INT. J. CONTROL, 1998, VOL. 71, NO. 4, 615–629

On the reconstruction of a finite sum of sinusoids from non-uniform periodic samples

G. C. GOODWIN†, A. FEUER‡* and S. J. LEE§

This paper considers the extension of known sufficient conditions for signal reconstruction from non-uniform periodic samples to the case where the signal comprises a finite combination of sinusoids. In this case, it is shown that the reconstruction can be achieved by a causal observer. The result hinges on the observability of the associated signal model. It is shown that the sufficient conditions for this observability to hold are the same as the band-limited condition which applies to the general non-causal reconstruction result.

1. Introduction

The problem of reconstructing band-limited signals from non-uniform periodic samples has been treated in the literature (Papoulis 1968, Jerri 1977). The associated construction is, however, non-causal, being based on an infinite weighted sum of sinc functions. Thus, these general results are essentially an existence theorem.

In this paper we consider the special case where the underlying signal comprises a finite combination of sinusoids. In this case, we show that the reconstruction can be achieved via a *causal finite-dimensional filter* designed using observer principles.

A key question in this context is the observability of the underlying signal model with non-uniformly sampled data. We show that observability of this model is guaranteed provided the sinewave frequencies satisfy the general constraint for reconstruction based on non-uniform samples as outlined in Papoulis (1968) and restated here.

The current result can be used in a practical sense to reconstruct signals that are well approximated by a finite Fourier representation. This gives additional insight into the formal existence results mentioned earlier. The results also have implication in determining the bandwidth of signals that can be dealt with in sampled data control systems where non-uniform sampling is employed. In fact, this study was motivated by an industrial application concerning a continuous galvanizing line where the instrumentation imposed a periodic measurement pattern (Goodwin *et al.* 1994).

Received in final form 16 April 1998. Communicated by Professor V. Kucera.

† Department of Electrical and Computer Engineering, The University of Newcastle, NSW 2308, Australia.

‡ Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel.

§ Department of Electrical Engineering, Chungnum National University, 220 Gung-Dong-Yuseong-Gu, Daejeon 305-764, South Korea.

* To whom correspondence should be addressed.

2. Non-causal reconstruction

There exist a number of results which extend the reconstruction of band-limited signals from uniformly sampled data to non-uniformly sampled data. These go back as far as 1968 (Papoulis 1968). An extensive treatment is presented in Jerri (1977).

We are specifically interested here in non-uniformly sampled data which has a periodic pattern. Namely, if $\{t_q : q = 0, \pm 1, \pm 2, \ldots$ are the sampling instances we have, for some integer M,

$$\left.\begin{aligned} t_0 &= 0 \\ t_q - t_{q-1} &= \Delta_q \end{aligned}\right\} \tag{2.1}$$

$$\Delta_{q+M} = \Delta_q \tag{2.2}$$

Let $y(t)$ be the underlying (continuous-time) signal and let $y[q] = y(t_q)$ be the corresponding sampled data. The question we address here is: when and how can one reconstruct $y(t)$ from $\{y[q]\}$?

2.1. *The sampling extension result* (Papoulis 1968)

Let $y(t)$ be a given continuous-time signal and $y[q] = y(t_q)$ the resulting sampled data sequence with periodic sampling as in equations (2.1), (2.2). Then, $y(t)$ can be uniquely recovered from $y[q]$ provided

$$Y(\omega) = 0 \quad \text{for} \quad |\omega| > \frac{M\pi}{T} \tag{2.3}$$

where $Y(\omega)$ is the Fourier transform of $y(t)$, and

$$T = \sum_{q=1}^{M} \Delta_q \tag{2.4}$$

The reconstruction formula is of the form

$$y(t) = \sum_{q=-\infty}^{\infty} \beta_q(t) \frac{\sin\left(\frac{\pi(t - t_q)}{T}\right)}{\frac{\pi(t - t_q)}{T}} y[q] \tag{2.5}$$

where $\beta_q(t)$ are time functions which depend only on the sampling pattern.

A proof of this result and the expression for $\beta_q(t)$ are given in the Appendix.

Note that the bandwidth limitation (2.3) has a simple intuitive interpretation, namely, that the usual Nyquist rate for uniform sampling should apply to the average sampling period. Clearly, if $M = 1$ we recover the well-known uniform sampling result.

The reconstruction given in (2.5) is non-causal and hence of limited practical value. In practice one would like to reconstruct the signal using only present and past data, namely, in a causal way. For a special class of signals which can be modelled as follows

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{2.6}$$

it is clear that, if the samples of the state $x(t)$ are available, namely $x[q] = x(t_q)$, the signal can be exactly reconstructed via

$$y(t) = C e^{A(t - t_q)} x[q], \quad t_q \leq t < t_{q+1} \tag{2.7}$$

This is clearly a causal process. Hence, going back to our original problem, we consider a two-stage reconstruction process: (i) use the data $\{y[q]\}$ to reconstruct $x[q]$, and (ii) use (2.7) to generate $y(t)$.

To carry out stage (i) we use an observer (which can be made, theoretically, to converge at any desired rate—even in a finite number of steps if we employ a dead-beat observer). Clearly, the key question to our ability to use an observer is whether $x[q]$ is indeed observable from $y[q]$.

The main purpose of this paper is to establish the connection between the condition in (2.3) and the observability question above. To do that we restrict attention to a special class of signals comprising a finite sum of sinusoids.

3. Problem set-up

Consider a continuous time signal given by

$$y(t) = \sum_{i=1}^{n} a_i \sin(\omega_i t + \phi_i) \tag{3.1}$$

The signal $y(t)$ is sampled in a non-uniform pattern, defined as follows.

Let $\{p[k]\}$ be a periodic sequence of zeros and ones such that for some integer $N \geq 1$

$$p[k + N] = p[k] \quad \text{for all}\, k \tag{3.2}$$

Denote

$$M = \sum_{k=1}^{N} p[k] \leq N \tag{3.3}$$

then, with k_q such that

$$p[k] = \begin{cases} 1 & \text{if} \quad k = k_q \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

we have

$$k_{q+M} = k_q + N \tag{3.5}$$

The sampled sequence is then given by

$$y[q] = y(k_q \Delta) \tag{3.6}$$

where Δ is an underlying fast sampling period. Or, if we denote

$$y_1[k] = y(k\Delta) \tag{3.7}$$

and

$$y_p[k] = \begin{cases} y[q] & \text{if} \quad k = k_q \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

we can write

$$y_p[k] = p[k]y_1[k] \tag{3.9}$$

Note that $y[q]$ and $y_p[k]$ contain the same data; hence, in the sequel, we will use either the one or the other as needed. *The problem we address here is*: Under what conditions can $y(t)$ be reconstructed from the data $\{y[q]\}$ (or, equivalently from $y_p[k]$) via a *finite-dimensional* observer using the two-stage method outlined in the previous section?

4. Signal model

The continuous time signal (3.1) can be modelled as in (2.6) where

$$A = \operatorname{diag}(A_i) \in \mathbb{R}^{2n\times 2n} \tag{4.1}$$

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix} \quad \text{for} \quad i = 1,2,\dots,n \tag{4.2}$$

and

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \dots \end{bmatrix} \in \mathbb{R}^{1\times 2n} \tag{4.3}$$

Without loss of generality we may assume that the frequencies are in increasing order of magnitude, i.e.

$$0 < \omega_1 < \omega_2 < \cdots < \omega_n \tag{4.4}$$

The extension to the case $\omega_1 = 0$ is straightforward but will not be considered here, for notational simplicity. Then, using (3.4), (2.6), (4.1) and (4.2), we see that the data $y_p[k]$ satisfies the equations

$$x[k+1] = A_S x[k] \tag{4.5}$$

$$y_p[k] = p[k]Cx[k] \tag{4.6}$$

where

$$\left.\begin{aligned} x[k] &= x(k\Delta) \\ y_1[k] &= y(k\Delta) \\ A_S &= \operatorname{diag}(A_{Si}) \end{aligned}\right\} \tag{4.7}$$

$$A_{Si} = \begin{bmatrix} \cos\omega_i\Delta & \dfrac{1}{\omega_i}\sin\omega_i\Delta \\ -\omega_i\sin\omega_i\Delta & \cos\omega_i\Delta \end{bmatrix} \tag{4.8}$$

Note that the eigenvalues of A_S are

$$\left.\begin{aligned} \lambda_{2i-1}^S &= e^{j\omega_i\Delta} \\ \lambda_{2i}^S &= e^{-j\omega_i\Delta} \end{aligned}\right\} \quad i = 1,2,\dots,n \tag{4.9}$$

With the model as in eqns. (4.5), (4.6) the question now is, *under what conditions is this system observable.*

5. Reconstruction from non-uniform periodic samples

We observe that the system in equations (4.5), (4.6) is a periodic linear system. To investigate its observability we could try a direct approach, calculating the observability Gramian, or we could transform the system into a time-invariant representation, using one of a number of existing possibilities (see Khargonekar *et al.* 1985, Flamm 1991, Misra 1996). We choose to transform the system into an LTI equivalent using a raising technique (Khargonekar *et al.* 1985).

To derive the 'raised system' we define

$$x_R[m] = x[mN] \tag{5.1}$$

$$y_R[m] = \begin{bmatrix} y_p[mN] \\ y_p[mN+1] \\ \vdots \\ y_p[mN+N-1] \end{bmatrix} \tag{5.2}$$

Then, using (4.5), (4.6), we can readily verify that these vectors satisfy

$$x_R[m+1] = A_R x_R[m] \tag{5.3}$$

$$y_R[m] = C_R x_R[m] \tag{5.4}$$

where

$$A_R = (A_S)^N \tag{5.5}$$

$$C_R = \begin{bmatrix} p[0]C \\ p[1]CA_S \\ \vdots \\ p[N-1]C(A_S)^{N-1} \end{bmatrix} \tag{5.6}$$

We note that the system of eqns. (5.3), (5.4), referred to as the 'raised system', is observable if and only if the original, periodic system, eqns. (4.5), (4.6), is observable.

The conditions under which the latter result holds are established in the following theorem, which is the main result of this paper.

Theorem 1: *The raised system* (5.3), (5.4) *is observable for any set of distinct frequencies* $\{\omega_i\}$, $i = 1, 2, \ldots, n$, *provided*

$$0 \le \omega_i < \frac{M\pi}{N\Delta} \tag{5.7}$$

Proof: To prove the claim we make use of the PBH test (see, e.g. Kailath 1980). Namely, the system is observable iff

$$\operatorname{rank}\begin{bmatrix} \lambda I - A_R \\ C_R \end{bmatrix} = 2n \tag{5.8}$$

for every eigenvalue λ of A_R.

Note first that, from (4.9) and (5.5), the eigenvalues of A_R are

$$\left.\begin{aligned} \lambda_{2i-1} &= \mathrm{e}^{jN\omega_i\Delta} \\ \lambda_{2i} &= \mathrm{e}^{-jN\omega_i\Delta} \end{aligned}\right\} \qquad i = 1,2,\ldots,n \tag{5.9}$$

Suppose now that for eigenvalue λ_ℓ, $1 \le \ell \le 2n$, there exists a vector $\underline{v} \in \mathbb{C}^{2n}$ such that

$$\begin{bmatrix} \lambda_\ell I - A_R \\ C_R \end{bmatrix} \underline{v} = 0 \tag{5.10}$$

Our objective is to show that any $\underline{v}$ satisfying (5.10) must be identically zero iff (5.7) holds.

Using (4.7), (5.5) and (5.6), eqn. (5.10) can be written as

$$\left[\lambda_\ell I - (A_{Si})^N\right]\begin{bmatrix} v_{2i-1} \\ v_{2i} \end{bmatrix} = 0; \quad i = 1,2,\ldots,n \tag{5.11}$$

$$\sum_{i=1}^{n} p[k]\left[1 \quad 0\right](A_{Si})^k \begin{bmatrix} v_{2i-1} \\ v_{2i} \end{bmatrix} = 0; \quad k = 0,1,\ldots,N-1 \tag{5.12}$$

Let us concentrate first on (5.11). Define the following two index sets:

$$I_1 = \left\{1 \le i \le n \,\middle|\, \mathrm{e}^{jN\omega_i\Delta} = \lambda_\ell\right\} \tag{5.13}$$

$$I_2 = \left\{1 \le r \le n \,\middle|\, \mathrm{e}^{-jN\omega_r\Delta} = \lambda_\ell\right\} \tag{5.14}$$

Then, for all $1 \le i \le n$ such that $i \notin I_1 \cup I_2$, the matrix $\left[\lambda_\ell I - (A_{Si})^N\right]$ is nonsingular; hence, from (5.11),

$$v_{2i-1} = v_{2i} = 0 \quad \forall i \notin I_1 \cup I_2 \tag{5.15}$$

Let i^* be the smallest integer in $I_1 \cup I_2$. Without loss of generality we assume $i^* \in I_1$. Then, by (5.13), for every $i \in I_1$ there exists an integer $L_i^1 \ge 0$ s.t.

$$N\omega_i\Delta = N\omega_{i^*}\Delta + 2\pi L_i^1 \tag{5.16}$$

and for every $r \in I_2$ there exists an integer $L_4^2 \ge 0$ s.t.

$$N\omega_r\Delta = 2\pi L_r^2 - N\omega_{i^*}\Delta \tag{5.17}$$

We consider two possible cases.

Case 1: $I_1 \cap I_2 = \emptyset$

Then, it can readily be shown from (5.13), (5.14) that $N\omega_i\Delta$ is not an integer multiple of π for all $i \in I_1 \cap I_2$. From (4.8) we have

$$(A_{Si})^k = \begin{bmatrix} \cos k\omega_i\Delta & \dfrac{1}{\omega_i}\sin k\omega_i\Delta \\ -\omega_i \sin k\omega_i\Delta & \cos k\omega_i\Delta \end{bmatrix} \tag{5.18}$$

and by (5.11), (5.13) and (5.14) we get

$$v_{2i} = j\omega_i v_{2i-1} \qquad \forall i \in I_1$$

$$v_{2r} = -j\omega_r v_{2r-1} \quad \forall r \in I_2$$

Substituting this and (5.18) in (5.12) we get

$$p[k]\left\{\sum_{i\in I_1} \mathrm{e}^{jk\omega_i\Delta} v_{2i-1} + \sum_{r\in I_2} \mathrm{e}^{jk\omega_r\Delta} v_{2r-1}\right\} = 0; \quad k = 0,1,\dots,N-1 \tag{5.19}$$

Denote by $0 \le k_1 < k_2 < \cdots < k_M < N$ the first M integers for which $p[k] = 1$. Then, using (5.16) and (5.17), we can rewrite (5.19) as

$$\begin{bmatrix} \mathrm{e}^{jk_1\omega_1 w_{i^*}\Delta} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{jk_M\omega_{i^*}\Delta} \end{bmatrix} \left\{ \sum_{i\in I_1} \begin{bmatrix} \mathrm{e}^{j2\pi\frac{L_i^1}{N}k_1} \\ \mathrm{e}^{j2\pi\frac{L_i^1}{N}k_2} \\ \vdots \\ \mathrm{e}^{j2\pi\frac{L_i^1}{N}k_M} \end{bmatrix} v_{2i-1} + \sum_{r\in I_2} \begin{bmatrix} \mathrm{e}^{j2\pi\frac{L_i^2}{N}k_1} \\ \mathrm{e}^{j2\pi\frac{L_i^2}{N}k_2} \\ \vdots \\ \mathrm{e}^{j2\pi\frac{L_i^2}{N}k_M} \end{bmatrix} v_{2r-1} \right\} = 0 \tag{5.20}$$

Let

$$\alpha_m = \mathrm{e}^{j2\pi\frac{k_m}{N}}; \quad m = 1,2,\dots,M$$

Then, since by (5.16) and (5.17) $0 \le L_i^1 + L_r^2 < M$ for all $i \in I_1$ and $r \in I_2$, the vectors in (5.20) are columns in the matrix product

$$\begin{bmatrix} (\alpha_1)^{L_{r^*}^2} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & (\alpha_M)^{L_{r^*}^2} \end{bmatrix} \begin{bmatrix} (\alpha_1)^{M-1} & (\alpha_1)^{M-2} & \dots & \alpha_1 & 1 \\ (\alpha_2)^{M-1} & (\alpha_2)^{M-2} & \dots & \alpha_2 & 1 \\ \vdots & \vdots & & \vdots & \\ (\alpha_M)^{M-1} & (\alpha_M)^{M-2} & \dots & \alpha_M & 1 \end{bmatrix} \tag{5.21}$$

where r^* is the largest integer in I_2 (hence, $L_{r^*}^2 \ge L_r^2$ for all $r \in I_2$). The matrix on the left is clearly nonsingular since $\alpha_m \ne 0$, and the matrix on the right is a Vandemonde matrix. As $0 \le k_m < M \le N$, we have $\alpha_\ell \ne \alpha_m$ for all $1 \le \ell \ne m \le M$, which guarantees the nonsingularity of the Vandemonde matrix (Muir 1960) and the independence of the vectors in (5.20). Hence, clearly

$$v_{2i-1} = v_{2r-1} = 0 \qquad \forall i \in I_1, r \in I_2$$

and the whole vector $\underline{v}$ must be zero.

Case 2: $I_n \cap I_2 \ne \varnothing$

From (5.13) and (5.14) it follows that $I_1 = I_2$ and $N\omega_i\Delta$, for all $i \in i_1$, is an integer multiple of π. Hence, (5.12) becomes

$$\sum_{i\in I_1}\left(\cos(k_\ell\omega_i\Delta) v_{2i-1} + \frac{1}{\omega_i}\sin(k_\ell\omega_i\Delta) v_{2i}\right) = 0 \qquad \ell = 1,2,\dots,M$$

or

$$\frac{1}{2}\sum_{i\in I}\left(e^{jk_\ell\omega_i\Delta}\left(v_{2i-1}+\frac{1}{j\omega_i}v_{2i}\right)+e^{-jk_\ell\omega_i\Delta}\left(v_{2i-1}-\frac{1}{j\omega_i}v_{2i}\right)\right)=0 \qquad \ell=1,2,\dots,M$$

Substituting (5.16), we have

$$\frac{1}{2}\begin{bmatrix} e^{jk_1\omega_{i*}\Delta} & & \\ & \ddots & \\ & & \ddots \\ & & & e^{jk_M\omega_{i*}\Delta}\end{bmatrix}\sum_{i\in I_1}\left\{\begin{bmatrix} e^{j2\pi\frac{L_i^1}{N}k_1} \\ \vdots \\ e^{j2\pi\frac{L_i^1}{N}k_M}\end{bmatrix}\left(v_{2i-1}+\frac{1}{j\omega_1}v_{2i}\right)\right.$$

$$\left.+\begin{bmatrix} e^{j2\pi\frac{L_i^1}{N}k_1} \\ \vdots \\ e^{-j2\pi\frac{L_i^1}{N}k_M}\end{bmatrix}\left(v_{2i-1}-\frac{1}{j\omega_i}v_{2i}\right)\right\}=0 \qquad (5.22)$$

Using similar arguments as in (5.20) and (5.21), we conclude that the vectors in the sum of (5.22) are linearly independent. Hence, we must have

$$\left.\begin{aligned} v_{2i-1}+\frac{1}{j\omega_i}v_{2i}&=0 \\ v_{2i-1}-\frac{1}{j\omega_i}v_{2i}&=0\end{aligned}\right\} \quad \forall i\in I_i$$

so that $v_{21-1}=v_{2i}=0\ \forall i\in I_1$; hence $\underline{v}=0$. □

The above result establishes the key observability condition.

At this point one could employ a (periodic) Kalman filter (see Bittanti *et al.* 1991). Alternatively, a time-invariant observer can be designed to reconstruct $x_R[m]$ (and through it $x[k]$). Then the recovery of $y(t)$ follows as given in (2.7).

Remark: We wish to point out two facts: (i) The observer is used to reconstruct $x[k]$ and through it $y_1[k]=Cx[k]=y(k\Delta)$. (ii) The choice of the sampling interval Δ is arbitrary so long as it satisfies (2.2).

These facts mean that, given some Δ which satisfies (2.2), one can choose any integer fraction, $\Delta_1=\Delta/N_1$ (N_1 integer), of this interval to reconstruct $y(k\Delta_1)$. Combining this with a simple ZOH allows one to obtain a reconstruction of $y(t)$ which is as accurate as one desires, using only an observer and ZOH. This point is demonstrated in the numerical example presented below.

6. Numerical example

The above method is illustrated by a simple example. Consider the continuous time signal

$$y(t)=0.5\sin\left(\frac{\pi t}{10}\right)+\sin\left(\frac{5\pi t}{10}+1\right)+2\sin\left(\frac{11\pi t}{10}+0.5\right)+\sin\left(\frac{29\pi t}{10}+1\right) \qquad (6.1)$$

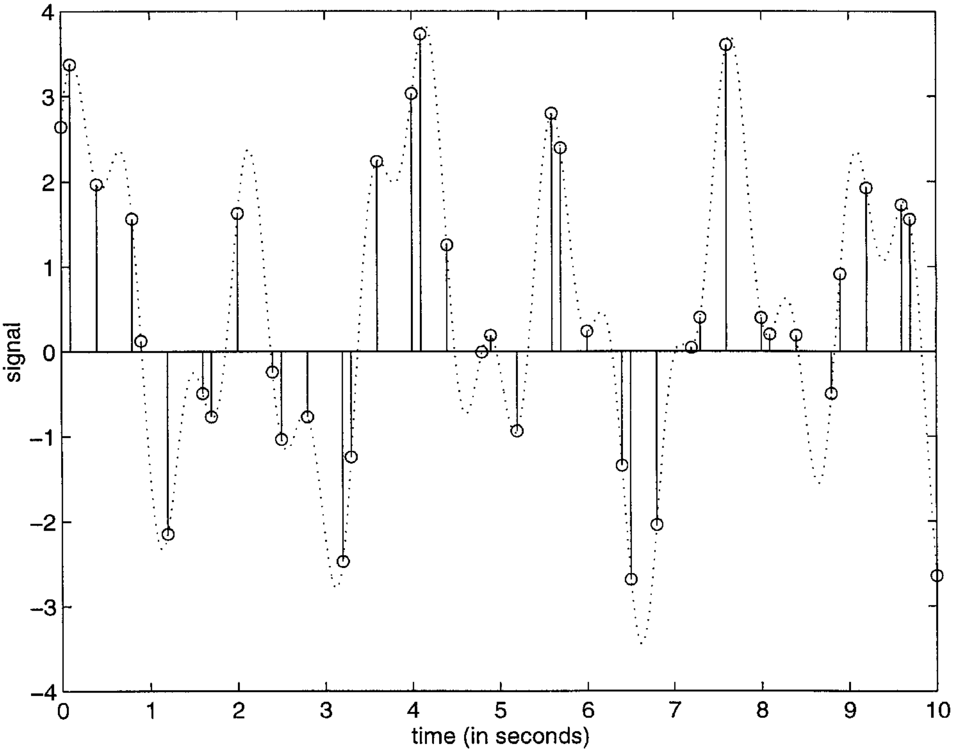

Figure 1. Continuous signal and the non-uniform sampling.

This signal is sampled using the following periodic pattern

$$\{t_q = 0, 0.1, 0.4, 0.8, 0.9, 1.2, 1.6, 1.7, 2.0, \ldots\}$$

to yield the sequence $\{y[q] = y(t_q)\}$. In figure 1 we show the continuous signal (dotted line) and the sampled data. Using the notation of Section 3, we note that $n = 4$ and $M = 3$ and, with $\Delta = 0.1$ s, $N = 8$.

Verifying that all four frequencies satisfy condition (5.7), we can use an observer to carry out Stage 1, namely, the recovery of $x[k]$. We then design a periodic observer (say using the method in Bittanti *et al.* (1991) with $Q = I$ and $R = I$). This result is

$$\hat{x}[k+1] = A_S \hat{x}[k] + L[k](y_p[k] - p[k]C\hat{x}[k]) \tag{6.2}$$

where C, A_S are as in (4.3), (4.7), (4.8),

$$y_p[k] = \begin{cases} y[q] & \text{for} \quad k = 0, 1, 4, 8, 9, 12, 16, 17, \ldots, \\ 0 & \quad\quad\ \text{otherwise} \end{cases} \tag{6.3}$$

$$p[k] = \{1, 1, 0, 0, 1, 0, 0, 0, \ldots,\} \tag{6.4}$$

and the vector $L[k]$ is periodic and given by

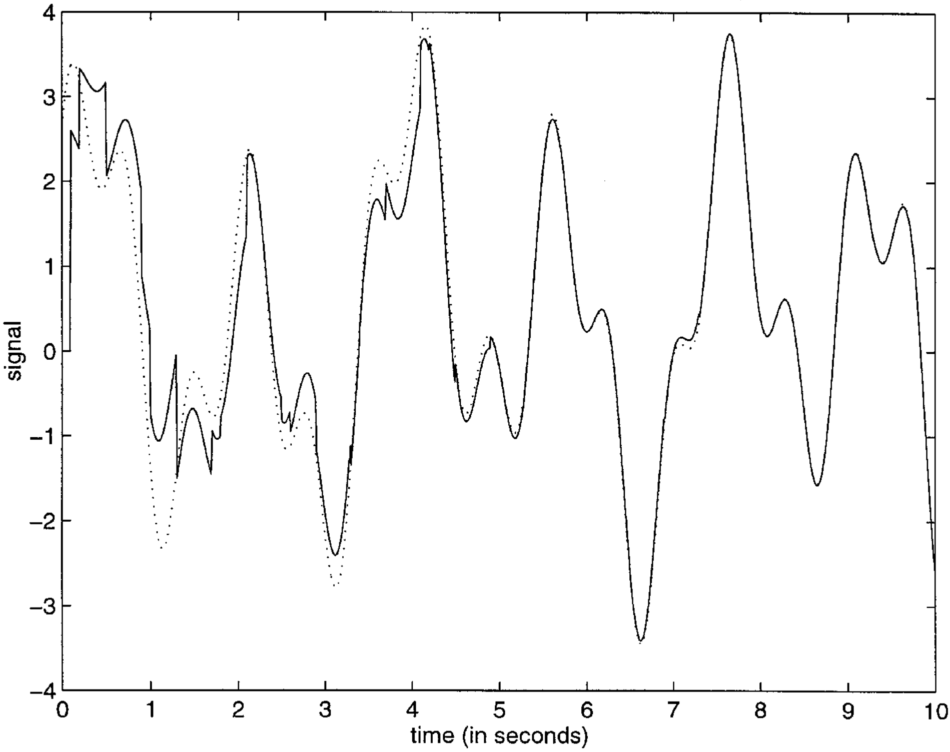

Figure 2. Continuous signal and reconstructed signal.

$$L[0] = \begin{bmatrix} 0.475 \\ 0.193 \\ 0.208 \\ 0.306 \\ 0.231 \\ 0.273 \\ 0.069 \\ -1.414 \end{bmatrix}; \quad L[1] = \begin{bmatrix} 0.533 \\ 0.243 \\ 0.153 \\ 0.472 \\ 0.140 \\ 1.097 \\ 0.425 \\ -0.222 \end{bmatrix}; \quad L[2] = L[3] - 0; L[4] \begin{bmatrix} 0.462 \\ 0.185 \\ 0.184 \\ 0.271 \\ 0.189 \\ 0.359 \\ 0.191 \\ -1.442 \end{bmatrix}$$

$$L[5] = L[6] = L[7] = 0$$

With $\hat{x}[k]$ available, Stage 2 of the reconstruction is given by

$$\hat{y}(t) = C e^{A(t-k\Delta)} \hat{x}[k] \qquad 0 \leq t - k\Delta < \Delta \tag{6.5}$$

The original continuous time signal (dotted line) and the reconstructed signal (solid line) are shown in figure 2. The step adjustments due to the discrete state updating are evident in the reconstructed signal. As anticipated, the continuous time signal is asymptotically recovered by the observer.

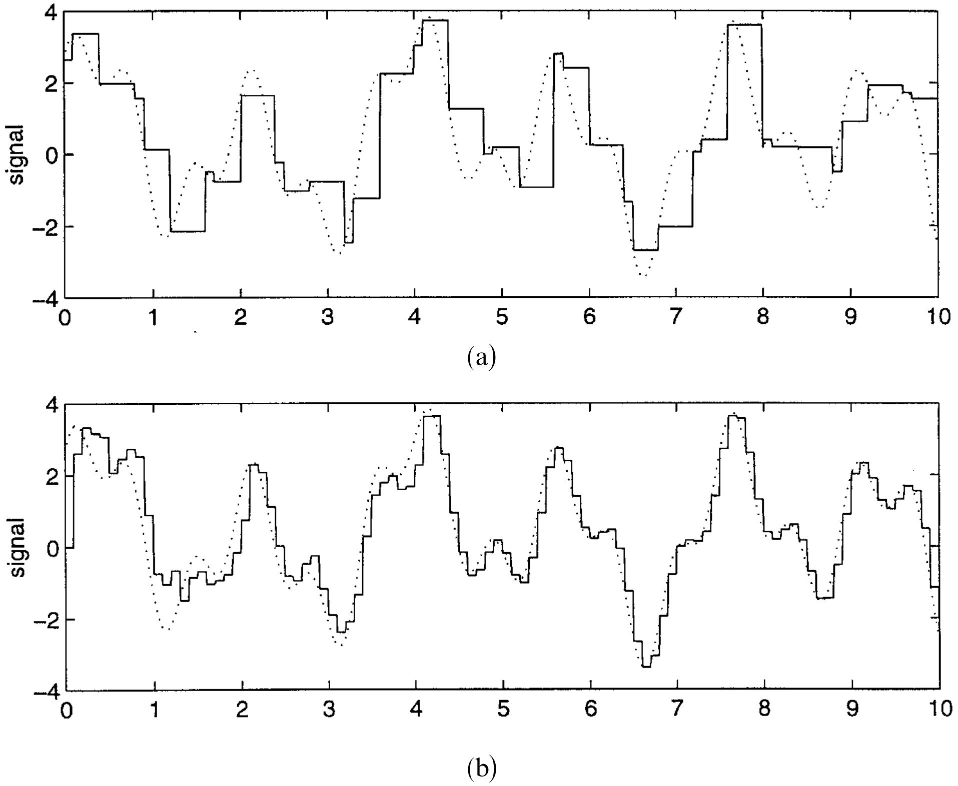

Figure 3. Reconstruction with periodic ZOH and with observer and ZOH. (a) Continuous signal and reconstructed with periodic ZOH. (b) Continuous signal and reconstructed with observer and ZOH.

Remark: As mentioned at the end of Section 5, an alternative in Stage 2 is to use, instead, a ZOH. This can be compared to using a periodic ZOH directly on $y_p[k]$. The results are shown in figure 3 where the benefit of using Stage 1 is evident. Furthermore, as pointed out in the previous section, the observer can be designed with any integer fraction of the base sampling period (0.1 s in our example) to get any desired accuracy (using the observer and ZOH only). We chose $\Delta_1 = 0.01$ s, and the results are shown in figure 4. We see that the results are now very close to the results obtained with the reconstruction obtained using Stage 2 as shown in figure 2.

7. Conclusion

This paper has shown that when a band-limited signal comprises a finite sum of sinusoids it can be reconstructed by a (causal) finite dimensional observer from non-uniform periodic samples. The sufficient condition for this reconstruction is shown to be the same as the known result for non-causal reconstruction of band-limited signals. A two-stage reconstruction is presented, leading to an exact reconstruction. We also show that recovery to any desired accuracy can be achieved using discrete time observer and ZOH only.

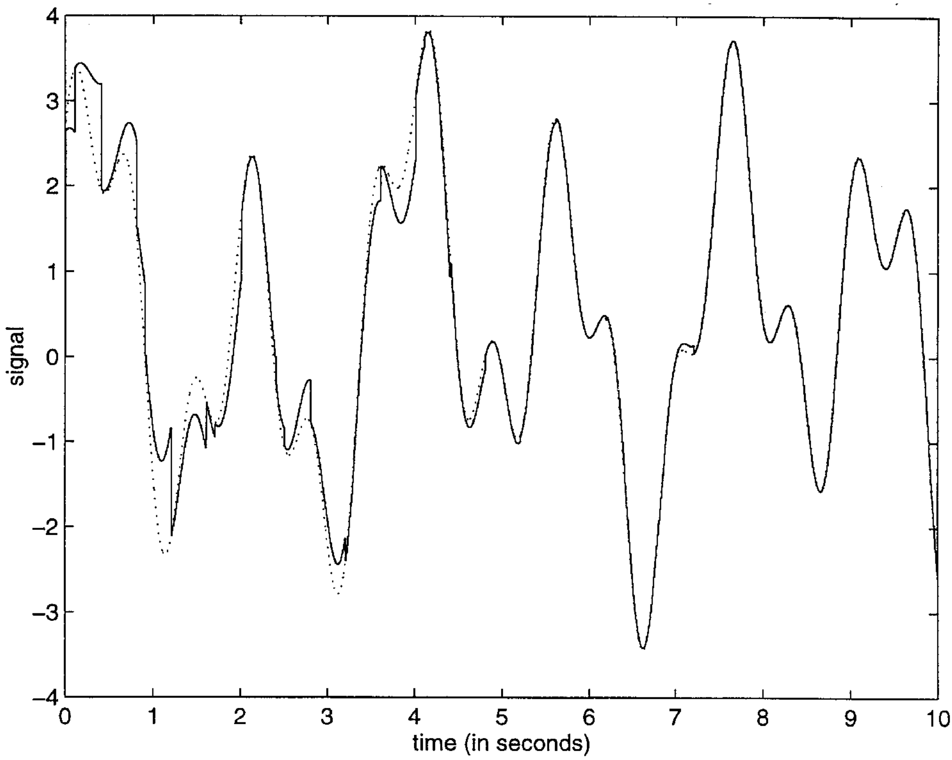

Figure 4. Reconstruction with observer and ZOH (sampling interval 0.01 s).

Appendix A

To prove the sampling extension result and derive a closed expression for $\beta_q(t)$ in equation (2.5), we assume an underlying common sampling interval. Namely, we assume

$$\Delta_q = M_q \Delta \tag{A 1}$$

for some Δ.

Let

$$Y^d(\omega) = \sum_{q=-\infty}^{\infty} \Delta_q y[q] \mathrm{e}^{-j\omega t_q} \tag{A 2}$$

be the spectrum associated with our sampled data. Also, let $Y_1^D(\omega)$ be the spectrum obtained from uniform sampling at period Δ, i.e.

$$Y_1^d(\omega) \sum_{k=-\infty}^{\infty} Y\left(\omega - k\frac{2\pi}{\Delta}\right) \tag{A 3}$$

where $Y(\omega)$ is the Fourier transform of $y(t)$ and $Y_1^d(\omega)$ is the Fourier transform of the sequence

$$y_1[k] = y(k\Delta) \tag{A 4}$$

Recovering $y(t)$ from $y[q]$ (or, equivalently, $Y(\omega)$ from $Y^d(\omega)$) will be done in two stages. First we will recover $y_1[k]$ from $y[q]$ (rather, $Y_1^d(\omega)$ from $Y^d(\omega)$), and then $y(t)$ from $y_1[k]$ (or, $Y(\omega)$ from $Y_1^d(\omega)$).

The second stage is straightforward and is well known. Namely, we simply use

$$Y(\omega) = H(\omega)\, Y_1^d(\omega) \tag{A 5}$$

with

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| < \dfrac{\pi}{\Delta} \\ 0 & \text{otherwise} \end{cases} \tag{A 6}$$

For the first stage we can write, from (A 2), (2.2) and (2.4),

$$Y^d(\omega)\frac{1}{T}\sum_{k=-\infty}^{\infty} a_k\, Y\left(\omega - k\frac{2\pi}{\Delta}\right) \tag{A 7}$$

where

$$a_k = \Delta \sum_{\ell=1}^{M} M_\ell \mathrm{e}^{jk\frac{2\pi}{T}t_\ell} \tag{A 8}$$

We can readily show that

$$a_{k+N} = a_k \qquad \text{for all } k \tag{A 9}$$

$$a_{nN} = a_0 = T \tag{A 10}$$

and

$$a_{N-r} = \bar{a}_r \quad \text{(complex conjugate) for} \quad r = 0, 1, \ldots, N \tag{A 11}$$

where

$$N = \sum_{\ell=1}^{M} M_\ell \tag{A 12}$$

Using these properties of a_k, (A 7) can be rewritten as

$$\begin{aligned} Y^d(\omega) &= \frac{1}{T}\sum_{m=-\infty}^{\infty}\sum_{\ell=1}^{N} a_{mN+\ell}\, Y\left(\omega - m\frac{2\pi}{\Delta} - \ell\frac{2\pi}{T}\right) \\ &= \frac{1}{T}\sum_{\ell=1}^{N} a_\ell \sum_{m=-\infty}^{\infty} Y\left(\omega - \ell\frac{2\pi}{T} - m\frac{2\pi}{\Delta}\right) \\ &= \frac{1}{T}\sum_{\ell=1}^{N} a_\ell\, Y_1^d\left(\omega - \ell\frac{2\pi}{T}\right) \end{aligned} \tag{A 13}$$

Or, since $Y_1^d(\omega)$ is periodic, using (A 9) we get

$$\underline{Y}^d(\omega) = A\underline{Y}_1^d(\omega) \tag{A 14}$$

where

$$\underline{Y}^d(\omega) = \left[Y^d(\omega), Y^d\left(\omega - \frac{2\pi}{T}\right), \ldots, Y^d\left(\omega - (N-1)\frac{2\pi}{T}\right) \right]^T \tag{A 15}$$

$$\underline{Y}_1^d(\omega) = \left[Y_1^d(\omega), Y_1^d\left(\omega - \frac{2\pi}{T}\right), \ldots, Y_1^d\left(\omega - (N-1)\frac{2\pi}{T}\right) \right]^T \tag{A 16}$$

and

$$A = \frac{1}{T}\begin{bmatrix} a_0 & a_1 & \ldots & a_{N-1} \\ a_{N-1} & a_0 & & a_{N-2} \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \ldots & a_0 \end{bmatrix} \tag{A 17}$$

Equation (A 14) is clearly a set of linear equations which we would like to solve for $\underline{Y}_1^d(\omega)$. Unfortunately, A is, in general, a singular matrix. Hence, to be able to solve the equations we will have to reduce the number of unknowns to be equal to the rank of A. This will be done using the band-limited property of $y(t)$ as given in (2.3).

From (2.3) it follows that

$$Y_1^d\left(\omega - \ell\frac{2\pi}{T}\right) = 0 \quad \text{for} \quad \ell = M-r, M-r+1, \ldots, N-r-1$$

$$\frac{M-2r-2}{T}\pi \le \omega < \frac{M-2r}{T}\pi \quad r = 0, 1, \ldots, M-1 \tag{A 18}$$

Let us define

$$A_r = E_r^T A E_r, \quad r = 0, 1, \ldots, M-1 \tag{A 19}$$

with

$$E_r = \left[e_1, e_2, \ldots, e_{M-r}, e_{N-r+1}, \ldots, e_K\right] \in \mathbb{R}^{N\times M} \tag{A 20}$$

e_i being the ith column of the $N \times N$ identity matrix. Using the properties of a_i and those of circulant matrices (see Muir 1960), it can be shown that A_r is nonsingular.

Using (A 18), we can write for (A 14)

$$E_r^T \underline{Y}^d(\omega) = A_r E_r^T \underline{Y}_1^d(\omega) \quad \text{for} \quad \frac{M-2r-2}{T}\pi \le \omega < \frac{M-2r}{T}\pi \quad r = 0, 1, \ldots, M-1 \tag{A 21}$$

Because of the nonsingularly of A_r, (A 21) can be uniquely solved for $E_r^T \underline{Y}_1^d(\omega)$. Namely, if we denote

$$b_\ell^r = e_1^T E_r A_r^{-1} E_r^T e_\ell \quad \text{for} \quad \ell = 1, 2, \ldots, N \quad r = 0, 1, \ldots, M-1 \tag{A 22}$$

we have, from (A 21),

$$Y_1^d(\omega) = \sum_{\ell=1}^{N} b_\ell^r Y^d\left(\omega - (\ell - 1)\frac{2\pi}{T}\right) \quad \text{for} \quad \frac{M-2r-2}{T}\pi \le \omega < \frac{M-2r}{T}\pi \tag{A 23}$$

Combining the solution in all subintervals of ω, we get the solution for the whole interval

$$\left[-\frac{M\pi}{T}, \frac{M\pi}{T}\right)$$

Namely,

$$Y_1^d(\omega) = \begin{cases} \sum_{r=0}^{M-1} H_r(\omega) \sum_{\ell=1}^{N} b_\ell^r Y^d\left(\omega - (\ell - 1)\frac{2\pi}{T}\right) & \text{for} \quad |\omega| < \frac{M\pi}{T} \\ 0 & \text{for} \quad \frac{N\pi}{T} < |\omega| < \frac{\pi}{\Delta} \end{cases} \tag{A 24}$$

where

$$H_r(\omega) = \begin{cases} 1 & \text{for} \quad \frac{M - 2r - 2}{T}\pi \le \omega < \frac{M - 2r}{T}\pi \\ 0 & \text{elsewhere} \end{cases} \tag{A 25}$$

and (A 24) is repeated periodically (with period $2\pi/\Delta$). This completes the first stage of the reconstruction. Then, combining (A 5) and (A 24), we get the reconstructed signal.

Applying the inverse Fourier transform to (A 24) we get, after some algebra, the result in equation (2.5) with

$$\beta_q(t) = \frac{1}{N} M_q \sum_{r=0}^{M-1} \sum_{\ell=1}^{N} e^{j(\ell-1)\frac{2\pi}{T}t_q} \cdot e^{j\frac{M-2r-1}{T}\pi(t-t_q)} \tag{A 26}$$

Clearly, $\beta_q(t)$ depends only on the sampling pattern (namely on $\{M_q\}$, Δ).

References

BITTANTI, S. P., COLANERI, P., and NICOLAE, G., 1991, The periodic Riccati equation. In S. P. Bittanti, Laub and Willems (Eds) *The Riccati Equation* (Berlin: Springer-Verlag).

FLAMM, S. D., 1991, A new shift-invariant representation for periodic linear systems. *System and Control Letters*, **17**, 9–14.

GOODWIN, G. C., LEE, S. J., CARLTON, A., and WALLACE, G., 1994, Application of Kalman filtering to zinc coating mass estimation. In *IFAC CCA '94*, Special Session on Steel Industry, Glasgow, UK.

JERRI, A. J., 1977, The Shannon sampling theorem—its various extensions and applications: a tutorial review. *Proceedings of the IEEE*, **65**, 1565–1596.

KAILATH, T., 1980, *Linear Systems* (New Jersey: Prentice-Hall).

KHARGONEKAR, P. P., POOLA, K., and TANNENBAUM, A., 1985, Robust control of linear time-invariant plants using periodic compensation. *IEEE Transactions on Automatic Control*, **AC-30**, 1088–1099.

MISRA, P., 1996, Time-invariant representation of discrete period systems. *Automatica*, **32**, 267–272.

MUIR, T., 1960, *A Treatise on the Theory of Determinants* (New York: Dover).

PAPOULIS, A., 1968, *Systems and Transforms with Applications in Optics* (New York: McGraw-Hill).