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Admissible sets in linear feedback systems with bounded controls†

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The concept of maximally admissible sets in regulator design for linear systems is defined and investigated, and efficient computational methods for finding maximally admissible sets are presented.

The significance of admissible set considerations in regulator design when control saturation constraints are present is illustrated via a practical synthesis example.

1. Introduction

In recent years, the synthesis problem of multivariable linear feedback systems received a great deal of attention in the literature, with particular emphasis on such problems as decoupling, pole assignment, output regulation, internal stabilization, etc. (see e.g. Wolovich (1974), Wonham (1974) and the literature cited therein).

Probably the simplest class of linear synthesis problems is the following: Consider the linear time invariant system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \tag{1.1}$$

where $x = x(t) \in \mathbf{R}^n$ is the state vector, $u = u(t) \in \mathbf{R}^m$ is the control vector, $y = y(t) \in \mathbf{R}^p$ is the output vector, and A, B, C and D are constant real matrices of appropriate dimensions. The feedback (control) law is given by an expression of the form

$$u = Fx + Gw \tag{1.2}$$

where $w = w(t) \in \mathbf{R}^m$ is the 'external control' and F and G are constant real matrices with G non-singular.

The 'closed loop' equations are then

$$\left.\begin{aligned} \dot{x} &= \hat{A}x + \hat{B}w \\ y &= \hat{C}x + \hat{D}w \end{aligned}\right\} \tag{1.3}$$

where $\hat{A} = A + BF$, $\hat{B} = BG$, $\hat{C} = C + DF$ and $\hat{D} = DG$. The *synthesis problem* is to find F and G (if they exist) so that the quadruple of matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ satisfies certain algebraic conditions (under which (1.3) has a predetermined dynamical behaviour).

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In practical situations, while of paramount importance, the above algebraic considerations are frequently incomplete because of the presence of (saturation) control constraints. Typically, the control $u(t)$ is constrained to take values in some bounded (and closed) region $U \subset \mathbf{R}^m$ where the boundaries of U are the saturation values of the control. Hence, the closed loop eqns. (1.3) are satisfied only so long as

$$Fx(t) + Gw(t) \in U \tag{1.4}$$

holds, and violation of this constraint will cause breakdown of the closed loop dynamics. To prevent such violation it is necessary to constrain the state $x(t)$ and the external control $w(t)$ to be maintained in some subsets $X \subset \mathbf{R}^n$ and $\Omega \subset \mathbf{R}^m$, respectively. The sets X and Ω are satisfactory for adequate feedback performance and are called *admissible sets* provided $Fx + Gw \in U$ for every pair $(x, w) \in X \times \Omega$.

While admissibility of a pair (X, Ω) is necessary, this property alone (being a pointwise property) does not necessarily imply that the pair (X, Ω) is also dynamically satisfactory. To this end it is required that for each initial state $x_0 = x(t_0) \in X$ there is a control function $w(t)$ $(t \geqslant t_0)$ taking values in Ω, such that the resultant state trajectory $x(t)$ remains in X for all $t \geqslant t_0$. A region X possessing this property is called *weakly Ω-invariant* (Feuer and Heymann (1976)). Thus, weak Ω-invariance is also required for adequate feedback performance.

Naturally, both the admissibility of a pair (X, Ω), and the weak Ω-invariance of X usually depend on the choice of the feedback matrices F and G. Since in most synthesis problems F and G are not unique, the freedom in their selection can be used to obtain admissible sets as large as possible so that the region of adequate feedback performance (without control constraint violation) is suitably maximized.

It is interesting to note that these important practical considerations of admissibility and weak invariance have not been discussed in the literature to date. This is especially the case in view of the fact that it is easy to demonstrate in many synthesis problems that certain algebraically suitable selections of F and G may render the closed loop system practically inoperable in that the region of admissibility is severely curtailed. In fact, it may often be the case that admissibility questions overshadow certain refined dynamical considerations.

The present paper is devoted to the computation of maximally admissible sets, i.e. admissible sets which, in a suitably defined way, cannot be enlarged. While the analysis is confined to the simple synthesis problem defined by (1.1) and (1.2), similar considerations are valid in more elaborate situations and the computations can be applied in analogous manner. It is shown how maximally admissible sets can be effectively computed, and in special (but quite common) situations how the evaluation can be carried out through closed form expressions. Hence the computation of maximally admissible sets can be easily incorporated into a synthesis algorithm and thus be used to find feedback matrices F and G for which adequate (or even optimal) admissible sets are obtained. Finally, it is shown via a practical synthesis example, how considerations of admissible sets may decisively effect a practical design problem. The problem of weak Ω-invariance is investigated elsewhere (Feuer and Heyman (1976)).

2. Preliminaries

The following notation will be used: Capital italic letters A, B, ..., will denote matrices and their corresponding linear transformations. $\mathscr{R}(A)$ and $\mathscr{N}(A)$ are the range of A and the null space of A, respectively. For a set $U \subset \mathbf{R}^k$, ∂U will denote its boundary and int (U) its interior.

Vectors are denoted by lower case italic letters $x, u, w, \ldots$, and superscripts are used for their enumeration (e.g. x^i, u^j), whereas subscripts are used for their coordinates (e.g. x_i, u_j). For a matrix H, H_k will denote its kth row. The empty set will be denoted by ϕ.

Assume now that a given synthesis problem has been solved with a control law (1.2) and closed loop eqns. (1.3). If (X, Ω) is a pair of admissible sets for this synthesis, then the admissibility conditions $x(t) \in X$ and $w(t) \in \Omega$ need to be constantly verified during the course of the closed loop system's operation. This verification may require quite cumbersome computations unless the sets X and Ω have very simple geometry and easily verified boundary conditions. Since one of the main objectives for control synthesis is to minimize the need for manual (or computational) supervision of the system's operation, it is, in practice, necessary to select admissible sets X and Ω of the simplest and most easily computable geometries. Hence, it will be assumed throughout that X and Ω are rectangular and take the form

$$X = \{x \in \mathbf{R}^n \mid \underline{x}_i \leqslant x_i \leqslant \bar{x}_i,\ i = 1, 2, \ldots, n\} \tag{2.1}$$

$$\Omega = \{w \in \mathbf{R}^m \mid \underline{w}_i \leqslant w_i \leqslant \overline{w}_i,\ i = 1, 2, \ldots, m\} \tag{2.2}$$

where $\underline{x}_i$ and $\bar{x}_i$ are the lower and upper saturation values of the ith component of the state vector, and $\underline{w}_i$ and $\overline{w}_i$ are the lower and upper saturation values of the ith component of the external control.

Since the magnitudes and relative ranges of operation are usually quite well known, it will also be assumed that

$$v_X \triangleq (\bar{x}_1 - \underline{x}_1, \ldots, \bar{x}_n - \underline{x}_n)^{\mathrm{T}} = s_X \xi \tag{2.3}$$

$$v_\Omega \triangleq (\overline{w}_1 - \underline{w}_1, \ldots, \overline{w}_m - \underline{w}_m)^{\mathrm{T}} = s_\Omega \eta \tag{2.4}$$

for some scalars s_X and s_Ω, where ξ and η are fixed (prescribed) unit vectors: (Euclidean norm equals 1) and where $(\cdot)^{\mathrm{T}}$ denotes transpose. Hence, the vectors ξ and η are considered as part of the problem's data, and in computation of the admissible sets it is only necessary to determine the scalars s_X, s_Ω and the 'centre points' $x_c = \frac{1}{2}(\bar{x} + \underline{x})$ and $w_c = \frac{1}{2}(\overline{w} + \underline{w})$ of the admissible sets X and Ω.

With the above framework in mind, we can now make the following:

Definition 2.1

Let F and G in (1.2) be fixed and let U, ξ and η be given. Then a pair (X, Ω) of the form (2.1) and (2.2) is *admissible* (with respect to the above data) if (i) $Fx + Gw \in U$ for each $x \in X$ and $w \in \Omega$, and (ii) there exist scalars s_X and s_Ω such that (2.3) and (2.4) are satisfied. The pair (X, Ω) is *maximally admissible* if it is admissible and if given any other admissible pair (X', Ω') that satisfies both $s_{X'} \geqslant s_X$ and $s_{\Omega'} \geqslant s_\Omega$, then $s_{X'} = s_X$ and $s_{\Omega'} = s_\Omega$.

If X and Ω are sets satisfying (2.1)–(2.4) then the *vertices* of these sets x^i, $i=1, \ldots, 2^n$ and w^j, $j=1, \ldots, 2^m$ are the vectors (appropriately enumerated) which satisfy the condition that for each component k, either $x_k{}^i = \overline{x}_k$ or $x_k{}^i = \underline{x}_k$ (respectively, $w_k{}^j = \overline{w}_k$ or $w_k{}^j = \underline{w}_k$). The diagonal directions of X and Ω are then given by $v_X{}^i = (x^i - x_c)/\|x^i - x_c\|$, $i=1, \ldots, 2^n$ and $v_\Omega{}^j = (w^j - w_c)/\|w^j - w_c\|$, $j=1, \ldots, 2^m$. These diagonal directions can, of course, be directly computed from the vectors ξ and η if the latter are known.

Assume now that ξ and η are fixed and that U is a given compact and convex restraint set. If (X, Ω) is an admissible pair satisfying (2.1)–(2.4), then $Fx^i + Gw^j \in U$ for every pair of vertices x^i and w^j, $i=1, \ldots, 2^n, j=1, \ldots, 2^m$. In order for (X, Ω) to be maximally admissible, it is clearly necessary that $Fx^i + Gw^j \in \partial U$ at least for some pair of vertices (x^i, w^j). This last fact will be used as a main tool in the computation of maximally admissible sets below.

It will be assumed throughout the paper that the set U is compact and convex and has non-empty interior.

3. Maximally admissible sets

We begin by considering the case where the external control takes on a fixed preassigned single value, i.e. $\Omega = \{w_0\}$ (for example, w_0 may be zero, which corresponds to the case of no external control at all). In this case clearly $s_\Omega = 0$ and the control u takes the form $u(t) = Fx(t) + Gw_0$.

Let U be the compact convex restraint set for the control and assume that U has non-empty interior. Then the translated set $\overline{U} = \{v \in \mathbf{R}^m \mid v + Gw_0 \in U\}$ is also compact convex with non-empty interior, and for fixed ξ, a set X satisfying (2.1) and (2.3) is admissible if and only if $X \subset X^0$, where $X^0 = \{x \in \mathbf{R}^n \mid Fx \in \overline{U}\}$. Since any admissible set is uniquely defined by its centre x_c and by the scalar s_X, these are then the values we seek for maximal admissibility.

Let $v_X{}^j$, $j=1, \ldots, 2^n$ be the diagonal directions of X determined by the given vector ξ, and consider the set $U^0 = U \cap \mathscr{R}(F)$. If $U^0 = \phi$ then the set X^0 is empty and no admissible set X exists (this can happen as a result of improper selection of w_0). Hence assume $U^0 \neq \phi$ and let $u \in U^0$ be any vector. By definition of U^0 there then exists $x \in X^0$ such that $u = Fx$ and if $v_X{}^j$ is any diagonal direction such that $v_X{}^j \notin \mathscr{N}(F)$, there exists a finite non-negative real number $\alpha_j(u)$ defined by

$$\alpha_j(u) = \max \{t \in \mathbf{R} \mid u + tFv_X{}^j \in \overline{U}\} \tag{3.1}$$

Since the vectors $v_X{}^j$, $j=1, \ldots, 2^n$ span $\mathbf{R}^n$, the set $I \triangleq \{j \mid v_X{}^j \notin \mathscr{N}(F)\}$ is non-empty whenever $F \neq 0$. In case $F=0$, $X = X^0 = \mathbf{R}^n$ is admissible and the maximal admissibility problem is trivial. Hence, we may assume $F \neq 0$ so that $I \neq \phi$ holds.

For each $j \in I$ we can now define the function $\alpha_j : U^0 \rightarrow \mathbf{R}$ where $\alpha_j(u)$ is as in (3.1). In view of the compactness of U the functions $\alpha_j(\cdot)$ are well defined and bounded.

Lemma 3.1

If U is compact and convex then for each $j \in I$ the function $\alpha_j : U^0 \rightarrow \mathbf{R}$ defined by (3.1) *is concave.*

Proof

Let $u_1, u_2 \in U^0$ be any two vectors and let

$$u_{01} = u_1 + \alpha_j(u_1) F v_X{}^j \in \overline{U}$$

$$u_{02} = u_2 + \alpha_j(u_2) F v_X{}^j \in \overline{U}$$

Then by the convexity of $\overline{U}$,

$$u_0(\lambda) \triangleq \lambda u_{01} + (1-\lambda) u_{02} = \lambda u_1 + (1-\lambda) u_2 + [\lambda \alpha_j(u_1) + (1-\lambda)\alpha_j(u_2)] F v_X{}^j \in \overline{U}$$

for every real $0 \leqslant \lambda \leqslant 1$. Also

$$\lambda u_{01} + (1-\lambda) u_{02} \in \overline{U}$$

$$\lambda u_1 + (1-\lambda) u_2 \in \overline{U}$$

and it follows immediately from the definition of $\alpha_j(\,\cdot\,)$ that

$$\lambda \alpha_j(u_1) + (1-\lambda)\alpha_j(u_2) \leqslant \alpha_j(\lambda u_1 + (1-\lambda) u_2)$$

Hence $\alpha_j(\,\cdot\,)$ is concave as claimed. ∎

Assume now that $F \neq 0$ and that U^0 is non-empty. Then I is a non-empty set, and define the function $\alpha : U^0 \to \mathbf{R}$ by

$$\alpha(u) = \min \{\alpha_j(u) \,|\, j \in I\} \tag{3.2}$$

Since the function $\alpha(\,\cdot\,)$ is defined as the minimum of a finite collection of concave functions, $\alpha(\,\cdot\,)$ is also concave, Rockafellar (1970), and hence attains a maximal value on the compact set U^0. We can now state the following result on maximally admissible sets :

Theorem 3.2

Let U be a given compact convex restraint set with non-empty interior. Let ξ, F, G and w_0 be fixed and let the feedback law be given by $u(t) = Fx(t) + Gw_0$. Assume that U^0 is non-empty and that $F \neq 0$, and let $u_c \in U^0$ be any vector which satisfies $\alpha(u_c) = \max \{\alpha(u) \,|\, u \in U^0\}$ where $\alpha(\,\cdot\,)$ is defined by (3.2). *Then there exists $x_c \in X^0$ such that $u_c = Fx_c$, and the set X determined by the centre x_c, by the vector ξ and the scalar $s_X = 2\alpha(u_c)$ is a maximal admissible set contained in X^0.*

Proof

We will first show that $X \subset X^0$. Let $v_X{}^j$, $j = 1, \ldots, 2^n$ be the diagonal directions determined by ξ. Then the vertices of X are given by

$$x^j = x_c + \alpha(u_c) v_X{}^j, \quad j = 1, 2, \ldots, 2^n$$

and every $x \in X$ can be written as a convex combination of these vertices, i.e. $x = \sum_{j=1}^{2^n} \lambda_j x^j$, $0 \leqslant \lambda_j \leqslant 1$, $\sum_{j=1}^{2^n} \lambda_j = 1$.

By (3.1) and (3.2) we have

$$u_c + \alpha(u_c) F v_X{}^j \in \overline{U} \quad \text{for all } j = 1, \ldots, 2^n$$

and since $\bar{U}$ is convex (and so is also U^0),

$$Fx = \sum_{j=1}^{2^n} \lambda_j [u_c + \alpha(u_c) F v_X{}^j] \in \bar{U}$$

and hence $x \in X^0$. Since $x \in X$ was arbitrarily chosen it follows that $X \subset X^0$.

Consider now any set $X' \subset X^0$ satisfying (2.1) and (2.3) with centre at $x_c{}'$ and $v_X{}' = s_X{}'\xi$. The vertices of X' are then $x'^j = x_c{}' + \frac{s_{X'}}{2} v_X{}^j$, $j = 1, \ldots, 2^n$, and

$$Fx'^j = Fx_c{}' + \frac{s_{X'}}{2} F v_X{}^j \in \bar{U}, \quad j \in I$$

Hence, $s_{X'}/2 \leqslant \alpha_j(u_c{}')$ for every $j \in I$ where $u_c{}' \triangleq Fx_c{}'$, and by the maximality of $\alpha(u_c)$ it follows that $s_X{}' \leqslant 2\alpha(u_c{}') \leqslant 2\alpha(u_c) = s_X$ which concludes the proof. ∎

The preceding analysis can now be summarized as follows. In order to compute a maximally admissible set X in X^0, one has to solve the problem

$$\left.\begin{array}{r} \text{Find } \max \alpha(u) \\ \\ u \in U \end{array}\right\} \tag{3.3}$$

and apply Theorem 3.2. In view of the fact that $\alpha(u)$ is a concave function on the compact convex set U, this maximization problem is a standard convex programming problem, Mangasarian (1969), for which many efficient computational algorithms are known.

Frequently, the set U is not an arbitrary convex set but rather is polyhedral and sometimes even symmetric. In these special cases, as we shall see below, the computation of maximally admissible sets is considerably simplified.

3.1. *U is polyhedral*

Assume now that the set U is given by $U = \{u \mid Hu \leqslant h\}$ where $h \in \mathbf{R}^p$ is a fixed real vector and H is a real matrix of appropriate dimensions. Assume further that U is bounded (and thus compact and convex). The set $\bar{U}$ is then given by

$$\bar{U} = \{v \in \mathbf{R}^m \mid Hv \leqslant h^0\}$$

where $h^0 \triangleq h - HGw_0$, and the sets X^0 and U^0, respectively, are defined by

$$X^0 = \{x \in \mathbf{R}^n \mid HFx \leqslant h^0\}$$

and

$$U^0 = \{u = Fx \mid x \in X^0\}$$

Let $v_X{}^j$, $j = 1, \ldots, 2^n$ be the diagonal directions of X. By (3.1), for each $u \in U^0$, the vector $v_j{}^0(u) \triangleq u + \alpha_j(u) F v_X{}^j$ is on the boundary of the set U^0 and hence lies in at least one of the hyperplanes $\mathscr{L}_k \triangleq \{v \in \mathbf{R}^m \mid H_k v = h_k{}^0\}$, $k = 1, \ldots, p$ (where H_k is the kth row of H and $h_k{}^0$ is the kth component of h^0). The assumption that U is bounded (and non-empty) implies that $\mathscr{N}(H) = 0$ and hence $HFv_X{}^j \neq 0$ if and only if $j \in I$. Thus, for $j \in I$ the vector $v_j{}^0(u)$ is in $\mathscr{L}_k$ whenever

$$H_k(u + \alpha_j(u) F v_X{}^j) = h_k{}^0$$

from which it follows that

$$\alpha_j(u) = \frac{h_k^0 - H_k u}{H_k F v_X^j}$$

We now define the list $S(j)$ by

$$S(j) \triangleq \{r \mid H_r F v_X^j > 0,\ r = 1, \ldots, p\}$$

and it is readily verified that for each $j \in I$

$$\alpha_j(u) = \min_{k \in S(j)} \left\{ \frac{h_k^0 - H_k u}{H_k F v_X^j} \right\} \tag{3.4}$$

and thus

$$\alpha(u) = \min_{j \in I} \left[\min_{k \in S(j)} \left\{ \frac{h_k^0 - H_k u}{H_k F v_X^j} \right\} \right] \tag{3.5}$$

Define now the list $P = \{k \mid H_k F \neq 0,\ k = 1, \ldots, p\}$ and relable the rows of H such that $k \in P$ for $k = 1, \ldots, \bar{p}$ $(\bar{p} \leqslant p)$. For each $k \in P$ define

$$s_k = \max_{j \in I} \{H_k F v_X^j\}$$

Clearly $s_k > 0$ for all $k \in P$ since for each vector v_X^j, the vector $(-)v_X^j$ is also a diagonal direction of X. The expression for $\alpha(u)$ can then be written as

$$\alpha(u) = \min_{k \in P} \left\{ \frac{h_k^0 - H_k u}{s_k} \right\}$$

for each $u \in U^0$, and thus

$$H_k u + s_k \alpha(u) \leqslant h_k^0\ ; \quad k \in P$$

Define now the $\bar{p} \times (m+1)$ matrix E and the vectors c, $y \in \mathbf{R}^{m+1}$ by

$$E = \begin{bmatrix} H_1 & s_1 \\ H_2 & s_2 \\ \vdots & \vdots \\ H_{\bar{p}} & s_{\bar{p}} \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} u \\ \alpha(u) \end{bmatrix}$$

and the computation of u_c and $\alpha(u_c)$ reduces to the following problem :

$$\begin{cases} \text{Find} & \max c^{\mathrm{T}} y \\ \text{subject to} & Ey \leqslant h^0 \end{cases}$$

This is a standard problem in linear programming and can readily be solved using the simplex method. This problem is clearly simpler than the convex programming problem (3.3).

3.2. *U symmetric*

Definition 3.3

A set $U \subset \mathbf{R}^m$ is called *symmetric with respect to a point of symmetry* $\tilde{u} \in \mathbf{R}^m$ if and only if $u \in U$ implies that $\bar{u} = 2\tilde{u} - u \in U$.

Proposition 3.4

Let $U \subset \mathbf{R}^m$ be a compact convex set, symmetric with respect to a point $\tilde{u} \in U$. Then $\alpha(\tilde{u}) = \max\{\alpha(u),\ u \in U\}$.

Proof

Let $u \in U$ be arbitrary and let $\bar{u} = 2\tilde{u} - u \in U$. For an arbitrary $j \in I$, let k be such that $v_X{}^k = -v_X{}^j$. Then $u + \alpha_j(u) F v_X{}^j \in U$ and $\bar{u} + \alpha_k(\bar{u}) F v_X{}^k \in U$ which in turn imply that $2\tilde{u} - [\bar{u} - \alpha_j(u) F v_X{}^j] \in U$ and $2\tilde{u} - [u - \alpha_k(\bar{u}) F v_X{}^k] \in U$. Since $\bar{u} - \alpha_j(u) F v_X{}^j = \bar{u} + \alpha_j(u) F v_X{}^k \in U$ and also $u - \alpha_k(u) F v_X{}^k = u + \alpha_k(u) F v_X{}^j \in U$ it is readily verified that $\alpha_j(u) \leqslant \alpha_k(\bar{u})$ and $\alpha_k(\bar{u}) \leqslant \alpha_j(u)$ implying that $\alpha_k(\bar{u}) = \alpha_j(u)$. It thus follows immediately that $\alpha(\bar{u}) = \alpha(u)$. By the concavity of $\alpha(\cdot)$ on U we have

$$\alpha(u) = \lambda\alpha(u) + (1-\lambda)\alpha(\bar{u}) \leqslant \alpha(\lambda u + (1-\lambda)\bar{u})$$

for all $0 \leqslant \lambda \leqslant 1$, and by taking $\lambda = \frac{1}{2}$ it follows that $\alpha(u) \leqslant \alpha(\tilde{u})$. Since $u \in U$ was chosen arbitrarily, the result follows. ∎

Proposition 3.4 implies that in case U is a symmetric set, any point of symmetry qualifies as u_c. By application of (3.1) and (3.2), $\alpha(u_c)$ can then be immediately computed. In case U is in addition a polyhedral set, the computation is further simplified since $\alpha(u_c)$ can then be computed by the formulas (3.4) and (3.5) which is a straightforward calculation.

So far in our discussion we assumed that the external control is required to take a fixed single value w_0. This assumption is not overly restrictive as we will now see. Suppose that Ω is a compact convex subset in $\mathbf{R}^m$ (specifically Ω could be chosen to be of the form (2.2), (2.4) with s_Ω and η prespecified). Define the set

$$Y = \{y \in \mathbf{R}^m \mid y + Gw \in U \text{ for every } w \in \Omega\}$$

If U is compact and convex, it is readily verified that so is also Y and substitution of $\bar{U}$ by Y in the previous analysis will yield the required maximally admissible sets.

Conversely, it might be assumed that the set X (compact and convex) is prespecified and it is desired to compute Ω (of the form (2.2) and (2.4)) such that the pair (X, Ω) is maximally admissible. In this case the set $\bar{U}$ can be replaced by the set

$$W = \{y \in \mathbf{R}^m \mid Fx + y \in U \text{ for every } x \in X\}$$

The computation beyond this substitution would proceed exactly as before. The need to prespecify one of the sets X or Ω is a consequence of the non-uniqueness of solutions to the admissibility problem because of the trade-off effect between external control capability and range of validity of the state equations.

The complexity of computation of Y or W will in general depend on the geometry of the control constraint set U. In case the set U is bounded by saturation constraints on the individual components the computation becomes extremely simple as discussed below.

Consider a compact set $V \subset \mathbf{R}^m$ and define

$$\left.\begin{aligned} &\mathrm{Sq}(V) = \{v \mid \underline{v} \leqslant v \leqslant \bar{v}\} \\ \text{where} \qquad &\underline{v}_i = \min_{v \in V} \{v_i\}, \quad \bar{v}_i = \max_{v \in V} \{v_i\}, \quad i = 1, \ldots, m \end{aligned}\right\} \tag{3.6}$$

The following properties are then immediate :

$$V \subset \mathrm{Sq}(V) \text{ for every compact set } V \tag{3.7}$$

If V_1 and V_2 are any two compact sets in $\mathbf{R}^m$ then

$$\mathrm{Sq}(V_1 + V_2) = \mathrm{Sq}(V_1) + \mathrm{Sq}(V_2) \tag{3.8}$$

$$V_1 \subset V_2 \text{ implies } \mathrm{Sq}(V_1) \subset \mathrm{Sq}(V_2) \tag{3.9}$$

In view of these properties we now have :

Proposition 3.5

Consider the set $U = \{u \mid \underline{u} \leqslant u \leqslant \bar{u}\} \subset \mathbf{R}^m$ *and a compact set* $Y \subset \mathbf{R}^m$. *Define* $Y^0 \subset \mathbf{R}^m$ *by*

$$Y^0 = \{y \mid \underline{y}^0 \leqslant y \leqslant \bar{y}^0\}$$

where $\underline{y}^0 = \underline{u} - \underline{y}$ *and* $\bar{y}^0 = \bar{u} - \bar{y}$, *and* $\underline{y}$ *and* $\bar{y}$ *are defined as in* (3.6). *Then*

$$y + y^0 \in U \text{ for every } y \in \mathrm{Sq}(Y) \text{ if and only if } y^0 \in Y^0$$

Proof

An easy application of (3.7)–(3.9).

4. Example : The regulation of a hovering helicopter

It was stated in the introduction that one of the chief objectives of this paper is to emphasize and illustrate the importance of admissible set considerations in practical control design. Below we shall compare the maximally admissible sets in two regulator designs for a hovering helicopter. Murphy and Narendra (1969) described the problem of regulating a hovering helicopter for which they designed a regulator using optimal control techniques. Subsequently Wolovich and Shirley (1970) considered the same problem for which they synthesized a regulator whose objective was to decouple the system and stabilize it. They also compared the two designs on a simulator in order to obtain a qualitative comparison based on the 'ease' of external regulation.

The (linearized) dynamical equations for the hovering helicopter are given by

$$A = \begin{bmatrix} -0{\cdot}016 & -0{\cdot}05 & 0{\cdot}0025 & 0{\cdot}0 & 0{\cdot}0 & -0{\cdot}0001 & 0{\cdot}0 & 0{\cdot}0 & -0{\cdot}0047 \\ 0{\cdot}0 & 0{\cdot}0 & 1{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 1{\cdot}97 & 0{\cdot}0 & -0{\cdot}542 & 1{\cdot}0 & 0{\cdot}0 & 0{\cdot}548 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}736 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}00018 & -0{\cdot}3242 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 1{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 2{\cdot}61 & 0{\cdot}0 & -1{\cdot}94 & -0{\cdot}163 & 0{\cdot}0 & -1{\cdot}96 & 0{\cdot}0 & 0{\cdot}01 & -7{\cdot}25 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 1{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}016 & 0{\cdot}0 & -0{\cdot}0083 & -0{\cdot}193 & 0{\cdot}0 & -0{\cdot}0043 & 0{\cdot}0 & -0{\cdot}303 & 5{\cdot}59 \\ 0{\cdot}0047 & 0{\cdot}0 & -0{\cdot}0024 & -0{\cdot}0007 & 0{\cdot}05 & -0{\cdot}0025 & 0{\cdot}0 & 0{\cdot}0009 & -0{\cdot}003 \end{bmatrix}$$

$$B=\begin{bmatrix} 0{\cdot}05 & 0{\cdot}005 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ -6{\cdot}15 & 0{\cdot}69 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & -0{\cdot}424 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & -2{\cdot}13 & 21{\cdot}81 & 0{\cdot}3465 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 5{\cdot}13 & 0{\cdot}174 & -7{\cdot}48 \\ 0{\cdot}0 & 0{\cdot}01 & 0{\cdot}05 & 0{\cdot}022 \end{bmatrix}$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(The reader is referred to Murphy and Narendra (1969) for an interpretation of the physical variables.)

Both in Murphy and Narendra (1969) and in Wolovich and Shirley (1970) the regulator equations were of the form

$$u=Fx+Gw$$

where in Murphy and Narendra (1969) F and G where selected to be

$$F^1=\begin{bmatrix} -2{\cdot}59 & 0{\cdot}28 & 0{\cdot}2 & 0{\cdot}12 & -0{\cdot}1 & 0{\cdot}0 & -0{\cdot}01 & 0{\cdot}0 & -1{\cdot}19 \\ 0{\cdot}05 & -0{\cdot}01 & 0{\cdot}0 & 0{\cdot}99 & 0{\cdot}01 & 0{\cdot}0 & 0{\cdot}22 & 0{\cdot}28 & 0{\cdot}58 \\ 0{\cdot}02 & 0{\cdot}0 & 0{\cdot}0 & 2{\cdot}15 & 0{\cdot}01 & 0{\cdot}0 & -0{\cdot}09 & -0{\cdot}15 & -0{\cdot}14 \\ 1{\cdot}16 & -0{\cdot}11 & 0{\cdot}01 & 0{\cdot}04 & -0{\cdot}2 & -0{\cdot}07 & -0{\cdot}02 & -0{\cdot}01 & -2{\cdot}66 \end{bmatrix}$$

$$G^1=I$$

and in Wolovich and Shirley (1970) F and G were chosen as

$$F^2=\begin{bmatrix} 0{\cdot}3203 & 1{\cdot}463 & 0{\cdot}6013 & 0{\cdot}3414 & 0{\cdot}0 & 0{\cdot}0891 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}1204 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0004 & 1{\cdot}594 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ -0{\cdot}1195 & 0{\cdot}0 & 0{\cdot}0888 & 0{\cdot}1461 & 0{\cdot}4122 & -0{\cdot}1044 & 0{\cdot}0 & -0{\cdot}0208 & 0{\cdot}3202 \\ -0{\cdot}0005 & 0{\cdot}0 & 0{\cdot}0012 & 1{\cdot}096 & -0{\cdot}009 & -0{\cdot}0029 & 0{\cdot}0 & 1{\cdot}296 & 0{\cdot}7540 \end{bmatrix}$$

$$G^2=\begin{bmatrix} 1{\cdot}0 & 0{\cdot}101 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}9 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}088 & 1{\cdot}0 & 0{\cdot}015 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}022 & 1{\cdot}0 \end{bmatrix}$$

For the problem at hand, reasonable values for the constraint set U (see Feuer (1973)) are given by

$$U=\{u \mid \underline{u} \leqslant u \leqslant \overline{u}\}$$

where

$$\bar{u} = -\underline{u} = \begin{bmatrix} 0{\cdot}2618 \\ 0{\cdot}0698 \\ 0{\cdot}2618 \\ 0{\cdot}1396 \end{bmatrix}$$

Wolovich and Shirley (1970) state in their discussion that a reasonable requirement (based on practical considerations) is that under no circumstances should more than 10% of the available control be consumed by feedback and that the remaining 90% be available for external manoeuvering. Hence we set

$$Y = \{y \mid \underline{u}^0 \leqslant y \leqslant \bar{u}^0\}$$

where

$$\bar{u}^0 = -\underline{u}^0 = \begin{bmatrix} 0{\cdot}02618 \\ 0{\cdot}00698 \\ 0{\cdot}02618 \\ 0{\cdot}01396 \end{bmatrix}$$

By practical considerations we are led to choose (Feuer 1973)

$$v_X{}^0 = \begin{bmatrix} 0{\cdot}1318 \\ 0{\cdot}0316 \\ 0{\cdot}079 \\ 0{\cdot}0079 \\ 0{\cdot}1054 \\ 0{\cdot}5271 \\ 0{\cdot}8276 \\ 0{\cdot}0211 \\ 0{\cdot}0216 \end{bmatrix}$$

Since U is symmetric with $u_c = 0$, $x_c = 0$ and $X = \{x \mid \underline{x} \leqslant x \leqslant \bar{x}\}$ so that $X \subset X^0 = \{x \mid Fx \in Y\}$ will be

$$\bar{x}^1 = -\underline{x}^1 = \begin{bmatrix} 0{\cdot}0041 \\ 0{\cdot}0010 \\ 0{\cdot}0025 \\ 0{\cdot}0003 \\ 0{\cdot}0033 \\ 0{\cdot}0165 \\ 0{\cdot}0260 \\ 0{\cdot}0007 \\ 0{\cdot}0010 \end{bmatrix}$$

for the controller of Murphy and Narendra (1969), and

$$\bar{x}^2 = \underline{x}^2 = \begin{bmatrix} 0{\cdot}0186 \\ 0{\cdot}0045 \\ 0{\cdot}0111 \\ 0{\cdot}0011 \\ 0{\cdot}0149 \\ 0{\cdot}0743 \\ 0{\cdot}1164 \\ 0{\cdot}0030 \\ 0{\cdot}0040 \end{bmatrix}$$

for the controller of Wolovich and Shirley (1970). Obviously, in comparing the two controller designs, the fact that the admissible set in Wolovich and Shirley (1970) is approximately four times larger (i.e. $s_X{}^2 \cong 4 s_X{}^1$) than the admissible set in Murphy and Narendra (1969) is of considerable practical importance in that the design in Wolovich and Shirley (1970) is much less sensitive to control saturation.

References

Feuer, A., 1973, M.Sc. Thesis, Technion, Haifa.
Feuer, A., and Heymann, M., 1976, *J. Math. Analysis applic.* **53,** No. 2.
Mangasarian, O. L., 1969, *Nonlinear Programming* (New York : McGraw-Hill).
Murphy, R. D., and Narendra, K. S., 1969, *J. Aircraft*, **6,** 129.
Rockaffellar, R. T., 1970, *Convex Analysis* (Princeton).
Wolovich, W. A., 1974, *Linear Multivariable Systems*, Applied Mathematical Sciences Series, Vol. 11 (New York : Springer Verlag).
Wolovich, W. A., and Shirley, R. S., 1970, *Proc. JACC*, Atlanta, Georgia, pp. 297–305.
Wonham, W. M., 1974, *Linear Multivariable Control*, Lecture Notes in Economics and Mathematical Systems, Vol. 101 (New York : Springer Verlag).