V. Concluding Remarks

In this note we have presented two algorithms for discrete time adaptive control which utilize prior information about the plant, including some known poles and zeros. If the plant is completely unknown, the algorithms are identical to those proposed by Goodwin *et al.* in [7]. However, the algorithms presented here have better transient performance and faster convergence rate when the system is partially known. In future work we will show the added robustness margins obtainable from our scheme.

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A Parametrization for Model Reference Adaptive Pole Placement

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Abstract—**A new parametrization for a linear system is presented. This parametrization is the basis for two approaches to design an adaptive controller for pole placement. One approach is based on parameter estimation and requires sufficient excitation, while the other, the model reference adaptive pole placement, uses a reference model and an augmented error. It is shown that the two distinct approaches result in identical error terms.**

I. Introduction

A major limitation of direct adaptive control is the limited numer of control problems to which it can be applied. The source of the problem is in what is often referred to in the literature as the "parametrization problem." In effect, it is the problem of presenting the plant model through the controller parameters so that these parameters can be directly estimated. So far, only the model matching control has a complete solution. Namely, an adaptive algorithm (MRAC) resulting in global stability and error convergence without requiring sufficient excitation. But this solution requires some very restrictive assumptions which limit its application. In some recent results (see [1], [2], [4], and [8]) a parametrization was presented and studied for the pole placement problem. However, the convergence of the resulting controller is dependent on sufficient excitation in the input and the use of the "block invariant" concept for the controller parameters. This requires that the controller parameter be updated only every predecided number of samples—for further information the reader is referred to the above references.

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In this note we present a new parametrization for the pole placement problem. This parametrization generalizes the result presented in [1] and can be presented in a form compatible with the parameter estimation form presented in [2]. Together with sufficiently exciting input and the "block invariant" concept, convergence is guaranteed. However, the same parametrization can be used, together with a reference model, as a basis for what we referred to as model reference adaptive pole placement (MRAPP). This approach has the potential of providing a controller which does not require sufficient excitation or the restrictive assumptions of the MRAC. It should be pointed out that the two approaches described in this note—one as a parameter estimation problem and the second the MRAPP with augmented error—result in fact in identical error terms to drive the corresponding adaptation algorithms.

II. The Basic Parametrization

Consider a discrete-time single-input single-output system modeled by a strictly proper transfer function

$$T_p(z)=\frac{r(z)}{p(z)} \tag{2.1}$$

where

$$r(z)=\sum_{i=1}^{n} r_i z^{n-i} \tag{2.2}$$

and

$$p(z)=z^n+\sum_{i=1}^{n} p_i z^{n-i}. \tag{2.3}$$

Using only the assumption that n is known, it is possible to describe the relationship between system input $u(t)$ and output $y(t)$ through an n-dimensional canonical system

$$\begin{aligned} x_p(t+1)&=A_0x_p(t)+b_0u(t)-b_0fx_p(t)\\ y(t)&=c_0x_p(t)-cx_p(t)\end{aligned} \tag{2.4}$$

where (A_0, b_0, c_0) is any n-dimensional canonical system preselected so that A_0 is strictly stable and f and c are row vectors of unknown parameters. The above follows directly from the following lemma which is a variation on results appearing in [5] and [6].

Lemma 2.1: There exist row vectors f and c such that

$$(c_0-c)[Iz-(A_0-b_0f)]^{-1}b_0=T_p(z). \tag{2.5}$$

Proof: Since (A_0, b_0) is a reachable pair there exists a row vector f such that

$$\text{char.pol. } (A_0-b_0f)=p$$

and the vector

$$p(z)[Iz-(A_0-b_0f)]^{-1}b_0$$

spans the $n-1$ order polynomial subspace.

Hence, there exists a row vector $\tilde{c}$ such that

$$p(z)\tilde{c}[Iz-(A_0-b_0f)]^{-1}b_0=r(z)$$

then choosing $c=c_0-\tilde{c}$ completes the proof. □

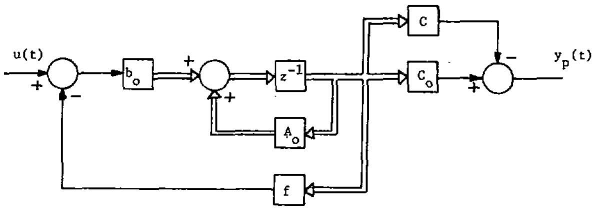

Fig. 1. Block diagram of system's basic parametrization.

Remark 2.1: We note in (2.4) the feedback and feedforward form in which f and c enter (see Fig. 1). The feedback part, with f, determines the poles of the system while the feedforward part, with c, the zeros. This observation is the basis for the model reference adaptive pole placement (MRAPP) configuration we introduce in the sequel.

We would like to use feedback control to place the system poles in some desired locations represented by the polynomial

$$p^*(z)=z^n+\sum_{i=1}^{n} p_i^* z^{n-i}. \tag{2.6}$$

For simplicity, we choose A_0 so that

$$\text{char.pol. } (A_0)=p^*(z) \tag{2.7}$$

then, clearly, if we could take the control $u^* = fx_p + v$, with v external command input, the pole placement would have been accomplished. However, we do not know f and cannot measure x_p, hence, we attempt to generate an estimate of the product fx_p. To do that, use is made of sensitivity function filters of the form

$$\partial_u^i(t+1)=A_i\partial_u^i(t)+b_iu(t)$$

$$\partial_y^i(t+1)=A_i\partial_y^i(t)+b_iy(t) \qquad i=1,\ 2 \tag{2.8}$$

where $(A_i,\ b_i)$ are reachable pairs and

$$\text{char.pol. } (A_i)=q^i(z) \qquad i=1,\ 2 \tag{2.9}$$

with

$$q^i(z)=z^n+\sum_{j=1}^{n} q_j^i z^{n-i} \tag{2.10}$$

stable polynomials.

Now we make use of the following lemma which again is an adaptation of results in [5] and [6].

Lemma 2.2: There exist vectors k_u, k_y, l_u, and l_y so that

$$fx_p(t)=k_u^T\partial_u^1(t)+k_y^T\partial_y^1(t) \qquad (\epsilon) \tag{2.11}$$

$$cx_p(t)=l_u^T\partial_u^2(t)+l_y^T\partial_y^2(t) \qquad (\epsilon) \tag{2.12}$$

where we write $(\ ) = (\ )\ (\epsilon)$ if each element of the difference $(\ ) - (\ )$ is a linear combination of decaying exponentials.

Proof: Since we assumed that $(A_0 - b_0f,\ b_0,\ c_0 - c)$ is canonical and $(A_1,\ b_1)$ is reachable, there exists a nonsingular matrix Q_1 and a vector h_1 so that

$$Q_1b_0=b_1$$

and

$$Q_1(A_0-b_0f)Q_1^{-1}=A_1+h_1c_1$$

where

$$c_1=(c_0-c)Q_1^{-1}.$$

Then for $x_1(t) = Q_1x_p(t)$, (2.4) and the above imply

$$x_1(t+1)=A_1x_1(t)+b_1u(t)+h_1y(t). \tag{2.13}$$

Now if $E_u^1 = [\partial_u^1, A_1\partial_u^1, \cdots, A_1^{n-1}\partial_u^1]B^{-1}$ and $E_y^1 = [\partial_y^1, A_1\partial_y^1, \cdots, A_1^{n-1}\partial_y^1]B^{-1}$ where B is the controllability matrix of $(A_1,\ b_1)$, then from (2.8) $E_u^1(t+1) = A_1E_u^1(t) + Iu(t)$ and $E_y^1(t+1) = A_1E_y^1(t) + Iy(t)$. This and (2.13) clearly imply that if $x = E_u^1b_1 + E_y^1h_1$

$$x_1(t)-x(t)=(A_1)^t(x_1(0)-x(0))$$

or, since A_1, is stable

$$x_1(t)=x(t) \qquad (\epsilon)$$

$$\begin{aligned} fx_p(t)&=fQ_1^{-1}x(t) \qquad (\epsilon) \text{ and} \\ &=fQ_1^{-1}[\partial_u^1,\ A_1\partial_u^1,\ \cdots,\ A_1^{n-1}\partial_u^1]B^{-1}b_1 \\ &\quad +fQ_1^{-1}[\partial_y^1,\ A_1\partial_y^1,\ \cdots,\ A_1^{n-1}\partial_y^1]B^{-1}h_1 \qquad (\epsilon). \end{aligned}$$

Then (2.11) follows directly for $k_u = F^TB^{-1}b_1$ and $k_y = F^TB^{-1}h_1$ where F is the observability matrix of (fQ_1^{-1}, A_1)[1]. A similar derivation will result in l_u and l_y to prove (2.12). □

Substitution of (2.11) and (2.12) into (2.4) will result in

$$x_p(t+1)=A_0x_p(t)+b_0u(t)-b_0k_u^T\partial_u^1(t)-b_0k_y^T\partial_y^1(t)$$

$$y(t)=c_0x_p(t)-l_u^T\partial_u^2(t)-l_y^T\partial_y^2(t) \qquad (\epsilon) \tag{2.14}$$

which together with (2.8) is a $5n$-dimensional realization of the transfer function (2.1) through the $4n$ parameters k_u, k_y, l_u, l_y. This parametrization provides a basis for the model reference adaptive pole placement scheme described next.

III. Model Reference Adaptive Pole Placement

Starting with (2.8) and (2.14) we proceed in two philosophically different approaches resulting, however, in the same error terms. The first, as a parameter estimation problem as was done in [1] and [2].

For this we define

$$z(t+1)=A_0z(t)+b_0u(t)$$

$$\bar{y}(t)=c_0z(t) \tag{3.1}$$

$$H(t+1)=A_0H(t)+b_0[\partial_u^1(t)^T,\ \partial_y^1(t)^T]$$

$$\varphi^1(t)^T=c_0H(t) \tag{3.2}$$

then (2.14) implies that $x_p(t) = z(t) - H(t)\left[\begin{smallmatrix}k_u\\ k_y\end{smallmatrix}\right]$ (ϵ) thus

$$y(t)=\bar{y}(t)-\varphi(t)^T\partial_0 \qquad (\epsilon)$$

or

$$\varphi(t)^T\partial_0=\bar{y}(t)-y(t) \qquad (\epsilon) \tag{3.3a}$$

where

$$\varphi(t)^T=[\varphi^1(t)^T,\ \partial_u^2(t)^T,\ \partial_y^2(t)^T] \tag{3.3b}$$

and

$$\partial_0^T=[k_u^T,\ k_y^T,\ l_u^T,\ l_y^T]. \tag{3.3c}$$

Using the delay operator notation used in [1] and [2] where $Dx(t) = x(t-1)$ and denoting for any polynomial $\alpha(D) = \alpha_0D^n + \alpha_1D^{n-1} + \cdots + \alpha_n$

$$\bar{\alpha}(D)=\alpha_0+\alpha_1(D)+\cdots+\alpha_{n-1}D^{n-1}+\alpha_nD^{n2} \tag{3.4}$$

[1] Note that in fact $k_u^T = f_1Q_1^{-1}$.

[2] If $\beta(D) = \beta_{n-m}D^m + \beta_{n-m-1}D^{m-1} + \cdots + \beta_n$, $m < n$ then $\overline{\beta(D)} = D^{n-m}(\beta_{n-m} + \beta_{n-m-1}D + \cdots + \beta_nD^m)$.

we get for (2.14) and (2.8) combined

$$y(t)=\frac{\bar{r}^*(D)}{\bar{p}^*(D)}\left[u(t)-\frac{\bar{k}(D)}{\bar{q}^1(D)}u(t)-\frac{\bar{h}(D)}{\bar{q}^1(D)}y(t)\right]-\left[\frac{\bar{\delta}(D)}{\bar{q}^2(D)}u(t)+\frac{\bar{\gamma}(D)}{\bar{q}^2(D)}y(t)\right] \quad (\epsilon) \quad (3.5)$$

where we used (2.8)–(2.10) to write

$$\frac{\bar{k}(D)}{\bar{q}^1(D)}u(t)=\frac{k(D^{-1})}{q^1(D^{-1})}u(t)=k_u^T\partial_u^1(t),\ \frac{\bar{h}(D)}{\bar{q}^1(D)}y(t)=\frac{k(D^{-1})}{q^1(D^{-1})}y(t)=k_y^T\partial_y^1(t)$$

$$\frac{\bar{l}(D)}{\bar{q}^2(D)}u(t)=\frac{\delta(D^{-1})}{q^2(D^{-1})}u(t)=l_u^T\partial_u^2(t),\ \frac{\bar{\gamma}(D)}{\bar{q}^2(D)}y(t)=\frac{\gamma(D^{-1})}{q^2(D^{-1})}y(t)=l_y^T\partial_y^2(t).$$

Equation (3.5) can be rewritten in the form

$$\begin{aligned}&[\bar{r}^*(D)\bar{q}^2(D)\bar{k}(D)+\bar{p}^*(D)q^1(D)\bar{\delta}(D)]u(t)\\&\quad+[\bar{r}^*(D)\bar{q}^2(D)\bar{h}(D)+\bar{p}^*(D)\bar{q}^1(D)\bar{\gamma}(D)]y(t)\\&=\bar{r}^*(D)\bar{q}^1(D)\bar{q}^2(D)u(t)-\bar{p}^*(D)\bar{q}^1(D)\bar{q}^2(D)y(t) \quad (\epsilon)\end{aligned} \quad (3.6)$$

which is compatible with (2.11) in [2]. Using the system equation in the form

$$\bar{p}(D)y(t)=\bar{r}(D)u(t)$$

in (3.6) will result in the following polynomial equation for the polynomials $\bar{k}(D)$, $\bar{h}(D)$, $\bar{\delta}(D)$, $\bar{\gamma}(D)$

$$\begin{aligned}&\bar{r}^*(D)\bar{q}^2(D)[\bar{p}(D)\bar{k}(D)+\bar{r}(D)\bar{h}(D)]\\&\quad+\bar{p}^*(D)\bar{q}^1(D)[\bar{p}(D)\bar{\delta}(D)+\bar{r}(D)\bar{\gamma}(D)]\\&=\bar{q}^1(D)\bar{q}^2(D)[\bar{r}^*(D)\bar{p}(D)-\bar{p}^*(D)\bar{r}(D)].\end{aligned} \quad (3.7)$$

It can readily be observed that $\bar{k}(D)$, $\bar{h}(D)$ which satisfy

$$\bar{p}(D)\bar{k}(D)+\bar{r}(D)\bar{h}(D)=\bar{q}^1(D)[\bar{p}(D)-\bar{p}^*(D)] \quad (3.8)$$

and $\bar{\delta}(D)$, $\bar{\gamma}(D)$ which satisfy

$$\bar{p}(D)\bar{\delta}(D)+\bar{r}(D)\bar{\gamma}(D)=\bar{q}^2(D)[\bar{r}^*(D)-\bar{r}(D)] \quad (3.9)$$

are also solutions of (3.7). Since we have assumed that $\bar{r}(D)$ and $\bar{p}(D)$ [or $r(D^{-1})$ and $p(D^{-1})$] are relatively prime, $\bar{k}(D)$ and $\bar{h}(D)$ are unique solutions of (3.8) and $\bar{\delta}(D)$, $\bar{\gamma}(D)$ are unique solutions of (3.9). However, for them to be unique solutions of (3.7) as well we must have that *$\bar{r}^*(D)\bar{q}^2(D)$ and $\bar{p}^*(D)\bar{q}^1(D)$ are also relatively prime.*

With the above conditions satisfied the approach proposed in [1] and further studied in [2] can be adopted. The control to be used is

$$u(t)=\hat{k}_u(t)^T\partial_u^1(t)+\hat{k}_y(t)^T\partial_y^1(t)+v(t) \quad (3.10)$$

where $\hat{k}_u(t)$ and $\hat{k}_y(t)$ are estimates of k_u and k_y generated by the parameter estimation algorithm. However, in the controller, they are not updated every sampling period but only after every finite number of steps. This scheme, termed in [2] "block invariant," together with sufficient exciting input $v(t)$, will guarantee parameter convergence with the desired pole placement accomplished. For detailed discussion and analysis of the scheme, the reader is referred to [2].

We would like, however, to point out that in any parameter estimation algorithm deployed here the error term used will be [see (3.1)]

$$\begin{aligned}\tilde{e}&=\varphi(t)^T\hat{\partial}(t)-\bar{y}(t)+y(t)\\&=\varphi(t)^T\tilde{\partial}(t) \quad (\epsilon)\end{aligned} \quad (3.11)$$

where $\hat{\partial}(t)$ is an estimate of ∂_0 and $\tilde{\partial}(t)=\hat{\partial}(t)-\partial_0$.

The second approach is based on providing a reference model and having the controlled system output track the reference model output. Since we are interested in pole placement the reference model will have the desired poles and a term reflecting estimates of system zeros (see Fig. 2). Specifically, the reference model will be

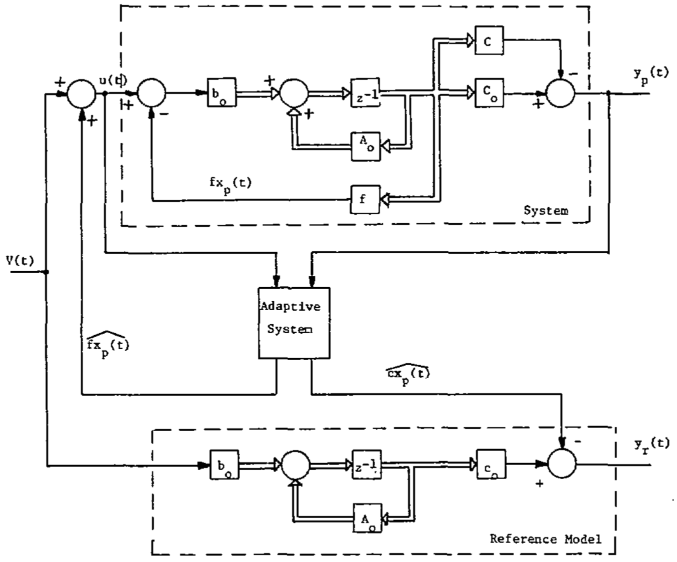

Fig. 2. The general configuration of model reference adaptive pole placement (MRAPP).

$$x_r(t+1)=A_0x_r(t)+b_0v(t)$$

$$y_r(t)=c_0x_r(t)-\hat{l}_u(t)^T\partial_u^2(t)-\hat{l}_y(t)^T\partial_y^2(t) \quad (3.12)$$

where the sum $\hat{l}_u(t)^T\partial_u^2(t)+\hat{l}_y(t)^T\partial_y^2(t)$ is an estimate of the feedforward term $cx_p(t)$ in (2.4).

Now, using the control defined in (3.10) together with (2.12) and (3.12), we can readily verify that the tracking error

$$e(t)=y(t)-y_r(t) \quad (3.13)$$

satisfies

$$e(t)=c_0x_e(t)+\bar{l}_u(t)^T\partial_u^2(t)+\bar{l}_y(t)^T\partial_y^2(t) \quad (\epsilon)$$

$$x_e(t+1)=A_0x_e(t)+b_0(\bar{k}_u(t)^T\partial_u^1(t)+\bar{k}_y(t)^T\partial_y^1(t)). \quad (3.14)$$

We make now use of the "augmented error" $\bar{e}(t)$ introduced by Monopoli [7] and defined through

$$\bar{e}(t)=e(t)+[\hat{k}_u(t)^T,\ \hat{k}_y(t)^T]\varphi^1(t)-c_0z_1(t) \quad (3.15)$$

$$z_1(t+1)=A_0z_1(t)+b_0(\hat{k}_u(t)^T\partial_u^1(t)+\hat{k}_y(t)^T\partial_y^1(t)) \quad (3.16)$$

where $\varphi^1(t)$ is defined in (3.2).

From (3.2), (3.14) and (3.16) one readily observes that

$$e(t)=c_0z_1(t)-[k_u^T,\ k_y^T]\varphi^1(t)+\bar{l}_u(t)^T\partial_u^2(t)+\bar{l}_y(t)^T\partial_y^2(t) \quad (\epsilon)$$

hence, substitution in (3.15) results in

$$\bar{e}(t)=[\bar{k}_u(t)^T,\ \bar{k}_y(t)^T]\varphi^1(t)+\bar{l}_u(t)^T\partial_u^2(t)+\bar{l}_y(t)^T\partial_y^2(t) \quad (\epsilon). \quad (3.17)$$

The above form is the basis for many of the model reference adaptive control (MRAC) algorithms. To complete the algorithm description the parameter adjustment law must be specified. One could use the projection algorithm (as was done in [6] for the model matching problem) or a recursive least squares algorithm. At this point we do not have a global stability proof for the approach with either adjustment law.

Remark 3.1: By comparing (3.11) to (3.17) and recalling (3.3b) and (3.3c) we clearly observe that, here, the estimation error $\tilde{e}(t)$ is identical to the augmented error $\bar{e}(t)$.

Remark 3.2: Every step of the parametrization described here carries through in continuous system resulting in the same equations (3.11) or (3.17).

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Nonparametric Fitting of Multivariate Functions

TOMASZ GAŁKOWSKI AND LESZEK RUTKOWSKI

Abstract—**A nonparametric algorithm, based on the multiple Fejér's sum, for fitting of multivariate functions is proposed. The mean square error convergence and the speed of convergence are investigated.**

I. INTRODUCTION

This note is concerned with nonparametric estimation of the function R in the model

$$y_r = R(x_r) + Z_r, \qquad r = 1, \cdots, n \tag{1}$$

relating the scalar output y_r, the input x_r (d-vector) selected by the experimeter, and the independent white measurement noise Z_r

$$EZ_r = 0, \; EZ_r^2 = \delta_r^2 < \delta^2, \qquad r = 1, \cdots, n. \tag{2}$$

Consider the d-dimensional space $Q_d = \{x : |x| \leqslant \pi\}$, x is the d-dimensional vector, $|x| = \max(|x_1|, |x_2|, \cdots, |x_d|)$. We shall use the complete orthonormal system defined on Q_d

$$\left\{ \begin{array}{c} g_k(x) = \dfrac{1}{(2\pi)^{d/2}} e^{ikx}, \; k = (k_1, \cdots, k_d), \\ k_j = 0, \; \pm 1, \; \pm 2, \; \cdots, j = 1, \; \cdots, d, \\ kx = \displaystyle\sum_{j=1}^{d} k_j x_j . \end{array} \right\}.$$

In the multidimensional case the Nth Fejér's sum σ_N of the function R is defined by

$$\sigma_N(x) = \sum_{|k| \leqslant N} \beta_k b_k e^{ikx}, \tag{3}$$

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where

$$\beta_k = \prod_{j=1}^{d} \left(1 - \frac{|k_j|}{N+1}\right), \tag{4}$$

$$b_k = \frac{1}{(2\pi)^d} \int_{Q_d} R(x) e^{-ikx}\, dx. \tag{5}$$

Generally, one has some problems with the convergence of the multidimensional expansions. Even in the unidimensional case, the condition for uniform convergence of different series are satisfied if the expanded functions are sufficiently smooth (see [7]). Fortunately, from the theory of multiple Fourier series (see, e.g., [4, Sect. 15.11]), it follows that the Nth Fejér's sum σ_N (3) of any continuous function R is uniformly convergent to R. This nice property is the reason for construction of the algorithm derived from multiple Fourier series. The algorithm presented in Section II is an extension to the multidimensional case of that proposed in [6]. It should be noted that the rule of selection of signals x_i by the experimenter is much more convenient from a computational point of view than that suggested in [5].

II. ALGORITHM

Let $n^{1/d}$ be an integer and $i_j = 1, \cdots, n^{1/d}, j = 1, \cdots, d$. Partition the interval $\langle -\pi, \pi\rangle$ on the jth axis into $n^{1/d}$ subsets $\Delta x_{j,i_j}$. Define the following Cartesian product:

$$\Delta x_{1,i_1} \times \Delta x_{2,i_2} \times \cdots \times \Delta x_{d,i_d} = Q_{d,i_1 \cdots i_d} = Q_{d,i}.$$

Let $Q_{d,i} \wedge Q_{d,l} = \emptyset$ for $i \neq l$ and $\cup Q_{d,i} = Q_d$. In the sequel we shall denote $\|\cdot\|$ the Euclidean norm and $\Delta \bar{x}_{j,i_j}$ the length of the interval $\Delta x_{j,i_j}$, $i_j = 1, \cdots, n^{1/d}, j = 1, \cdots, d$.

Relation (1) can be expressed in another way

$$y_i = R(x_i) + Z_i \tag{6}$$

where the outputs $y_i \in R$, and the inputs x_i are selected so that $x_i \in Q_{d,i}$.

Fig. 1 presents the partition of $Q_d = [-\pi, \pi]^2$ for $n = 16$ and equal intervals $\Delta x_{j,i_j}$, $j = 1, 2$, $i_j = 1, 2, 3, 4$. We propose the following algorithm

$$\hat{R}_n(x) = \sum_{|k| \leqslant N} \beta_k \hat{b}_k e^{ikx} \tag{7}$$

where

$$\hat{b}_k = \frac{1}{(2\pi)^d} \sum_{|i| \leqslant n^{1/d}} y_i \int_{Q_{d,i}} e^{-ikt}\, dt \tag{8}$$

and N depends on the number of observations n, i.e., $N = N(n)$. From properties of the multiple Fejér's sum (see [4]) it follows that the computation of algorithm (7) can be realized by

$$\hat{R}_n(x) = \frac{1}{\pi^d} \sum_{|i| \leqslant n^{1/d}} y_i \int_{Q_{d,i}} \Phi_N(t-x)\, dt$$

where

$$\Phi_N(t-x) = \prod_{j=1}^{d} \left[1/2 + \sum_{l_j=1}^{N} \left(1 - \frac{l_j}{N+1}\right) \cos l_j(t_j - x_j) \right].$$

III. CONVERGENCE

Theorem (Mean Square Error Convergence): If

$$|R(x) - R(y)| \leqslant L\|x - y\| \tag{9}$$