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# Generalized Sample Hold Functions—Frequency Domain Analysis of Robustness, Sensitivity, and Intersample Difficulties

Arie Feuer and Graham C. Goodwin

*Abstract*—**Over the past 5 years, there has been substantial interest in the use of generalized sample hold functions for control. In this correspondence, we use a tool, which is novel in this context, namely, Amplitude Modulation Theory. We employ this tool to analyze the quantitative and qualitative features of the intersample behavior in a frequency domain setting. This offers new theoretical and practical insights into the method. Our conclusion is that the perceived benefits come at substantial cost which makes its practical use questionable.**

## I. Introduction

In practice, most sampled data controllers are implemented using zero-order hold functions. This raises the issue, however, of whether or not any advantage can be gained from the use of other types of hold functions.

This problem has been widely studied under the heading of Generalized Sample Hold Functions. For example, Kabamba [1] establishes that by using these functions, one can achieve many desirable properties for the associated sampled response, including: arbitrary input–output transfer functions up to the order of the original system, simultaneous pole assignment for several systems, optimal noise rejection, and decoupling. Many related results have been studied in the literature [2–[6]. Several papers, however, such as [5], have questioned the intersample behavior.

Indeed, this has become a significant focus of recent literature on sampled data control for the zero-order hold case [7]–[9].

Here we present a novel approach to the analysis of the intersample response of sampled data control systems based on frequency domain modulation theory. The use of this approach is a natural consequence of the periodic nature of the hold function. We apply this methodology to the case of generalized sample hold functions. This mode of analysis gives new frequency domain insights into the performance of generalized sample hold controllers, including a thorough analysis of the intersample behavior and associated robustness and sensitivity issues.

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We view the action of the generalized sample hold function as a form of amplitude modulation. When viewed in the frequency domain, this means that additional high-frequency components are generated centered on multiples of the fundamental generalized sample hold frequency (i.e., the sampling frequency). When the output is sampled, these high-frequency components are folded back into the base-band frequency range resulting in a modified sampled frequency response. The action of the generalized sampled hold policy is then clearly revealed. Specifically, one only need choose the various continuous time frequency components so that when they are superimposed at the base band by the sampling action, they result in the desired sampled frequency response. On the other hand, this line of reasoning also reveals the inherent drawbacks of this approach. In particular, one sees that the continuous time output response necessarily contains nonnegligible high-frequency components centered on multiples of the sampling frequency. Moreover, the more one demands of the generalized sampled hold function, the larger these high-frequency components must be. This difficulty becomes more pronounced if one considers the input signal. Since, in most cases, the gain of a process will decrease with frequency, a necessary consequence of having nonnegligible output power at high frequencies is that there must be even greater input power at the same frequencies. Moreover, this high-frequency input power must increase with the sampling rate.

Turning to the issue of sensitivity, the above arguments suggest that any disturbance injected into the loop will result in nonnegligible high-frequency continuous time output components and large-amplitude high-frequency continuous time input components. Thus, if sensitivity is defined in terms of the magnitude of the disturbance response (irrespective of frequency), then it is clear that the use of generalized sample hold functions will generally give very poor sensitivity performance. For example, even a constant disturbance leads to large high-frequency continuous time components which do not decay with time!

On the issue of robustness, we see that the fidelity of the folded base-band response relies upon the fidelity of the system's high-frequency continuous time response. In practice, the high-frequency response of a system is usually difficult to accurately define. For example, unmodeled dynamics such as small time delays, high-frequency poles, etc., will significantly change the continuous time system's high-frequency response, and this will almost certainly destroy the resultant folded behavior. This must be considered as unacceptable robustness behavior.

In this correspondence, we make the above arguments precise by quantifying the intersample behavior using novel frequency domain arguments based on amplitude modulation theory.

## II. The Generalized Sample Hold Function Approach to Control

In the literature, there are slight variants of the generalized sample hold approach. These all lead to basically the same end result. Thus, to be specific, we will follow the approach in [1].

Consider a linear time-invariant single input–single output continuous time system given by

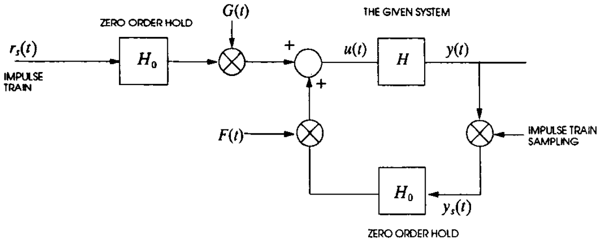


Fig. 1. The GSHF control scheme.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) \tag{2.1}$$

where $x$ is an $n$-dimension state vector.

We define the control law as follows (see Fig. 1):

$$u(t) = F(t)y(k\Delta) + G(t)r(k\Delta) \qquad k\Delta \le t < (k+1)\Delta \tag{2.2}$$

where $\Delta$ is a predetermined time interval [to be used as the sampling period for (2.1)] and $F(t)$, $G(t)$ are periodic functions of period $\Delta$. The signals $y(t)$ and $r(t)$ are the output and reference input, respectively, and $y_s(t)$ and $r_s(t)$ are their impulse sampled forms.

The sampled state response of (2.1) is given by

$$x((k+1)\Delta) = e^{A\Delta}x(k\Delta) + \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta-\tau)}Bu(\tau)\,d\tau. \tag{2.3}$$

Substituting (2.2) into (2.3) and using the periodicity of $F(t)$ and $G(t)$ gives

$$x((k+1)\Delta) = \left[e^{A\Delta} + \int_0^{\Delta} e^{A(\Delta-\tau)}BF(\tau)\,d\tau\, C\right]x(k\Delta) + \int_0^{\Delta} e^{A(\Delta-\tau)}BG(\tau)\,d\tau\, r(k\Delta). \tag{2.4}$$

Assuming (2.1) is controllable, it can readily be shown that $F(t)$ and $G(t)$ can be chosen so that

$$\int_0^{\Delta} e^{A(\Delta-\tau)}BF(\tau)\,d\tau = f \tag{2.5}$$

$$\int_0^{\Delta} e^{A(\Delta-\tau)}BG(\tau)\,d\tau = g \tag{2.6}$$

where $f$ and $g$ are arbitrary $n$ vectors. Using (2.5), (2.6) in (2.4) gives

$$x((k+1)\Delta) = [e^{A\Delta} + fC]x(k\Delta) + gr(k\Delta). \tag{2.7}$$

Combining (2.7) with (2.1), we see that the sampled input–output frequency response is given by

$$H_S(\omega) = C(e^{j\omega\Delta} - e^{A\Delta} - fC)^{-1}g = \frac{C(e^{j\omega\Delta}I - e^{A\Delta})^{-1}g}{1 - C(e^{j\omega\Delta}I - e^{A\Delta})^{-1}f}. \tag{2.8}$$

From (2.8), it is evident that if the system is both observable and controllable, then, provided pathological values of $\Delta$ are avoided, $f$ and $g$ [and hence $F(t)$, $G(t)$] can be chosen so as to arbitrarily assign the numerator and denominator of the system discrete time transfer function. Full details of this approach are given in [1]. Note that the solutions to (2.5) and (2.6) are not unique. This flexibility has been used to achieve multiple objectives such as simultaneous pole assignment for several systems [1].

Some of the approaches that have been suggested in the literature for evaluating $F(t)$, $G(t)$ are: 1) to use piecewise constant functions (e.g., $F(t) = F_i$; $(i-1)\frac{\Delta}{n} \le t < i\frac{\Delta}{n}$); or 2) to define

$$W = \int_0^{\Delta} e^{A\tau}BB^Te^{A^T\tau}\,d\tau \tag{2.9}$$

and then to put

$$\left.\begin{aligned} F(t) &= B^Te^{A^T(\Delta-t)}W^{-1}f \\ G(t) &= B^Te^{A^T(\Delta-t)}W^{-1}g \end{aligned}\right\} \quad 0 \le t < \Delta. \tag{2.10}$$

The solution given in (2.9), (2.10) can be shown to result in minimum power in the generalized sample hold functions [6].

Note that the above analysis only considers the sampled behavior. In the next section, we examine the associated intersample behavior using amplitude modulation theory.

## III. Frequency Domain Analys

Since $G(t)$ and $F(t)$ are periodic functions with period $\Delta$, it is natural to expand them in a Fourier series of the form

$$F(t) = \sum_{p=-\infty}^{\infty} a_p^f e^{jp\omega_0 t} \tag{3.1}$$

$$G(t) = \sum_{p=-\infty}^{\infty} a_p^g e^{jp\omega_0 t} \tag{3.2}$$

where

$$\omega_0 = \frac{2\pi}{\Delta}. \tag{3.3}$$

We then have the following result, which summarizes many of the observations made in Section II using the frequency domain representation (3.1), (3.2).

*Lemma 3.1:*

a) When $F(t)$, $G(t)$ are defined as in (3.1), (3.2), then (2.5), (2.6) are equivalent to

$$\sum_{p=-\infty}^{\infty} a_p^f B_p = f \tag{3.4}$$

$$\sum_{p=-\infty}^{\infty} a_p^g B_p = g \tag{3.5}$$

where $\{B_p\}$ denotes a set of basis vectors given by

$$B_p = \int_0^{\Delta} e^{(A-jp\omega_0 I)\sigma}\,d\sigma\, B = \left[\sum_{i=0}^{\infty} \frac{(A-jp\omega_0 I)^i \Delta^{i+1}}{(i+1)!}\right]B. \tag{3.6}$$

b) Provided the system is completely controllable, then any set of $n$ vectors $\{B_{p_i}\}(p_i \ne p_j)$ will be linearly independent and hence span $\mathbb{C}^n$. (If $A$ is singular $B_0$ must be included in this set).

*Proof:*

a) By direct substitution.

b) We will distinguish two cases.

*Case 1 (A Nonsingular):* Generically, (3.6) can be rewritten as

$$B_p = [jp\omega_0 I - A]^{-1}[I - e^{A\Delta}]B \qquad \text{for all } p. \tag{3.7}$$

Suppose $\{B_{p_i}\} i = 1, \cdots, n$ are linearly dependent. Then, there exists a vector $0 \neq v \in \mathbb{C}^n$ such that

$$v^T B_{p_i} = 0, \qquad i = 1, \cdots, n.$$

This implies, in view of (3.7), that $jp_i\omega_0$ are zeros of the transfer function

$$v^T[sI - A]^{-1}[I - e^{A\Delta}]B. \tag{3.8}$$

Since, however, $A$ is nonsingular and $(A, B)$ is a controllable pair, $v^T[I - e^{A\Delta}] \neq 0$, $[I - e^{A\Delta}]B \neq 0$, so that the transfer function in (3.8) has at most $(n - 1)$ zeros. This is a contradiction which implies $v = 0$.

*Case 2 (A Singular):* In this case, (3.7) holds for all $p \neq 0$. It can be readily shown, using (3.6), however, that if $v^T B_0 = 0$ as assumed here, $v^T[I - e^{A\Delta}] \neq 0$. Also, in this case, the transfer function in (3.8) has a pole zero cancellation at the origin. So, $v^T B_{p_i}, i = 1, \cdots, n-1$ would imply that this transfer function has $n - 1$ zeros while only $n - 2$ are possible after the cancellation. Thus, the conclusion is the same as in case 1. □

Referring to Fig. 1, since $r_s(t)$ is a sampled signal, the corresponding transformed signal must be periodic; i.e., $R_s(\omega + \omega_0) = R_s(\omega)$, where $\omega_0 = \frac{2\pi}{\Delta}$ and $R_s(\omega) \triangleq \mathcal{F}\{r_s(t)\}$.

The (zero-order) hold block has impulse response given by

$$h_0(t) = \begin{cases} 1 & 0 \leq t < \Delta \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

so that its frequency response is given by

$$\begin{aligned} H_0(\omega) &= e^{-j\omega\frac{\Delta}{2}}\left[\frac{2\sin\left(\omega\frac{\Delta}{2}\right)}{\omega}\right] \\ &= \frac{1 - e^{-j\omega\Delta}}{j\omega}. \end{aligned} \tag{3.10}$$

Also, from (3.1) and (3.2), we have the Fourier transforms

$$\tilde{G}(\omega) = \mathcal{F}\{G(t)\} = 2\pi \sum_{p=-\infty}^{\infty} a_p^g \delta(\omega - p\omega_0) \tag{3.11}$$

$$\tilde{F}(\omega) = \mathcal{F}\{F(t)\} = 2\pi \sum_{p=-\infty}^{\infty} a_p^f \delta(\omega - p\omega_0). \tag{3.12}$$

Next we consider the modulation phase.

From Fig. 1, we have (using the modulation property of Fourier transforms),

$$\begin{aligned} \mathcal{F}\{u(t)\} = U(\omega) = \frac{1}{2\pi}\{\tilde{G}(\omega)^*[H_0(\omega)R_s(\omega)] \\ + \tilde{F}(\omega)^*[H_0(\omega)Y_s(\omega)]\} \end{aligned} \tag{3.13}$$

where $^*$ denotes convolution and $Y_s(\omega)$ denotes the transform of the sampled output signal, $y_s(t)$. Also, from Fig. 1, we have

$$\begin{aligned} Y(\omega) &= H(\omega)U(\omega) \\ &= \frac{1}{2\pi}H(\omega)\{\tilde{G}(\omega)^*[H_0(\omega)R_s(\omega)] \\ &\quad + \tilde{F}(\omega)^*[H_0(\omega)Y_s(\omega)]\}. \end{aligned} \tag{3.14}$$

Substituting (3.10), (3.11), and (3.12) in (3.14) we obtain

$$\begin{aligned} Y(\omega) = H(\omega)\Bigg\{ &\sum_{p=-\infty}^{\infty} a_p^g H_0(\omega - p\omega_0)R_s(\omega - p\omega_0) \\ &+ \sum_{p=-\infty}^{\infty} a_p^f H_0(\omega - p\omega_0)Y_s(\omega - p\omega_0)\Bigg\} \end{aligned} \tag{3.15}$$

and since both $R_s(\omega)$ and $Y_s(\omega)$ are periodic with period $\omega_0$, we have

$$\begin{aligned} Y(\omega) = H(\omega)\Bigg[ &\sum_{p=-\infty}^{\infty} a_p^g H_0(\omega - p\omega_0)R_s(\omega) \\ &+ \sum_{p=-\infty}^{\infty} a_p^f H_0(\omega - p\omega_0)Y_s(\omega)\Bigg]. \end{aligned} \tag{3.16}$$

Also, we have that

$$Y_s(\omega) = H_s(\omega)R_s(\omega) \tag{3.17}$$

where $H_s(\omega)$ is as in (2.8).

Substituting (3.17) in (3.16), we get

$$\begin{aligned} Y(\omega) = \Bigg[ &\sum_{p=-\infty}^{\infty} a_p^g H_0(\omega - p\omega_0)H(\omega) \\ &+ \sum_{p=-\infty}^{\infty} a_p^f H_0(\omega - p\omega_0)H(\omega)H_s(\omega)\Bigg] R_s(\omega). \end{aligned} \tag{3.18}$$

We summarize the above development in the following result.

*Theorem 3.1 (Continuous Time Frequency Response of Sampled Data System with GSHF's):*

a) Under the conditions depicted in Fig. 1, the frequency content of the *continuous time* output, $y(t)$, is given by

$$Y(\omega) = \tilde{H}(\omega)R_s(\omega) \tag{3.19}$$

where

$$\tilde{H}(\omega) = \left[\sum_{p=-\infty}^{\infty} (a_p^g + a_p^f H_s(\omega))H_0(\omega - p\omega_0)H(\omega)\right]. \tag{3.20}$$

b) The sampled output, $y_s(t)$, has a periodic transform given by (3.17), where

$$H_s(\omega) = \frac{\frac{1}{\Delta}\sum_{k=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} a_p^g H_0(\omega - (p+k)\omega_0)H(\omega - k\omega_0)}{1 - \frac{1}{\Delta}\sum_{k=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} a_p^f H_0(\omega - (p+k)\omega_0)H(\omega - k\omega_0)} \tag{3.21}$$

$$= \frac{\sum_{p=-\infty}^{\infty} a_p^g C(e^{j\omega\Delta}I - e^{A\Delta})^{-1}B_p}{1 - \sum_{p=-\infty}^{\infty} a_p^f C(e^{j\omega\Delta}I - e^{A\Delta})^{-1}B_p} \tag{3.22}$$

$$= \frac{C[e^{j\omega\Delta}I - e^{A\Delta}]^{-1}g}{1 - C[e^{j\omega\Delta}I - e^{A\Delta}]^{-1}f}. \tag{3.23}$$

*Proof:*

a) Immediate from (3.18).

b) We obtain $y_s(t)$ from $y(t)$ by impulse sampling. This produces frequency folding as outlined in the Introduction.

Pulse sampling $Y(\omega)$ and using (3.20), we obtain

$$\begin{aligned} Y_s(\omega) = \frac{1}{\Delta}\sum_{i=-\infty}^{\infty} Y(\omega - k\omega_0) = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty} H(\omega - k\omega_0) \\ \cdot \Bigg[\sum_{p=-\infty}^{\infty} a_p^g H_0(\omega - (p+k)\omega_0)R_s(\omega - k\omega_0) \\ + \sum_{p=-\infty}^{\infty} a_p^f H_0(\omega - (p+k)\omega_0)Y_s(\omega - k\omega_0)\Bigg] \end{aligned}$$

$$= \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} a_p^g H_0(\omega - (p+k)\omega_0) \cdot H(\omega - k\omega_0) R_s(\omega) + \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} a_p^f \cdot H_0(\omega - (p+k)\omega_0) H(\omega - k\omega_0) Y_s(\omega) \quad (3.24)$$

and hence see (3.25) at the bottom of the page. Equation (3.21) follows immediately. Equation (3.22) is then a direct consequence of (2.8), (3.4), (3.5), and (3.7). ∇∇∇

As we shall see in the next section, the above theorem allows us to calculate the intersample behavior as well as giving insight to the robustness and sensitivity behavior.

## IV. Intersample Behavior

If we use only zero-order hold functions, then this is a special case of GSHF's with

$$a_p^g = k_1 \quad \text{for } p = 0; \quad = 0 \quad \text{otherwise} \quad (4.1)$$

$$a_p^f = k_2 \quad \text{for } p = 0; \quad = 0 \quad \text{otherwise.} \quad (4.2)$$

Substituting (4.1), (4.2) into (3.22) shows that, in this case, the sampled data frequency response has the form

$$H_s^n(\omega) = \frac{k_1 C(e^{j\omega\Delta} I - e^{A\Delta})^{-1} B_0}{I - k_2 C(e^{j\omega\Delta} I - e^{A\Delta})^{-1} B_0}; \qquad k_1, k_2 \in \mathbb{R}. \quad (4.3)$$

Let us assume that we want to use the generalized sampled hold approach to modify this sampled frequency response to achieve some other desired value $H_s^*(\omega)$.

For example, we may seek to achieve

$$H_s^*(\omega) = \frac{C(e^{j\omega\Delta} I - e^{A\Delta})^{-1} g^*}{I - C(e^{j\omega\Delta} I - e^{A\Delta})^{-1} f^*}. \quad (4.4)$$

If $H_s^*(\omega)$ does not lie in the set $\{H_s^n(\omega)\colon k_1, k_2 \in \mathbb{R}\}$ as in (4.3), then it follows immediately from (3.4), (3.5) that

$$\left\| \sum_{\substack{p=-\infty \\ p\neq 0}}^{\infty} a_p^f B_p \right\| \geq \min_{k_1 \in \mathbb{R}} \|f^* - k_1 B_0\| \quad (4.5)$$

$$\left\| \sum_{\substack{p=-\infty \\ p\neq 0}}^{\infty} a_p^g B_p \right\| \geq \min_{k_2 \in \mathbb{R}} \|g^* - k_2 B_0\|. \quad (4.6)$$

Thus, the total high-frequency power in the generalized sampled hold function must be, at least, equal to the square of the amount that $f^*$ and $g^*$ are shifted from values which lie in the set (4.1), (4.2). Since the vectors $B_p$ are of the same order of magnitude as $|H(p\omega_0)|$, it follows from (3.20) that the term $|\tilde{H}(\omega)|$ will have nonnegligible power outside the band $\left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$. Since $R_s(\omega)$ in (3.19) has a periodic Fourier transform, it is clear that $Y(\omega)$ must have nonnegligible high-frequency power which is of the same order as the periodic energy in $R_s(\omega)$ multiplied by the magnitude of the amount that $f^*$ and $g^*$ are shifted from the (4.1), (4.2). Further, the input energy in the high-frequency range necessary to achieve this response at the output must be scaled up by the inverse of $|H(\omega)|$. Since most systems will be strictly proper, this means that the input energy needs to increase as higher and higher frequencies are needed.

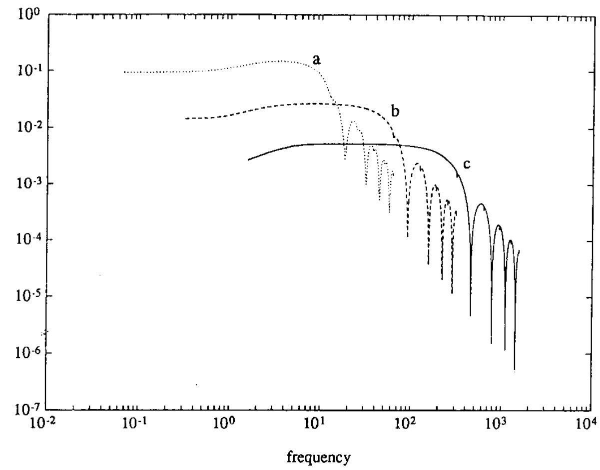


Fig. 2. Continuous frequency response $|\overline{H}(\omega)|$ for (a) $\Delta = 0.5$ s, (b) $\Delta = 0.1$ s, and (c) $\Delta = 0.02$ s.

Even for a sampled dc input, we have

$$R_s(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0).$$

In this case,

$$Y(\omega) = \frac{\Delta}{2\pi} \sum_{k=-\infty}^{\infty} [a_k^g + a_k^f H_s(0)] H(k\omega_0) \delta[\omega - k\omega_0]. \quad (4.7)$$

We observe that $Y(\omega)$ contains high-frequency components even for this constant set point!

In summary, we see that the generalized hold approach depends upon the generation of high-frequency components in the continuous time output which are folded when the output is sampled. These folded components are then superimposed at the base band to give the desired sampled frequency response.

*Remark 4.1:* The disturbance sensitivity of the approach follows immediately from the above discussion. Any disturbance injected into the lop will generate additional nonnegligible high-frequency components in the output, due to the modulation action. These components will be of the order of the shift from the zero-order hold response.

*Remark 4.2:* The robustness of the scheme can also be easily inferred from the frequency domain perspective. In practice, the high-frequency response of any dynamic system is difficult to precisely quantify. For example it is quite common to have high-frequency unmodeled dynamics including high-frequency unmodeled poles or small unmodeled time delays. These modify the high-frequency response of the system. Usually, in any design, one aims to make the system insensitive to these unmodeled components. The generalized sampled hold approach *intrinsically* depends on the folding of the high-frequency continuous time response back to the base band so

$$Y_s(\omega) = \frac{\frac{1}{\Delta}\sum_{k=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} a_p^g H_0(\omega - (p+k)\omega_0) H(\omega - k\omega_0) R_s(\omega)}{1 - \frac{1}{\Delta}\sum_{k=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} a_p^f H_0(\omega - (p+k)\omega_0) H(\omega - k\omega_0)}. \quad (3.25)$$

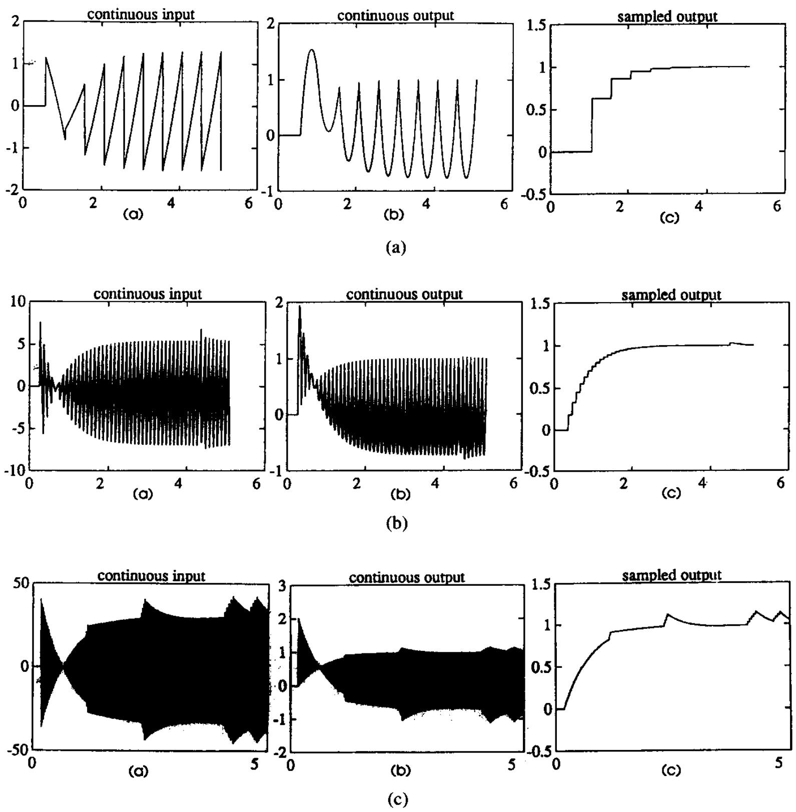


Fig. 3. Continuous and sampled responses: (a) $\Delta = 0.5$ s, (b) $\Delta = 0.1$ s, and (c) $\Delta = 0.02$.

as to construct the desired sampled frequency response. Hence, any variation in the high-frequency response of the system will be directly reflected in changes to the resulting sampled frequency response. This will not only significantly change the sampled performance, but could well lead to instability.

## V. Simulation Examples

Consider a system of the form (2.1), where $A$, $B$, $C$ take the following values:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 7.33 \\ 2.66 \end{bmatrix} \tag{5.1}$$

$$C = [1 \quad 1]. \tag{5.2}$$

Note that this system has continuous time poles at $-2$, $+1$ and a continuous time zero at 0.2. (This choice is reflected in the noninteger values in the vector $B$.)

Our strategy will be to use the generalized sample hold approach (with *minimum energy* as described in [6]) to shift both the unstable pole and zero to $e^{-\Delta}$ which is the discrete location corresponding to $-1$ in the continuous domain. This is repeated for 3 different sampling periods, namely, $\Delta = 0.5, 0.1, 0.02$, respectively.

Fig. 2 shows $|\tilde{H}(\omega)|$ as a function of normalized frequency $(\omega/\omega_0)$ for the 3 different sampling rates. We observe, in all cases, the presence of significant high-frequency components as predicted in Section IV. We also see that as $\Delta$ decreases, the larger the high-frequency components are.

Fig. 3 shows the unit step response of the system. In a) we show the continuous input, in b) the continuous output, and in c) the sampled output. Observe that, while the sampled output response is always an ideal exponential rising to the set point, the actual continuous time output has strong oscillatory behavior even in steady state. This is precisely as predicted in (4.7). We also see that the magnitude of the oscillations on the output are roughly independent of the sampling rate, but their frequency content obviously increases with $\omega_0$. Also, as predicted in Section IV, the magnitude of the input oscillations increase as $\frac{1}{\Delta}$.

Fig. 4 corresponds to the same conditions for $\Delta = 0.5$, but where an unmodeled time delay of 0.01 s has been added to the continuous time system. Note that this delay is much smaller than the dominant poles of the system. From the figure, we see that this very small perturbation is sufficient to destabilize the system! This is exactly as predicted in Remark 4.2. An extensive range of other simulations has been carried out. All of the results are in accord with the predictions made in Section IV. Some additional results are reported by us in [10].

## VI. Conclusion

This correspondence has given a quantitative and qualitative analysis of the robustness, sensitivity, and intersample properties of control based on generalized sample hold functions. Our mode of analysis uses amplitude modulation theory as in communication systems. Our conclusion is that while generalized sample hold control seems to offer great promise when viewed in purely sampled data terms,

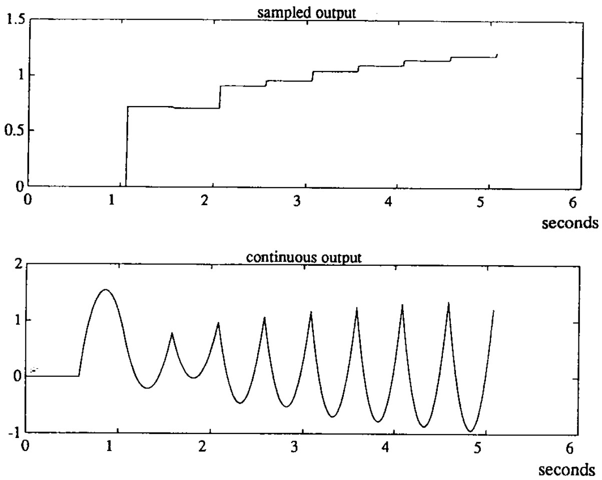


Fig. 4. Step response with $\Delta = 0.5$ s, and unmodeled time delay of $\Delta = 0.01$ s.

the continuous time performance is of such a nature to render the technique of limited practical use. Also, the method of analysis indicates that these difficulties are inescapable consequences of the modulation process, and hence cannot be fixed by different choices of the generalized function.

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# Utilization of Automatic Differentiation in Control Algorithms

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***Abstract*—Symbolic languages are increasingly being used in the analysis and implementation of control algorithms. Many of these control procedures involve some type of differentiation or Jacobian formulation. Automatic differentiation provides an alternative means of computing this information which is rarely considered in the control literature. This correspondence will discuss the use of automatic differentiation. Numerical results comparing the use of automatic differentiation and symbolic approaches in the context of a particular nonlinear control system will be given.**

## I. Introduction

Many control algorithms involve some differentiation of the equations defining the system. Among these are system inversion, some path following algorithms, and construction of observability and controllability matrices [1], [6], [8], [12], [14]–[16], [18], [19]. In addition, optimization routines and solving nonlinear equations require the construction of Jacobians. Nonlinear descriptor systems often use differentiation for regularization purposes [2].

The use of symbolic languages such as MAPLE, MACSYMA, MATHEMATICA, and REDUCE has had a revolutionary impact on the development of control algorithms. Larger and more complex models can be considered. Less model simplification needs to be done. For complicated expressions or where higher derivatives are required, however, symbolic languages are known to be slow and to often produce large expressions which can become unmanageable.

Automatic differentiation (AD) is a method for evaluating first and higher partial derivatives of multivariable functions. In symbolic differentiation, formulas are derived and then differentiated symbolically, with the result being a formula that can be subsequently evaluated. Automatic differentiation is fundamentally different. Automatic differentiation can handle functions defined by a large variety of procedures or subroutines. An explicit formula is not required. AD also tends to be faster than symbolic approaches. A numeric value is returned for the quantity of interest at a given value of the variables.

The idea of automatic differentiation has been around for some time [13], [17]. The recent development of general-purpose automatic differentiation codes combined with the increasing interest in larger and more sophisticated control problems makes a consideration of the use of automatic differentiation in control problems appropriate [9], [10], [16].

In Section II, we briefly discuss AD codes. Section III will describe a problem that has many features in common with the control problems mentioned above. Section IV will describe how the results returned by an AD program can be converted to a more familiar form. Section V will present the results of our comparison study of the use of the AD code ADOL-C [10] and a symbolic code written in MAPLE V on the problem of Section III. A secondary point of this study is to reinforce the idea that what is computationally practical is rapidly

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