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A state-space technique for the evaluation of diagonalizing compensators

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Abstract

This note develops a technique for designing diagonalizing compensators based on state-space descriptions. The relationship with stable decoupling invariants is also explored. © 1997 Elsevier Science B.V.

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1. Introduction

An important problem in multi-input–multi-output control system design is the determination of pre- or post-compensators which diagonalize a given plant. There has been on-going interest in this problem in the control literature; see, e.g., [1–3, 6, 7, 9].

The majority of this literature deals with the question of existence of diagonalizing compensators rather than the methods for evaluating them. Also, most of these references use state feedback to achieve decoupling. Our interest is on strict pre- or post-compensation, namely, the determination of K_1 (or K_2) such that, for a given plant, P, the product PK_1 (K_2P) is diagonal. We will primarily focus on diagonalizing pre-compensators for a stable plant (with stable K), since the extension to post-compensation is then straightforward.

The main contribution of the current paper is a state-space approach to the design of pre- or post-compensators for a given stable plant. In addition, we relate this methodology to stable decoupling invariants, showing that the designed compensator achieves the simplest diagonal transfer function in terms of the multiplicity of non-minimum phase zeros and zeros at infinity. Finally, an extension to unstable plants is presented. In the latter case, cancellation of unstable poles is an issue that must be carefully addressed.

2. Pre- and post-compensation of stable transfer matrices

Consider a p-input–p-output n-dimensional LTI and stable system described by the following (minimal) state-space realization:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

where the transfer matrix $H = C(sI - A)^{-1}B + D$ has full normal rank. (We work with continuous time systems, but the results can be applied, *mutatis mutandis*, to the discrete time case.)

We will first focus on pre-compensator design. We thus wish to design a stable pre-compensator

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G_c described by the following (minimal) state-space realization:

$$\dot{x}_c = A_c x_c + B_c r, \qquad u = C_c x_c + D_c r, \tag{2}$$

such that the transfer matrix HG_c from r to y is diagonal.

Let ξ_r denote the right interactor matrix [8] for the system (1), so that

$$\begin{aligned}\bar{H} \doteq H\xi_r &= [C(sI - A)^{-1}B + D]\xi_r \\ &= C(sI - A)^{-1}\bar{B} + \bar{D}\end{aligned} \tag{3}$$

is a biproper transfer matrix (i.e., $\bar{D}$ is non-singular). The right interactor is a natural extension of the interactor introduced in [8]. Expressions to evaluate an interactor based on state-space methods are provided in [4], where dual results for the right interactor are briefly explored.

Let

$$\begin{aligned} A_0 &= A - B\bar{D}^{-1}C, \qquad B_0 = \bar{B}\bar{D}^{-1}, \\ C_0 &= -\bar{D}^{-1}C, \qquad D_0 = \bar{D}^{-1}. \end{aligned} \tag{4}$$

From the above definition it can be readily concluded that

$$[C(sI - A)^{-1}\bar{B} + \bar{D}][C_0(sI - A_0)^{-1}B_0 + D_0] = I. \tag{5}$$

Since we are interested in finding a diagonalizing pre-compensator, it may appear from (5) that we have, basically, solved the problem. However, since the system (1) is not necessarily minimum phase, the system (C_0, A_0, B_0, D_0) could be unstable. We will proceed by building on the system (C_0, A_0, B_0, D_0), by introducing modifications in such a way that its diagonalizing property is preserved.

We define the following subsystems:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i \tilde{u}_i, \\ \bar{u}_i &= C_i x_i + D_i \tilde{u}_i, \end{aligned} \qquad i = 1, \ldots, p, \tag{6}$$

as the minimal realizations of $(C_0, A_0, B_0 e_i, D_0 e_i)$, where e_i is the ith column of the $p \times p$ identity matrix.

Since the system (6) can be unstable, let

$$\tilde{u}_i = -K_i x_i + \tilde{r}_i, \quad i = 1, \ldots, p, \tag{7}$$

where the K_i's are such that $(A_i - B_i K_i)$ is stable for all i. Furthermore, define

$$\bar{u} = \sum_{i=1}^{p} \bar{u}_i. \tag{8}$$

With the above definitions, we are in a position to establish the following result:

Lemma 2.1. (a) *The transfer matrix from* $\tilde{r} = [\tilde{r}_1, \ldots, \tilde{r}_p]^T$ *to* $\bar{u}$ *is given by*

$$\begin{aligned} G &= [C_0(sI - A_0)^{-1}B_0 + D_0] \\ &\quad \times \operatorname{diag}\{[1 + K_i(sI - A_i)^{-1}B_i]^{-1}\}. \end{aligned} \tag{9}$$

(b) *$\bar{H}G$ is diagonal.*

Proof. (a) From definitions (6) and (7) the transfer matrix from $\tilde{r}_i$ to $\bar{u}_i$ is given by

$$G_i = (C_i - D_i K_i)(sI - A_i + B_i K_i)^{-1}B_i + D_i. \tag{10}$$

Using the matrix inversion lemma and (6) we obtain

$$\begin{aligned} G_i &= [C_0(sI - A_0)^{-1}B_0 + D_0] \\ &\quad \times e_i[1 + K_i(sI - A_i)^{-1}B_i]^{-1}. \end{aligned} \tag{11}$$

Then, the transfer matrix from $\tilde{r}$ to $\bar{u} = \sum_{i=1}^{p} \bar{u}_i$ is given by

$$\begin{aligned} G = \sum_{i=1}^{p} G_i e_i^T &= [C_0(sI - A_0)^{-1}B_0 + D_0] \\ &\quad \times \operatorname{diag}\{[1 + K_i(sI - A_i)^{-1}B_i]^{-1}\}. \end{aligned} \tag{12}$$

(b) Immediate from (5) and (12). □

Lemma 2.1 will form the basis for the construction of a stable and proper diagonalizing pre-compensator for H. The following result shows how this is achieved.

Theorem 2.1. (a) *A diagonalizing pre-compensator for H can be obtained as a minimal realization of*

$$G_c = \xi_r G \operatorname{diag}\{1/p_i\}, \tag{13}$$

where G has state-space realization as defined by (6)–(8), ξ_r is as in (3), and p_i is any stable monic polynomial of degree q_i. Here, each q_i is the highest degree occurring among the degrees of all entries of ξ_{li}, where ξ_{li} is the ith column of the left interactor for H, which is defined by $\xi_l H = \hat{H}$ ($\hat{H}$ biproper).

(b) *With G_c as in (a) we obtain $HG_c = D$, where D is given by*

$$D = \operatorname{diag}\{d_i\}, \tag{14}$$

$$d_i = 1/p_i[1 + K_i(sI - A_i)^{-1}B_i]^{-1}. \tag{15}$$

Also, the set of integers given by the number of zeros at infinity (q_i) *plus the number of non-minimum phase zeros of* d_i *corresponds to the set of* stable decoupling invariants [2].

Proof. (a) With this choice, we see from Lemma 2.1 that

$$\begin{aligned} HG_c &= \bar{H}G\,\mathrm{diag}\{1/p_i\} \\ &= \mathrm{diag}\{[1+K_i(sI-A_i)^{-1}B_i]^{-1}\}\,\mathrm{diag}\{1/p_i\}, \end{aligned} \tag{16}$$

which is diagonal as required.

We note that G is, by construction, stable. Hence, G_c is stable. To check that G_c is proper, we observe that

$$\begin{aligned} G_c &= \xi_r[C_0(sI-A_0)^{-1}B_0+D_0] \\ &\quad\times \mathrm{diag}\{[1+K_i(sI-A_i)^{-1}B_i]^{-1}\}\,\mathrm{diag}\{1/p_i\} \\ &= \xi_r\xi_r^{-1}H^{-1}\,\mathrm{diag}\{[1+K_i(sI-A_i)^{-1}B_i]^{-1}\} \\ &\quad\times \mathrm{diag}\{1/p_i\} \\ &= H^{-1}\,\mathrm{diag}\{[1+K_i(sI-A_i)^{-1}B_i]^{-1}\} \\ &\quad\times \mathrm{diag}\{1/p_i\} \\ &= \hat{H}^{-1}\xi_1\,\mathrm{diag}\{1/p_i\} \\ &\quad\times \mathrm{diag}\{[1+K_i(sI-A_i)^{-1}B_i]^{-1}\}, \end{aligned}$$

which is proper because of the choice made for the degree of p_i, $i=1,\dots,p$.

(b) We note that

$$HG_c = \mathrm{diag}\{1/p_i\}\,\mathrm{diag}\{[1+K_i(sI-A_i)^{-1}B_i]^{-1}\}. \tag{17}$$

The definition of the integers q_i ensures that $\mathrm{diag}\{1/p_i\}$ is the simplest diagonal matrix that makes $H^{-1}\,\mathrm{diag}\{1/p_i\}$ proper and, hence, $\{q_i\}$ is the set of decoupling invariants as defined in [2], i.e., they characterize the minimal relative degree achievable for each diagonal entry of any diagonalized system.

Next, consider the expression

$$d_i = 1/p_i[1+K_i(sI-A_i)^{-1}B_i]^{-1}. \tag{18}$$

Clearly, the non-minimum phase zeros of d_i are the unstable eigenvalues of A_i, with corresponding multiplicities. Say a_{ij} is an unstable eigenvalue of A_i with multiplicity $m_{a_{ij}}$. Since A_i has as its eigenvalues the poles of a minimal realization of the ith column of $\bar{H}^{-1}=\xi_r^{-1}H^{-1}$, it is clear that the highest multiplicity of a_{ij}, as a pole, among the entries of the ith column of H^{-1} is exactly $m_{a_{ij}}$. It is, thus, clear that $m_{a_{ij}}$ is the minimal multiplicity that a_{ij} must have as a non-minimum phase zero of the ith diagonal entry of any diagonal matrix D such that the ith column of $H^{-1}D$ does not have unstable poles at a_{ij}. By repeating this argument with the remaining unstable eigenvalues of each matrix A_i, and using a similar argument as was used for the zeros at infinity above, it follows as in [2] that the set of integers corresponding to the number of non-minimum phase zeros plus the zeros at infinity of each $\mathrm{diag}\{d_i\}$ is precisely the set of stable decoupling invariants of H. □

Part (b) of Theorem 2.1 establishes that the proposed pre-compensator is such that the resulting diagonalized system has the least possible multiplicity for each non-minimum phase zero and zero at infinity. This is a very important property, because non-minimum phase zeros are a source of limitations on achievable performance in feedback systems. Therefore, it is of interest to know that the proposed design introduces only those additional zeros at infinity and/or non-minimum phase zeros which are strictly necessary.

Theorem 2.1 shows how to find a diagonalizing pre-compensator for H. The related problem of finding a post-compensator for H can be easily solved as follows: First, we find a pre-compensator K for H^{T} and then, the post-compensator for H is obtained as $\bar{G}_c = K^{\mathrm{T}}$.

3. Extension to unstable transfer matrices

We will show here how the results of Section 2 can be employed (whenever this is possible) to design a pre-compensator K for an unstable plant, such that when K is used in a one-degree-of-freedom unity feedback structure, the resulting closed loop is both internally stable and diagonally decoupled.

The design method uses coprime factorizations over the ring of proper and stable transfer functions. Thus, we will express the system H as

$$H = \tilde{D}_h^{-1}\tilde{N}_h = N_h D_h^{-1}, \tag{19}$$

where $\tilde{D}_h$ and $\tilde{N}_h$ are left coprime, and D_h and N_h are right coprime.

The method described below is based on recent results given in [5]. There it was shown that part of the conditions needed to achieve decoupling is the existence of stable transfer matrices $\bar{X}_1$ and $\bar{X}_2$ such that

$$N_h\bar{X}_1 \text{ and } \bar{X}_2\tilde{D}_h \text{ are diagonal.} \tag{20}$$

We note that N_h and $\tilde{D}_h$ are stable. We, thus, recognize this sub-problem as the problem of finding stable pre- and post-compensators that diagonalize stable transfer matrices. This is precisely the problem addressed in Section 2.

In addition to (20), it is also necessary that the following equation be satisfied [5]:

$$N_h\bar{X}_1 + \bar{X}_2\tilde{D}_h = I. \tag{21}$$

In an effort to verify if this is indeed possible, and if a diagonalizing compensator K can be obtained from $\bar{X}_1$ and $\bar{X}_2$, one can proceed in the following way:

- The procedure described in Section 2 is first used to find a diagonalizing pre-compensator, R_{Nh}, for N_h and a diagonalizing post-compensator, $\tilde{R}_{Dh}$, for $\tilde{D}_h$. Thus, we obtain $S_{Nh} = N_hR_{Nh}$ and $\tilde{S}_{Dh} = \tilde{R}_{Dh}\tilde{D}_h$.
- We then attempt to solve the following equation:

$$S_{Nh}X_{Nh}^{\text{diag}} + X_{Dh}^{\text{diag}}\tilde{S}_{Dh} = I \tag{22}$$

  for diagonal matrices X_{Dh}^{diag} and X_{Nh}^{diag}.
  Since all the matrices in (22) are diagonal, this equation can be solved if and only if S_{Nh} and $\tilde{S}_{Dh}$ are coprime. [1]
  As was shown in Section 2, the matrices S_{Nh} and $\tilde{S}_{Dh}$ are the simplest achievable diagonal matrices (for N_h and $\tilde{D}_h$, respectively) in terms of the multiplicity of non-minimum phase zeros and zeros at infinity. Thus, if S_{Nh} and $\tilde{S}_{Dh}$ are not coprime, then it is not possible to find a decoupling compensator for H.
- If a solution to (22) exists, we can construct $D_k = X_{Dh}^{\text{diag}}\tilde{R}_{Dh}$ and $\tilde{N}_k = R_{Nh}X_{Nh}^{\text{diag}}$. Thus, with $\bar{X}_1 = \tilde{N}_k$ and $\bar{X}_2 = D_k$, (20) is clearly satisfied. Furthermore, D_k and $\tilde{N}_k$ have the property that

$$N_h\tilde{N}_k + D_k\tilde{D}_h = I. \tag{23}$$

  Thus, a possible decoupling compensator is given by the following alternative fractional representations:

$$K = \tilde{D}_k^{-1}\tilde{N}_k, \quad \text{where } \tilde{D}_k = (I - \tilde{N}_kN_h)D_h^{-1},$$

  or

$$K = N_kD_k^{-1}, \quad \text{where } N_k = D_h\tilde{N}_k\tilde{D}_h^{-1}.$$

  Note that the above expressions are left and right fractional representations of the same compensator K. In fact, from the above definitions of $\tilde{D}_k$ and N_k and Eq. (23), it is easily shown that $\tilde{D}_kN_k = \tilde{N}_kD_k$.
- Finally, it remains to check that either $\tilde{D}_k$ or N_k are stable [5]. If one is stable then so is the other. Also, in this case, the compensator proposed in the previous dot-point is a decoupling compensator, and the fractional representations above are, respectively, left and right coprime factorizations of K. However, if they are not stable then, as explained in [5], no decoupling compensator exists. Finally, in the special case when H has no coincident unstable poles and zeros, $\tilde{D}_k$ and N_k are always stable, and hence, the compensator proposed in the previous dot-point is guaranteed to be a decoupling compensator.

4. Conclusions

This paper has studied the problem of evaluating diagonalizing compensators for square transfer matrices. A state-space approach has been considered, which in the case of stable transfer matrices directly leads to the diagonalizing compensators. Extensions to unstable transfer matrices have been briefly explored. The relationship between the proposed design and the concept of *stable decoupling invariants* has also been discussed.

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[1] Since S_{Nh} and $\tilde{S}_{Dh}$ are diagonal matrices, one can test the coprimeness of these matrices by examining the coprimeness of each diagonal entry. It can be easily shown that if they are left coprime, then they are also right coprime and vice versa. Thus, we simply use the term coprime.

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