# Correspondence

## Conditioning of LMS Algorithms with Fast Sampling

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*Abstract*—**The LMS algorithm is very commonly used in signal processing. Its convergence properties depend primarily on the step size chosen and the condition number of an information matrix associated with the system. In most applications today, the LMS uses a regression vector based on the shift operator (including the ubiquitous tapped delay line). In this correspondence, we demonstrate that generically, when fast sampling is employed, these regression vectors lead to poorly conditioned LMS. By comparison, delta operator based regression vectors lead with rapid sampling to improved condition numbers, hence, to better performance.**

### I. Introduction

All linear estimations are based on a model of the form

$$y[k] = X[k]^{\mathrm{T}} W_0 + n[k] \tag{1}$$

where $y[k]$ is measured and $X[k] \in \mathbb{R}^n$, frequently referred to as a regression vector, is an $n$-dimensional vector formed from measured variables. $W_o$ is a constant vector to be estimated, and $n[k]$ is a "noise" sequence. Depending on the way $X[k]$ is formed, (1) represents different models. One common choice is the tapped delay line, where $X[k]^{\mathrm{T}} = \left[x[k-1], \cdots, x[k-n]\right]$, with $x[k]$ a scalar sequence.

The well-known least mean square (LMS) algorithm is a recursive algorithm to estimate $W_0$ in (1) using mean square error (MSE) as the optimality criterion. The MSE is given by

$$\begin{aligned}\mathrm{MSE} &= E\left\{\left(y[k] - X[k]^{\mathrm{T}} W\right)^2\right\} \\ &= E\left\{y[k]^2\right\} - 2P^{\mathrm{T}} W + W^T R W \end{aligned} \tag{2}$$

where

$$P = E\left\{y[k]X[k]\right\}$$

and

$$R = E\left\{X[k]X[k]^{\mathrm{T}}\right\}. \tag{3}$$

Roughly speaking, the LMS is a gradient search algorithm on the surface defined in (2) and attempts to converge to its unique minimum at $R^{-1}P$. It is quite clear that the performance of this search algorithm depends solely on the shape of this surface which in turn, depends on the matrix $R$. Equi-MSE surfaces are ellipsoids. How narrow these ellipsoids are is determined by the relative values of the eigenvalues of $R$. Specifically, consider the condition number of $R$, $S$, which is defined as

$$S = \frac{\lambda_{\max}[R]}{\lambda_{min}[R]} \tag{4}$$

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where $\lambda_{\max}[R]$ and $\lambda_{\min}[R]$ are the largest and smallest eigenvalues of $R$. Then, large $S$ implies narrow ellipsoids, when $S = 1$ (all eigenvalues of $R$ are equal) the ellipsoids become spheres. On the other hand, it is well known that when these ellipsoids are narrow the gradient search becomes more difficult and proceeds "cautiously." In other words, the search algorithm will take longer to converge to a predetermined neighborhood of the optimum.

The above observations appear in many references (see e.g., [1]–[4]) and are known in practical applications of the LMS algorithm. With this in mind, we wish to highlight in this paper a tradeoff in the choice of $X[k]$ between simplicity in implementation and performance.

Since typically the measured data sequences result from sampling a continuous time process, we start our discussion with a continuous time process $x(t)$. It is assumed to be stationary with autocorrelation function

$$r(\tau) = E\left\{x(t+\tau)x(t)\right\}. \tag{5}$$

$x(t)$ is passed through an anti-aliasing filter and then sampled at a period $\Delta$. For simplicity, we consider a simple "integrate and reset" anti-aliasing filter. So, if $x[k]$ denotes the sampled sequence, we have

$$x[k] = \frac{1}{\Delta}\int_{(k-1)\Delta}^{k\Delta} x(t)\,dt. \tag{6}$$

The autocorrelation sequence for $x[k]$ will then be

$$\begin{aligned} r[m] &\triangleq E\left\{x[k+m]x[k]\right\} \\ &= \frac{1}{\Delta^2}\int_0^{\Delta}\int_0^{\Delta} r(m\Delta + \sigma - \zeta)\,d\sigma\,d\zeta. \end{aligned} \tag{7}$$

The following can now be established:

*Lemma 1:* For any autocorrelation function $r(\tau)$ of $x(t)$, which is continuous at $\tau = 0$, we have for any fixed integer $m$

$$\lim_{\Delta\to 0} r[m] = r(0). \tag{8}$$

*Proof:* From (7)

$$r[m] - r[0] = \frac{1}{\Delta^2}\int_0^{\Delta}\int_0^{\Delta}\left(r(m\Delta + \sigma - \zeta) - r(0)\right)d\sigma\,d\zeta. \tag{9}$$

Given any $\epsilon > 0$, the continuity of $r(\tau)$ at $\tau = 0$ implies that there exists $\delta_1 > 0$ such that given $|\tau| < \delta_1$ we have $|r(\tau) - r(0)| < \epsilon$. Let $\delta = \frac{\delta_1}{|m|+1}$, then given $|\Delta| < \delta$ we have for $0 \le \sigma, \zeta \le \Delta$ $|m\Delta + \sigma - \zeta| \le (|m|+1)\Delta < \delta_1$ so $|r(m\Delta + \sigma - \zeta) - r(0)| < \epsilon$ and, by (9) also $|r[m] - r(0)| < \epsilon$. This establishes (8). □

Next, we consider two general ways of generating the regression vector $X[k]$ and discuss the significance of these choices as far as the condition number of the resulting $R$ is concerned.

### II. Generic Shift Operator Results

In this section, we consider a generic shift operator formulation, in which the regressor $X_q[k]$ is generated via

$$\begin{aligned} qX_q[k] &\triangleq X_q[k+1] \\ &= A_q X_q[k] + B_q x[k] \end{aligned} \tag{10}$$

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where $(A_q, B_q)$ is a controllable pair and $\Big\{|\lambda_i(A_q)| < 1 \quad i = 1, 2, \cdots, n\Big\}$ ($\lambda_i(A_q)$ denotes the $i$th eigenvalue of $A_q$).

Note that the commonly used tapped delay line corresponds to the choice

$$A_q = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_q = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{11}$$

where the regression vector is $X_q[k]^{\mathrm{T}} = \Big[x[k-1], \cdots, x[k-n]\Big]$, and is therefore a special case of (10).

We then have the following result:

*Lemma 2:* For any fixed (i.e., independent of sampling interval) pair $(A_q, B_q)$, $\Big\{|\lambda_i(A_q)| < 1, \quad i = 1, 2, \cdots, n\Big\}$ we have

$$\lim_{\Delta \to 0} R_q = r(0)(I - A_q)^{-1} B_q B_q^{\mathrm{T}} (I - A_q^{\mathrm{T}})^{-1} \tag{12}$$

where

$$R_q \triangleq E\Big\{X_q[k] X_q[k]^{\mathrm{T}}\Big\}.$$

*Proof:* From (10)

$$X_q[k] = \sum_{m=0}^{\infty} (A_q)^m B_q x[k-1-m]$$

hence

$$R_q = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} (A_q)^m B_q B_q^{\mathrm{T}} (A_q^{\mathrm{T}})^l r[m-l]. \tag{13}$$

Since $A_q$ has all its eigenvalues inside the unit circle, the infinite summations in (13) converge uniformly, and so:

$$\lim_{\Delta \to 0} R_q = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} (A_q)^m B_q B_q^{\mathrm{T}} (A_q^{\mathrm{T}})^l \Big(\lim_{\Delta \to 0} r[m-l]\Big).$$

$$\begin{aligned}(\text{Using Lemma 1.1}) &= r(0) \Big(\sum_{m=0}^{\infty} (A_q)^m\Big) B_q B_q^{\mathrm{T}} \Big(\sum_{l=0}^{\infty} (A_q^{r})^l\Big) \\ &= r(0)(I - A_q)^{-1} B_q B_q^{\mathrm{T}} (I - A_q^{r})^{-1}.\end{aligned}$$

□

Since $B_q B_q^{\mathrm{T}}$ has rank one, Lemma 2 clearly shows that as the sampling rate increases, the shift operator correlation matrix $R_q$ tends to a singular matrix for all $n > 1$. Thus, its condition number increases with the sampling rate to infinity.

## III. Difference Operator Results

As an alternative to the shift operator-based generation of the regression vector, we suggest here using the difference operator. In [5], a difference operator based discrete time system calculus is proposed. It is shown to be a numerically superior alternative to shift operator based calculus. The proposed numerical advantages are shown to be particularly evident at rapid sampling rates.

The difference operator used, called the delta operator, is defined as follows:

$$\delta \triangleq \frac{q-1}{\Delta} \tag{14}$$

or

$$\delta X[k] \triangleq \frac{1}{\Delta} \Big(X[k+1] - X[k]\Big). \tag{15}$$

Using the delta operator we can generate the regression vector via

$$\delta X_\delta[k] = A_\delta X_\delta[k] + B_\delta x[k]. \tag{16}$$

From (15), (16) can be rewritten as

$$q X_\delta[k] = \Big(I + \Delta A_\delta\Big) X_\delta[k] + \Delta B_\delta x[k]. \tag{17}$$

Thus, we see that the delta operator form is equivalent to a shift operator form, except that an explicit, structured dependence of the matrices $A$ and $B$ on the sampling period has been incorporated.

We now proceed to our main result for the delta operator case:

*Lemma 3:* Let $X_\delta[k]$ be generated as in (16), and define

$$R_\delta \triangleq E\Big\{X_\delta[k] X_\delta[k]^{\mathrm{T}}\Big\}. \tag{18}$$

Then, assuming $\Big\{\mathrm{Re}\, \lambda_i(A_\delta) < 0, \quad i = 1, 2, \cdots, n\Big\}$,

$$\lim_{\Delta \to 0} R_\delta = \int_0^\infty \int_0^\infty e^{A_\delta \sigma} B_\delta B_\delta^{\mathrm{T}} e^{A_\delta^{\mathrm{T}} \zeta} r(\sigma - \zeta)\, d\sigma\, d\zeta. \tag{19}$$

*Proof:* In view of (17), (13), and (7) we have

$$\begin{aligned} R_\delta &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \Delta^2 \Big(I + \Delta A_\delta\Big)^m B_\delta B_\delta^{\mathrm{T}} \Big(I + \Delta A_\delta^{\mathrm{T}}\Big)^l r[m-l] \\ &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \Big\{(I + \Delta A_\delta)^m B_\delta B_\delta^{\mathrm{T}} \Big(I + \Delta A_\delta^{r}\Big)^l \\ &\quad \cdot \int_{l\Delta}^{(m+1)\Delta} \int_{l\Delta}^{(l+1)\Delta} r(\sigma - \zeta)\, d\sigma\, d\zeta \\ &= \int_0^\infty \int_0^\infty \Big(I + \Delta A_\delta\Big)^{[\frac{\sigma}{\Delta}]} B_\delta B_\delta^{\mathrm{T}} \Big(I + \Delta A_\delta^{\mathrm{T}}\Big)^{[\frac{\zeta}{\Delta}]} \\ &\quad \cdot r(\sigma - \zeta)\, d\sigma\, d\zeta \end{aligned} \tag{20}$$

where "$[\omega]$" denotes the greatest integer less than or equal to the real number $\omega$. For sufficiently small $\Delta \mathrm{Re}\,\{\lambda_i(A)\} < 0$ implies $|1 + \Delta \lambda_i(A)| < 1$, and so the integral in (20) converges uniformly. Therefore, it can readily be shown that

$$\lim_{\Delta \to 0} \Big(I + \Delta A_\delta\Big)^{[\frac{\sigma}{\Delta}]} = e^{A_\delta \sigma}$$

and

$$\begin{aligned} \lim_{\Delta \to 0} R_\delta &= \int_0^\infty \int_0^\infty \lim_{\Delta \to 0} \Big\{\Big(I + \Delta A_\delta\Big)^{[\frac{\sigma}{\Delta}]} B_\delta B_\delta^{\mathrm{T}} \Big(I + \Delta A_\delta^{\mathrm{T}}\Big)^{[\frac{\zeta}{\Delta}]}\Big\} \\ &\quad \cdot r(\sigma - \zeta)\, d\sigma\, d\zeta \\ &= \int_0^\infty \int_0^\infty e^{A_\delta \sigma} B_\delta B_\delta^{\mathrm{T}} e^{A_\delta^{\mathrm{T}} \zeta} r(\sigma - \zeta)\, d\sigma\, d\zeta. \end{aligned}$$

□

Under mild conditions on $A_\delta$, $B_\delta$ and the auto-correlation function $r(\tau)$, it can be shown that the right-hand side of (19) is nonsingular. For example, it suffices that $A_\delta$, $B_\delta$ be controllable, $\Big\{\mathrm{Re}\, \lambda_i(A_\delta) < 0, \quad i = 1, 2, \cdots, n\Big\}$ and the spectrum of $x(t)$ has a support at least at $n$ isolated points (see [2]). One should note that when the condition on the spectrum of $x(t)$ does not hold there is no sampling rate and no regressor structure that will give a full rank correlation matrix. That is, the "problem" is fundamentally singular and the dimension of the model must be reduced.

## IV. Discussion

The condition number of the correlation matrix in LMS algorithms is an important factor in the algorithm's performance. Lemma 2 indicates that with rapid sampling, generically, shift operator-based LMS algorithms (including the common tapped delay line implementations)

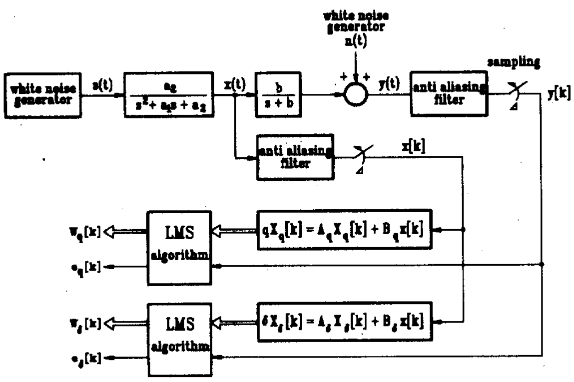


Fig. 1. Experiment set-up.

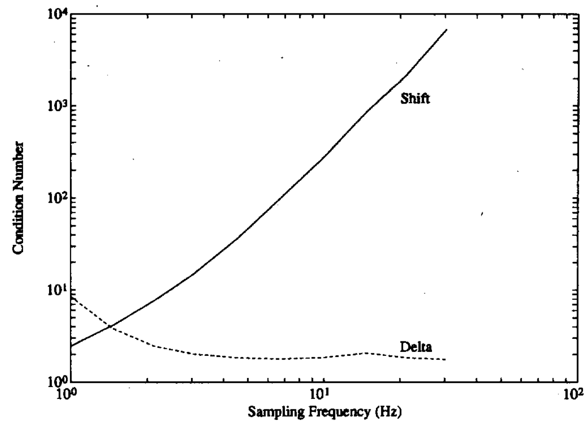


Fig. 2. Condition numbers versus sampling frequency.

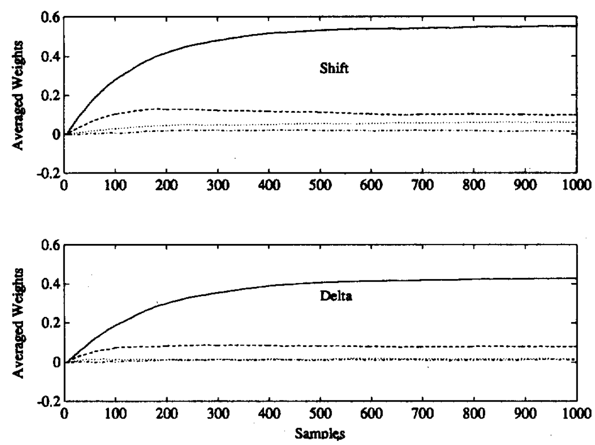


Fig. 3. Weight behavior for 1 Hz sampling.

will be poorly conditioned. On the other hand, as seen in Lemma 3, delta operator-based implementations are generically well conditioned at fast sampling rates. These observations imply that when compared, one can expect an improved convergence rate when the delta form is used.

Of course, this improvement in convergence rate comes at some price—an increase in the number of computations required. Generating the regression vector via a tapped delay line, which is the simplest shift form, will require no additions and no multiplications. On the other hand, the simplest delta form (using a canonical representation of $A_\delta$, $B_\delta$), will require $2n$ additions and $n$ multiplications.

## V. Simulation Results

To demonstrate and test the analysis and conclusions in the earlier sections we have conducted extensive simulation experiments. A sample of their results is presented here. The experiment setup is described in Fig. 1. Zero-mean Gaussian white noise $s(t)$ with incremental variance $\sigma_s$, is passed through a second-order filter to generate the continuous process $x(t)$. $x(t)$ is then passed through a first-order filter and corrupted by additive zero-mean Gaussian white noise $n(t)$ with variance $\sigma_n$, to result in the continuous process $y(t)$. Both $x(t)$ and $y(t)$ are passed through anti-aliasing filters and sampled at period $\Delta$ as described in (6). $x[k]$ is then fed into two distinct regression vector generators, one shift operator based and the other delta operator based according to (10) and (16), respectively. The resulting regression vectors $X_q[k]$ and $X_\delta[k]$ together with $y[k]$ are then used each in an LMS algorithm providing the estimates $W_q[k]$, $W_\delta[k]$ and the errors $e_q[k]$, $e_\delta[k]$, respectively. This experiment was repeated for several sampling periods and the results are enclosed.

The parameters chosen are:

$$a_1 = 12,\ a_2 = 25,\ b = 1,\ \sigma_s = 1,\ \sigma_n = 0.01$$

$$A_q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad B_q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\quad \text{tapped delay line}$$

$$A_\delta = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ -1 & -0.5 & 0 & 0 \\ -1 & -1 & -0.5 & 0 \\ -1 & -1 & -1 & -0.5 \end{bmatrix},\quad B_\delta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The step-sizes for each LMS were chosen to get the fastest convergence. After numerous experiments they were set at

$$\mu_q = 0.02,\ \mu_\delta = \frac{0.01}{\sqrt{\Delta}}.$$

In Fig. 2 we observe the dependence of each condition number on the sampling rate. Clearly, beyond about 1.5 Hz ($\Delta$ =0.67 s) the shift operator condition number takes off and grows while the condition number corresponding to the delta operator converges to a constant value as predicted by our analysis. The curves in Fig. 2 as well as all other results were generated by Monte Carlo runs repeated one hundred times.

In Fig. 3, we see the behavior of the respective estimate vectors for 1 Hz sampling. Clearly, since the difference in the condition numbers is small (see Fig. 2), it is not noticeable in the rates of convergence of the respective vector estimates. Neither is there any discernible difference in the MSE behavior of each algorithm as seen in Fig. 4.

The picture changes drastically when the sampling rate is increased to 33 Hz ($\Delta$ =0.03 s). In Fig. 5 (top) we see that after about 800 samples (the algorithm was turned on only after 5 s) the estimate vector is still far from convergence in the shift LMS. On the other hand, in Fig. 5 (bottom) the weight vector in the delta LMS converges after about 500 samples and does not differ much from the 1 Hz sampling case. This clearly agrees with the conclusions from our analysis.

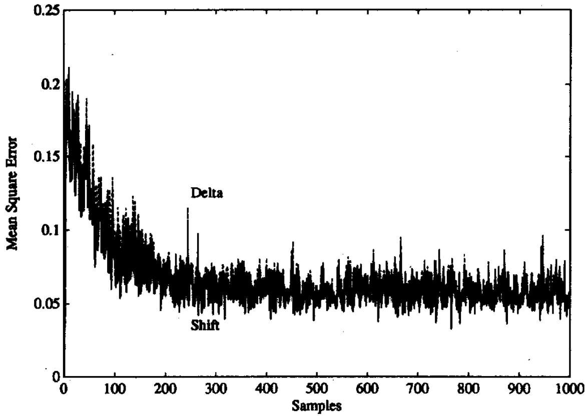


Fig. 4. MSE behavior at 1 Hz sampling rate.

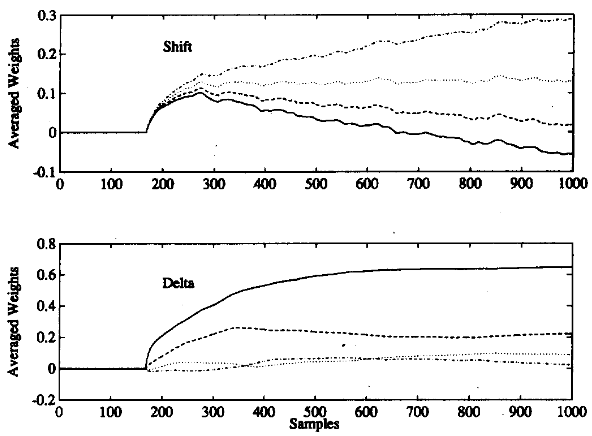


Fig. 5. Weight behavior for 33 Hz sampling rate.

In Fig. 6 the respective MSE are compared as well. The advantage of delta LMS is striking—after 800 samples there is an order of magnitude difference in the respective MSE's.

## VI. Concluding Remarks

Both the analysis and even more so the simulation experiments conducted indicate that the use of the delta-based regression vector used for LMS has a clear advantage as far as performance over the commonly used shift-based regression vector. This advantage becomes more and more significant as the sampling rate is increased. There are a number of applications where the sampling rates are higher than the required Nyquist rates (e.g., when the sampling rate is predetermined or when "rough" processing is done in the analog form while "fine" processing is done digitally). In cases like these our recommendation is quite clear—consider using the delta operator even if it means increased computational load.

With this in mind we wish to point out that our choice of $(A_\delta, B_\delta)$ in the experiment represents a choice we generally recommend

$$A_\delta = \begin{bmatrix} -a & 0 & 0 & 0 \\ -2a & -a & 0 & 0 \\ -2a & -2a & -a & 0 \\ -2a & -2a & -2a & -a \end{bmatrix}, \quad B_\delta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

with "$a$" as a design parameter.

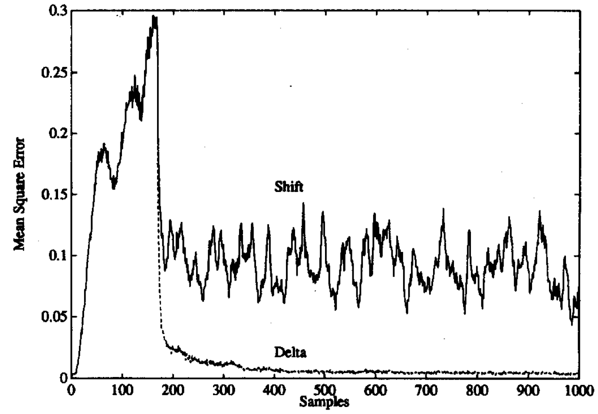


Fig. 6. MSE behavior at 33 Hz sampling rate.

Note that this structure corresponds (modulo scaling of $B_\delta$) to the Laguerre polynomial form discussed in, for example, [6]. The parameter "$a$" should be chosen to be about $\frac{1}{\tau}$ where $\tau$ is the dominant time constant of the system of interest. Note that in the example, we deliberately selected "$a$" so as to have an error of a factor of two compared with the "correct" value ($\tau \approx 1$ second), to realistically represent uncertainty in the time constant. Also, although it may appear that the Laguerre structure requires $O(n^2)$ computations for the regression vector, this is not so. In fact, the structure can be implemented with $O(n)$ by noting that (16) can be implemented as

$$\delta X_{\delta,1}[k] = -aX_{\delta,1}[k] + x[k]$$
$$\delta X_{\delta,i}[k] = -aX_{\delta,i}[k] + \delta X_{\delta,i-1}[k] - aX_{\delta,i-1} \quad \text{for } i = 2, \cdots, n$$

($X_{\delta,i}[k]$ is the $i$-th component of $X_\delta[k]$).

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