On the Necessity of Papoulis' Result for Multidimensional GSE

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***Abstract*—As is well known, Papoulis' generalized sampling expansion (GSE) can be carried out if a set of linear equations has a solution. This enables the reconstruction of a signal from its filtered and downsampled versions. In this letter, we show that the existence of this solution is also a necessary condition for the reconstruction of the signal from the available data. Special attention is given to the case of recurrent multidimensional sampling.**

***Index Terms*—Multidimensional sampling, nonuniform sampling.**

I. Introduction

THIS LETTER consists of two parts. In the first part, we introduce a multidimensional version of Papoulis' result [1]. **au: vol. & issue no./month for [1]?** While the configuration is only somewhat more general than previously done (see [2]), our main contribution is the proof of a *necessary* as well as sufficient condition for signal reconstruction.

One of the applications for the one-dimensional (1-D) Papoulis' generalized sampling expansion (GSE) has been for the problem of reconstruction from recurrent irregular sampling—e.g., see [3] and [4]. In the second part of this letter, we relate the above multidimensional result to the multidimensional recurrent sampling problem. The necessary and sufficient condition in this case is restated and discussed.

We start with some notation. Let $\Lambda = \{U\mathbf{k} : \mathbf{k} \in \mathbb{Z}^D\}$ ($U \in R^{D\times D}$ is a nonsingular matrix) be a D-dimensional lattice and Λ^* its reciprocal lattice, namely, $\Lambda^* = \{2\pi U^{-T}\mathbf{k} : \mathbf{k} \in \mathbb{Z}^D\}$. Let $\mathcal{P}(\Lambda^*)$ denote the unit cell for Λ^*. Then, we have $(\mathcal{P}(\Lambda^*) + \omega_1) \cap (\mathcal{P}(\Lambda^*) + \omega_2) = \emptyset$ for all $\omega_1 \neq \omega_2 \in \Lambda^*$ and $\bigcup_{\omega\in\Lambda^*} (\mathcal{P}(\Lambda^*) + \omega) = \mathbb{R}^D$ (see [5]).

II. On the Necessity of Papoulis' Result

We are ready now to formally state the problem we are interested in.

Consider a band limited signal $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^D$. Namely, its Fourier transform $\widehat{f}(\omega)$ *exists* and has a *finite support*. Then, for a given D-dimensional lattice Λ there exists a finite set $\{\mathbf{c}_m\}_{m=1}^M \subset \Lambda^*$ such that

$$\text{support}\left(\widehat{f}(\omega)\right) \subseteq \mathcal{S} = \cup_{m=1}^M \left(\mathcal{P}(\Lambda^*) + \mathbf{c}_m\right) \tag{1}$$

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Suppose now we generate a data set $\{\{g_m(U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$ as described in Fig. 1. Namely, $f(\mathbf{x})$ is passed through the filters $\widehat{h}_m(\omega)$ (we assume the $\widehat{h}_m(\omega)$ to be continuous within $\mathcal{S}$) with outputs $g_m(\mathbf{x})$ each of which is sampled on the lattice Λ. The question is *when* and *how* can $f(\mathbf{x})$ be reconstructed from this data set. As we pointed out, Cheung [2] has discussed this problem. Note, however, that our formulation is slightly more general than the one presented by Cheung. The result in [2] can readily be generalized to our formulation.

For the configuration in Fig. 1 the reconstruction formula of $f(\mathbf{x})$ from the data $\{\{g_m(U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$ is as follows:

$$f(\mathbf{x}) = \sum_{m=1}^{M}\sum_{\mathbf{k}\in\mathbb{Z}^D} g_m(U\mathbf{k})\varphi_m(\mathbf{x} - U\mathbf{k}) \tag{2}$$

where

$$\varphi_m(\mathbf{x}) = \frac{1}{|\det(U)|}\int_{\mathcal{P}(\Lambda^*)} \mathbf{\Phi}_m(\omega, \mathbf{x})e^{j\omega^T\mathbf{x}}d\omega \tag{3}$$

$\{\mathbf{\Phi}_m(\omega, \mathbf{x})\}_{m=1}^M$ are the solutions (*if they exist*) of the equations

$$H(\omega)\begin{bmatrix}\mathbf{\Phi}_1(\omega,\mathbf{x})\\ \mathbf{\Phi}_2(\omega,\mathbf{x})\\ \vdots\\ \mathbf{\Phi}_M(\omega,\mathbf{x})\end{bmatrix} = \begin{bmatrix}e^{j\mathbf{c}_1^T\mathbf{x}}\\ e^{j\mathbf{c}_2^T\mathbf{x}}\\ \vdots\\ e^{j\mathbf{c}_M^T\mathbf{x}}\end{bmatrix} \quad \text{for } \omega \in \mathcal{P}(\Lambda^*) \text{ and all } \mathbf{x} \tag{4}$$

and the matrix $H(\omega)$ is

$$H(\omega) = \begin{bmatrix}\widehat{h}_1(\omega+\mathbf{c}_1) & \widehat{h}_2(\omega+\mathbf{c}_1) & \cdots & \widehat{h}_M(\omega+\mathbf{c}_1)\\ \widehat{h}_1(\omega+\mathbf{c}_2) & \widehat{h}_2(\omega+\mathbf{c}_2) & \cdots & \widehat{h}_M(\omega+\mathbf{c}_2)\\ \vdots & \vdots & \ddots & \vdots\\ \widehat{h}_1(\omega+\mathbf{c}_M) & \widehat{h}_2(\omega+\mathbf{c}_M) & \cdots & \widehat{h}_M(\omega+\mathbf{c}_M)\end{bmatrix} \tag{5}$$

for $\omega \in \mathcal{P}(\Lambda^*)$

Following the discussion in [3], it can readily be observed that the nonsingularity of $H(\omega)$ on $\mathcal{P}(\Lambda^*)$ (or rather, the existence of its bounded inverse) is a *sufficient* condition for the posed reconstructability problem. We now make the following claim.

Theorem 1: Let $H(\omega)$ be defined as in (5). Then, if $H(\omega)$ is singular for some $\omega \in \mathcal{P}(\Lambda^*)$, $f(\mathbf{x})$ cannot be uniquely reconstructed from the data $\{\{g_m(U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$.

Proof: We will prove this theorem by constructing a function $f^o(\mathbf{x})$ (not identically zero) which when fed into the configuration of Fig. 1 results in samples $\{\{g_m(U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D} = \{0\}$.

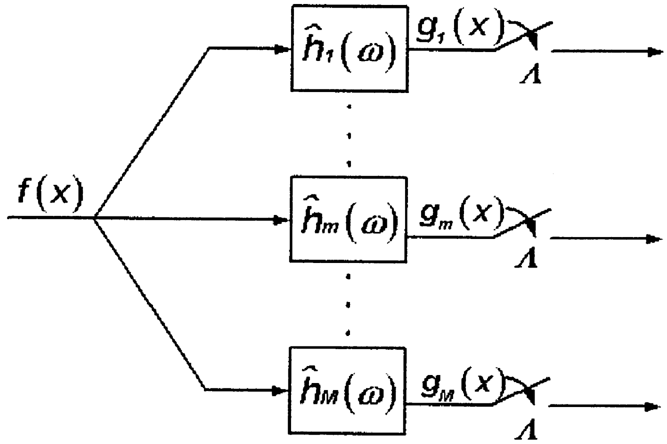

Fig. 1. M-channel configuration.

Suppose $H(\omega_0)$ is singular for some $\omega \in \mathcal{P}(\Lambda^*)$. Then, there exist $\{f_n^o\}_{n=1}^M$ such that

$$[f_1^o \quad f_2^o \quad \cdots \quad f_M^o]\, H(\omega_0) = [0 \quad 0 \quad \cdots \quad 0] \tag{6}$$

Then, let us define $f^o(\mathbf{x})$ through its Fourier transform

$$\widehat{f^o}(\omega) = \sum_{n=1}^{M} f_n^o \delta(\omega - \omega_0 - \mathbf{c}_n) \tag{7}$$

Note that $f^o(\mathbf{x}) = (1/2\pi)^D \sum_{n=1}^{M} f_n^o e^{j\mathbf{c}_n^T \mathbf{x}}$ which is clearly, not identically zero and $\widehat{f^o}(\omega)$ is band limited to $\mathcal{S}$. Then, using (1), (6), (7), and the unit cell properties, we get

$$\begin{aligned} g_m^o(U\mathbf{k}) &= \left(\frac{1}{2\pi}\right)^D \int_{\mathcal{S}} \widehat{f^o}(\omega)\widehat{h}_m(\omega) e^{j\omega^T U\mathbf{k}} d\omega \\ &= \left(\frac{1}{2\pi}\right)^D \sum_{n=1}^{M} \int_{\mathcal{P}(\Lambda^*)+\mathbf{c}_n} \sum_{i=1}^{M} (f_i^o \delta\omega - \omega_0 - \mathbf{c}_i) \\ &\quad \cdot \widehat{h}_m(\omega) e^{j\omega^T U\mathbf{k}} d\omega \\ &= \left(\frac{1}{2\pi}\right)^D \sum_{n=1}^{M} \int_{\mathcal{P}(\Lambda^*)} \sum_{i=1}^{M} f_i^o \delta(\omega + \mathbf{c}_n - \omega_0 - \mathbf{c}_i) \\ &\quad \cdot \widehat{h}_m(\omega + \mathbf{c}_n) e^{j(\omega+\mathbf{c}_n)^T U\mathbf{k}} d\omega \\ &= \left(\frac{1}{2\pi}\right)^D e^{j\omega_0^T U\mathbf{k}} \sum_{i=1}^{M} f_i^o \widehat{h}_m(\omega_0 + \mathbf{c}_i) \\ &\quad \cdot \int_{\mathcal{P}(\Lambda^*)} \delta(\omega - \omega_0) d\omega \\ &= 0 \end{aligned} \tag{8}$$

for all $\mathbf{k} \in \mathbb{Z}^D$ and $m = 1, 2, \ldots, M$. Namely, $f(\mathbf{x})$ and $f(\mathbf{x}) + f^o(\mathbf{x})$ result in the same data $\{\{g_m(U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$ hence, unique reconstruction is impossible. ■

Remark 2: Because of the properties of $\widehat{h}_m(\omega)$, the matrix $H(\omega)$ is either singular over a finite number of ω's or for all frequencies. The necessity proof holds for both possibilities. Above, we have carried it out for the first possibility. Since for the recurrent sampling discussed in the next section, it is necessarily the latter, we rederive the proof for the second possibility in the next section.

Remark 3: We have assumed that the number of filters is the same as the number of vectors in the set $\{\mathbf{c}_m\}_{m=1}^M$. This leads to a square $H(\omega)$ matrix. One may have a case however, where the number of filters is larger than M. It is then, necessary and sufficient for perfect reconstruction, that the matrix $H(\omega)$ has full row rank. Then, one has (infinitely) many solutions of the set of linear equations, leading to different sets of reconstruction functions $\{\varphi_m(\mathbf{x})\}$. Each set can be used for perfect reconstruction.

III. Multidimensional Recurrent Sampling

In the sequel, we will restrict ourselves to a special case of the above formulation which arises in the recurrent sampling problem. Specifically, suppose our data is $\{\{f(\mathbf{x}_m + U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$ for a set $\{\mathbf{x}_m\}_{m=1}^M$. We would like to know when can one reconstruct $f(\mathbf{x})$ from it. Without loss of generality we will assume in the sequel that $\mathbf{x}_1 = \mathbf{0}$.

We observe that by defining $\widehat{h}_m(\omega) = e^{j\omega^T \mathbf{x}_m}$ we do get $g_m(U\mathbf{k}) = f(\mathbf{x}_m + U\mathbf{k})$ and we can apply the results above to get the reconstruction if the corresponding $H(\omega)$ is nonsingular on $\mathcal{P}(\Lambda^*)$. However, in this case the condition is simpler and we restrict ourselves here to finite energy signals.

Theorem 4: The reconstruction of $f(\mathbf{x}) \in \mathcal{L}^2(\mathbb{R}^D)$ from the data $\{\{f(\mathbf{x}_m + U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$ is possible if and **only if** $H(\mathbf{0})$ is nonsingular.

Proof: Let us first write the matrix $H(\omega)$ in our case

$$\begin{aligned} H(\omega) &= \begin{bmatrix} e^{j(\omega+\mathbf{c}_1)^T\mathbf{x}_1} & e^{j(\omega+\mathbf{c}_1)^T\mathbf{x}_2} & \cdots & e^{j(\omega+\mathbf{c}_1)^T\mathbf{x}_M} \\ e^{j(\omega+\mathbf{c}_2)^T\mathbf{x}_1} & e^{j(\omega+\mathbf{c}_2)^T\mathbf{x}_2} & \cdots & e^{j(\omega+\mathbf{c}_2)^T\mathbf{x}_M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(\omega+\mathbf{c}_M)^T\mathbf{x}_1} & e^{j(\omega+\mathbf{c}_M)^T\mathbf{x}_2} & \cdots & e^{j(\omega+\mathbf{c}_M)^T\mathbf{x}_M} \end{bmatrix} \\ &= \begin{bmatrix} e^{j\mathbf{c}_1^T\mathbf{x}_1} & e^{j\mathbf{c}_1^T\mathbf{x}_2} & \cdots & e^{j\mathbf{c}_1^T\mathbf{x}_M} \\ e^{j\mathbf{c}_2^T\mathbf{x}_1} & e^{j\mathbf{c}_2^T\mathbf{x}_2} & \cdots & e^{j\mathbf{c}_2^T\mathbf{x}_M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j\mathbf{c}_M^T\mathbf{x}_1} & e^{j\mathbf{c}_M^T\mathbf{x}_2} & \cdots & e^{j\mathbf{c}_M^T\mathbf{x}_M} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} e^{j\omega^T\mathbf{x}_1} & 0 & \cdots & 0 \\ 0 & e^{j\omega^T\mathbf{x}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{j\omega^T\mathbf{x}_M} \end{bmatrix} \\ &= H(\mathbf{0})\mathrm{diag}\left\{e^{j\omega^T x_m}\right\}. \end{aligned} \tag{9}$$

Since $\mathrm{diag}\{e^{j\omega^T\mathbf{x}_m}\}$ is nonsingular for all ω, $H(\omega)$ is nonsingular for all ω if and only if $H(\mathbf{0})$ is nonsingular. Hence, the reconstruction procedure outlined in (2)–(4) applies here when, indeed, $H(\mathbf{0})$ is nonsingular, which completes the sufficiency part.

Suppose now that $H(\mathbf{0})$ is singular. Then, there exist $\{f_n^o\}_{n=1}^M$ not all zero such that

$$[f_1^o \quad f_2^o \quad \cdots \quad f_M^o]\, H(\mathbf{0}) = \mathbf{0} \Rightarrow \sum_{n=1}^{M} f_n^o e^{j\mathbf{c}_n^T\mathbf{x}_m} = 0 \quad \text{for } m = 1, 2, \ldots, M \tag{10}$$

Let us define the signal $f^o(\mathbf{x})$ band-limited to $\mathcal{P}(\Gamma^*)$ such that

$$\widehat{f^o}(\omega) = f_n^o \text{ for } \omega \in (\mathcal{P}(\Lambda^*) + c_n); \quad n = 1, \ldots, M \tag{11}$$

Clearly, $f^o(\mathbf{x}) \in \mathcal{L}^2(\mathbb{R}^D)$. Then, we have, using again the fact that $\mathbf{c}_n^T U\mathbf{k}$ is an integer multiple of 2π and (10)

$$\begin{aligned}
&f^o(\mathbf{x}_m + U\mathbf{k})\\
&= \left(\frac{1}{2\pi}\right)^D \int_{\mathcal{S}} \widehat{f^o}(\omega) e^{j\omega^T(\mathbf{x}_m + U\mathbf{k})} d\omega\\
&= \left(\frac{1}{2\pi}\right)^D \sum_{n=1}^{M} \int_{\mathcal{P}(\Lambda^*)+\mathbf{c}_n} \widehat{f^o}(\omega) e^{j\omega^T(\mathbf{x}_m + U\mathbf{k})} d\omega\\
&= \left(\frac{1}{2\pi}\right)^D \sum_{n=1}^{M} f_n^o \int_{\mathcal{P}(\Lambda^*)+\mathbf{c}_n} e^{j\omega^T(\mathbf{x}_m + U\mathbf{k})} d\omega\\
&= \left(\frac{1}{2\pi}\right)^D \sum_{n=1}^{M} f_n^o \int_{\mathcal{P}(\Lambda^*)} e^{j(\omega+\mathbf{c}_n)^T(\mathbf{x}_m + U\mathbf{k})} d\omega\\
&= \left(\frac{1}{2\pi}\right)^D \sum_{n=1}^{M} f_n^o e^{j\mathbf{c}_n^T(\mathbf{x}_m + U\mathbf{k})} \int_{\mathcal{P}(\Lambda^*)} e^{j\omega^T(\mathbf{x}_m + U\mathbf{k})} d\omega\\
&= \left(\frac{1}{2\pi}\right)^D \sum_{n=1}^{M} f_n^o e^{j\mathbf{c}_n^T \mathbf{x}_m} \int_{\mathcal{P}(\Lambda^*)} e^{j\omega^T(\mathbf{x}_m + U\mathbf{k})} d\omega\\
&= 0
\end{aligned}$$

for all $\mathbf{k} \in \mathbb{Z}^D$ and $m = 1, 2, \ldots, M$. This means that $f(\mathbf{x}) \neq f(\mathbf{x}) + f^o(\mathbf{x}) \in \mathcal{L}^2(\mathbb{R}^D)$, while $\{\{f(\mathbf{x}_m + U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D} = \{\{f(\mathbf{x}_m + U\mathbf{k}) + f^o(\mathbf{x}_m + U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$. Namely, unique reconstruction is not possible, which completes the proof. ■

The above result is quite meaningful for the multidimensional recurrent sampling problem. Given the set of sampled data $\{\{f(\mathbf{x}_m + U\mathbf{k})\}_{m=1}^M\}_{\mathbf{k}\in\mathbb{Z}^D}$ one could ask what family of (band limited) signals, $f(\mathbf{x})$, could be reconstructed from this sampling configuration. Specifically, given U and M, every choice of matrix $N \in \mathbb{Z}^{D\times D}$ such that $|\det N| = M$ determines a lattice $\Gamma_N = \{UN^{-1}\mathbf{n} : \mathbf{n} \in \mathbb{Z}^D\}$, the corresponding unit cell $\mathcal{P}(\Gamma^*{}_N)$ and set $\{\mathbf{c}_m\}_{m=1}^M$. These determine the matrix $H(\mathbf{0})$. Now, using the result above, we can make the following statement: All $f(\mathbf{x}) \in \mathcal{L}^2(\mathbb{R}^D)$ for which support $(\widehat{f}(\omega)) \subset \mathcal{P}(\Gamma^*{}_N)$ can be reconstructed from their samples on the given sampling pattern if $H(\mathbf{0})$ is nonsingular. This would be of much interest for various super resolution problems with global motion (see [6] for the 1-D case).

IV. Conclusion

We have considered in this letter the problem of reconstructing a multidimensional signal from a set of its filtered versions, each sampled at rate below the signal Nyquist rate. We have shown that the commonly known sufficient condition for the reconstruction (using Papoulis' GSE) is *necessary* as well. Special attention was given to the multidimensional recurrent sampling problem resulting in a much simplified condition.

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