, we plot the ROC curve for the different algorithms. It can be seen how the proposed architecture is able to achieve higher probability of detection for the same false alarm probability as compared to the CAMP algorithm. Further tests would be required to establish which algorithm performs better and under which conditions.

V. Conclusions

In this paper, we address target detection from compressive sensing radar measurements corrupted by additive white Gaussian noise. By taking a detection point-of-view, we generalize the MBMP algorithm proposed in [7], such that the number of targets is now one of the unknowns. The resulting architecture for the sparse recovery problem is fully adaptive, i.e., it does not require knowledge of the number of targets or the noise variance. In addition, we analyze the false alarm and detection probabilities for the proposed architecture. Using numerical simulations, the proposed algorithm is compared against a state-of-the-art compressive sensing algorithm for target detection.

VI. Appendix

Proof of Theorem 1 - Because of the limited space, here we provide a sketch of the proof. Following the same steps as [2, Pag. 371-372], it can be shown that the GLRT can be written as $\left(\hat{\sigma}_0^2 - \hat{\sigma}_1^2\right) / \hat{\sigma}_1^2$ where $\hat{\sigma}_l^2$ is the MLE of the noise level σ^2 under $\mathcal{H}_{i,l}$ for $l = 1, 2$. Extending [9, Pag. 176-177], it can be shown that $\hat{\sigma}_1^2 = \left\|\Pi_{\mathbf{A}_S}^{\perp}\mathbf{Y}\right\|_F^2 / MNP$, while, following similar steps as [2, Appendix 7B], $\hat{\sigma}_0^2 - \hat{\sigma}_1^2$ reduces to $\left\|\hat{\mathbf{X}}(i,:)\right\|_2^2 / MNP\left[\left(\mathbf{A}_S^H\mathbf{A}_S\right)^{-1}\right]_{i,i}$ where $\hat{\mathbf{X}} \triangleq \mathbf{A}_S^{\dagger}\mathbf{Y}$ is the MLE of $\tilde{\mathbf{X}}$ under $\mathcal{H}_{i,1}$. Therefore, (6) can be written as $(MN - |S|)\left(\hat{\sigma}_0^2 - \hat{\sigma}_1^2\right)/\hat{\sigma}_1^2$, which, except from a scaling factor, is the GLRT. Now, consider the random variable $\left\|\hat{\mathbf{X}}(i,:)\right\|_2^2 / \left(\left[\left(\mathbf{A}_S^H\mathbf{A}_S\right)^{-1}\right]_{i,i} \frac{\sigma^2}{2}\right)$. Under $\mathcal{H}_{i,1}$ and assuming $\beta = \left|\left[\tilde{\mathbf{X}}\right]_{i,t}\right|$, for every t, it has a non-central Chi-Squared distribution with $2P$ degrees of freedom and non-centrality parameter given by η_i, and, under $\mathcal{H}_{i,0}$, it has a central Chi-Squared distribution with $2P$ degrees of freedom. Equivalently, it can be shown that the random variable $\left\|\Pi_{\mathbf{A}_S}^{\perp}\mathbf{Y}\right\|_F^2 / \frac{\sigma^2}{2}$ has a Chi-Squared distribution with $2P\left(MN - |S|\right)$ degrees of freedom under either $\mathcal{H}_{i,0}$ and $\mathcal{H}_{i,1}$. These two random variables are independent. Notice that (6) can be obtained by normalizing each of the random variables by the number of degrees of freedom, and by taking their ratio. By definition, (6) follows a $F'_{2P,2P(MN-|S|)}(\eta_i)$ distribution, under $\mathcal{H}_{i,1}$, and a $F_{2P,2P(MN-|S|)}$ distribution, under $\mathcal{H}_{i,0}$.

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