An Unstable Dynamical System Associated with Model Reference Adaptive Control

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Abstract—**It is shown that a certain system of differential equations of importance to the proof of stability of the adaptive system proposed in [1], admit unbounded solutions. The implication of this result is that a much more elaborate argument than heretofore thought necessary is required to prove that the adaptive system of [1] is stable.**

In studying the asymptotic behavior of the adaptive control system proposed in [1], one encounters equations of the form

$$\dot{\eta} = -\eta + \phi(t)\delta \tag{1}$$

$$\dot{\delta} = -\phi(t)\eta \tag{2}$$

$$\dot{w}_1 = -w_1 + \phi^2(t)\eta \tag{3}$$

where as in [1], η, δ, w_1, and ϕ are the augmented error, parameter error, auxiliary signal, and sensitivity function, respectively, of the adaptive system. These particular equations result if one assumes (for simplicity) that $D_m(p) = (p+1)D_w(p)$, $D_f(p) = p+1$, $N=4$, $\delta_0(t) \equiv \delta_1(t) \equiv \delta_4(t) \equiv 0$, and $\delta(t) = \delta_3(t)$, where D_m, D_w, D_f, N and the δ_i are as defined in [1].

To prove that the adaptive system of [1] is stable, it is necessary to show that η, δ, and w are bounded. Since the structure of the adaptive system makes it difficult to deduce very much about ϕ unless η, δ, and w_1 are known *a priori* to be bounded, the approach in [1] and elsewhere has been to try to establish the boundedness of η, δ, w_1 without first assuming that ϕ is bounded. To get some idea of what is involved, observe that for continuous ϕ the time function

$$\alpha = \tfrac{1}{2}(\eta^2 + \delta^2) \tag{4}$$

satisfies

$$\dot{\alpha} = -\eta^2 \tag{5}$$

from which boundedness of η and δ directly follow. This and (2) imply that the output of any stable first-order linear system with input $\phi\eta$, is bounded. It is thus reasonable to expect that w_1, the output of a stable first-order linear system forced by $\eta\phi^2$, will be bounded as well. The following counterexample shows that this is not the case.

Proposition: If

$$\phi = \dot{\theta} + (\sin\theta)(\cos\theta) \tag{6}$$

where

$$\theta = e^{-t}\sin^2(e^{6t}) \tag{7}$$

then there exists an unbounded solution to (1)–(3).

Since the sensitivity function ϕ actually generated by the adaptive system of [1] is not, in fact, arbitrary, the preceding is *not* a counterexample to the claim of stability of the adaptive system proposed in [1]. On the other hand, the example does imply that a much more elaborate argument involving the differential equations which generate ϕ is required to prove that the adaptive system is stable.

To prove the proposition, first observe from (1), (2), and (7), with $\eta(0) = \sin(\sin 1)$ and $\delta(0) = \cos(\sin 1)$, that $\eta = \zeta\sin\theta$ where $\delta = \zeta\cos\theta$ and

$$\zeta(t) = e^{-\int_0^t \sin^2\theta(\tau)\,d\tau}. \tag{8}$$

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Hence,

$$\eta\phi^2 = \zeta\phi^2\sin\theta. \tag{9}$$

The definition of θ in (5) implies that

$$\sin\theta \geqslant \theta^2/2 \geqslant 0 \tag{10}$$

and that $\theta^2 \leqslant e^{-2t}$; from the last inequality, the trigonometric relation $\sin^2\theta \leqslant \theta^2$ and (8) it follows that $\zeta(t) \geqslant c_1 \equiv e^{-1/2}$. This (9) and (10) thus yield $\eta\phi^2 \geqslant c_1\phi^2\sin\theta$. Using (4) to substitute for ϕ, we obtain

$$\eta\phi^2 \geqslant c_1\sin\theta\left(\dot{\theta}^2 + 2\dot{\theta}\,(\sin\theta)(\cos\theta) + (\sin^2\theta)(\cos^2\theta)\right). \tag{11}$$

Now observe that from (5), $\dot{\theta} + \theta = 6e^{5t}\sin(2\gamma)$ where

$$\gamma = e^{6t}. \tag{12}$$

Hence,

$$\dot{\theta}^2\sin\theta = (6e^{5t}\sin 2\gamma)^2\sin\theta - (2\theta\dot{\theta} + \theta^2)(\sin\theta). \tag{13}$$

If we now define

$$\left.\begin{aligned} b_1 &= \left((\sin^2\theta)(\cos^2\theta) - \theta^2\right)\sin\theta\right)c_1 \\ b_2 &= \left(2/3\sin^3\theta + 2(\theta\cos\theta - \sin\theta)\right)c_1 \end{aligned}\right\} \tag{14}$$

then using (11) and (12),

$$\eta\phi^2 \geqslant c_1\sin\theta\,(6e^{5t}\sin 2\gamma)^2 + b_1 + \dot{b}_2. \tag{15}$$

From (10), and then (7), and (12)

$$\begin{aligned} \sin\theta\,(6e^{5t}\sin 2\gamma)^2 &\geqslant 18\theta^2e^{10t}\sin^2 2\gamma \\ &= 18e^{8t}(\sin^4\gamma)(\sin^2 2\gamma) \\ &= 18e^{8t}(1-\cos^2\gamma)^2(\sin^2 2\gamma) \\ &\geqslant 18e^{8t}(1-2\cos^2\gamma)(\sin^2 2\gamma) \\ &= -18e^{8t}(\cos 2\gamma)(\sin^2 2\gamma) \\ &= -\frac{1}{2}e^{2t}\frac{d}{dt}(\sin^3 2\gamma). \end{aligned}$$

This and (15) thus show that

$$\eta\phi^2 \geqslant -c_2e^{2t}\frac{d}{dt}(\sin^3 2\gamma) + b_1 + \dot{b}_2 \tag{16}$$

where $c_2 = c_1/2 > 0$.

From the easily verified identities

$$-\int_0^t e^{3\tau}\frac{d}{d\tau}(\sin^3 2\gamma)\,d\tau = -e^{3t}\sin^3 2\gamma + 3\int_0^t e^{3t}\sin^3 2\gamma\,d\tau$$

and

$$3e^{3t}\sin^3 2\gamma = \frac{3}{4}b_3 + \frac{1}{4}\dot{b}_3$$

where

$$b_3 \equiv e^{-3t}\left(\frac{1}{3}\cos^3 2\gamma - \cos 2\gamma\right) \tag{17}$$

it follows that

$$\int_0^t e^{3\tau}\frac{d}{d\tau}(\sin^3 2\gamma)\,d\tau = -e^{3t}\sin^3 2\gamma + \frac{1}{4}\int_0^t\left(3b_3 + \dot{b}_3\right)d\tau.$$

Thus, from (16)

$$\int_0^t e^{(\tau-t)}\eta(\tau)\phi^2\,d\tau \geqslant -c_2e^{2t}\sin^3 2\gamma + b(t) \tag{18}$$

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where

$$b(t) = -\frac{e^{-t}}{4} c_2 \int_0^t (3b_3+\dot{b}_3)\,d\tau + \int_0^t e^{(\tau-t)}\left(b_1+\dot{b}_2\right)d\tau.$$

Since (14) and (17) show that b_1, b_2 and b_3 are bounded, $b(t)$ is a bounded function as well.

Thus, if we take $w_1(t)$ to be the zero initial condition solution to (3), then from (18), and (12)

$$w_1(t) \geq -c_2 e^{2t} \sin^3(2e^{6t}) + b(t).$$

Clearly $w_1(t)$ is unbounded. □

References

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Parametric Identification of Unstable Linear Systems

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***Abstract*—Identification of an unstable subsystem operating in a stable closed-loop in the presence of noise, is considered. The Equation Error–Input Covariance (EEIC) method is shown to be applicable. The method can be implemented on-line except for the case where the identified system has poles or zeros on the imaginary axis. A simulated example demonstrates the results.**

I. Introduction

The difficulty in the direct identification of unstable systems obviously results from the difficulty of obtaining and utilizing divergent data in conventional open-loop identification algorithms. Unstable systems however, invariably operate in a closed-loop together with additional stabilizing feedback loops or networks. In general, the dynamics of these stabilizing loops are known. In [1] and [2] the closed-loop identification problem is considered in conjunction with Least Squares Estimation. It is shown in this paper that direct unbiased estimation of an unstable system can be obtained by means of the EEIC method, described in [3], recently developed by the authors. Overall closed-loop stability is assumed. The method is shown to be directly applicable except for the case in which the identified unstable system has poles or zeros on the imaginary axis, which is treated separately. A simulated example demonstrates the results for two levels of noise.

II. Direct Identification of Open-Loop Dynamics

We consider the identification of the open-loop dynamics of the subsystem $G(s)$ which may have right-hand poles. In terms of Laplace transforms it is given by

$$G(s) = \frac{N(s,\boldsymbol{b})}{D(s,\boldsymbol{a})} = \frac{\sum_{i=0}^{m} b_i s^i}{1+\sum_{j=1}^{n} a_j s^j}. \tag{1}$$

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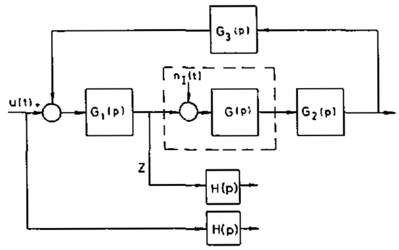

Fig. 1. Closed-loop system.

It is a part of a closed-loop system (Fig. 1) incorporating other known dynamical systems. The system is excited by a stationary random input, and uncorrelated additive noise is present in the loop. The following assumptions regarding the input, system, and noise are made:

1) The input $u(t)$ is a sample of a random stationary mean square bounded ergodic process. Its spectral distribution guarantees persistent excitation of all the modes of $G(s)$.

2) The closed-loop system, denoted by $T(s)$, is stable and time invariant.

3) The noise $n_I(t)$ is a zero mean stationary process uncorrelated with $u(t)$.

We also assume that the subsystem $G(s)$ has no poles or zeros on the imaginary axis. This assumption will be relaxed later.

It is required to identify the order and form (n,m) and parameters $(\boldsymbol{a},\boldsymbol{b})$ of $G(s)$ from $u(t)$ and the closed-loop system output $z(t)$ (Fig. 1). We now show that the method described in [3] is also valid for an unstable $G(s)$. The closed-loop transfer function relating $z(t)$ to $u(t)$ is

$$T(s) = G_1(s)\left[1+G_1(s)G_2(s)G_3(s)G(s)\right]^{-1}. \tag{2}$$

In time domain $z(t)$ is given by

$$z(t) = T(p)u(t) + n(t) \tag{3}$$

where $p^i \triangleq d^i/dt^i$, and

$$n(t) = -G_2(p)G_3(p)G(p)T(p)n_I(t). \tag{4}$$

The known subsystems $G_i(s) \triangleq N_i(s)/D_i(s)$, $i=1,2,3$ and the filtered "states" $z_i(t)$, $i=0,1,\cdots,l$; $u_j(t)$, $j=0,1,\cdots,h$ are defined in [3]. h and l determine the higher order open-loop model [4]. The open-loop parameters a_j, b_i in (1) are defined by the model parameters $\Gamma \triangleq \mathrm{col}(\alpha_1,\alpha_2,\cdots,\alpha_l,\beta_0,\cdots,\beta_h)$ and by means of the "composite filtered states" [3] defined by

$$\eta_i(t) \triangleq F_2(p)z_i(t) - F_1(p)u_i(t) \qquad i=0,1,\cdots,l$$

and

$$\xi_j(t) \triangleq F_3(p)z_j(t) \qquad j=0,1,\cdots,h$$

where $F_1(p)$, $F_2(p)$ and $F_3(p)$ are defined by

$$F_1(p) \triangleq N_1(p)D_2(p)D_3(p),$$

$$F_2(p) \triangleq D_1(p)D_2(p)D_3(p)$$

and

$$F_3(p) \triangleq N_1(p)N_2(p)N_3(p).$$

In terms of η_i and ξ_j the equation error is given by

$$e(t) = \eta_0(t) + \sum_{i=1}^{l} \alpha_i \eta_i(t) + \sum_{j=0}^{h} \beta_j \xi_j(t) = \eta_0(t) + \rho^T(t)\Gamma$$

$$\rho(t) \triangleq \mathrm{col}\{\eta_1(t),\eta_2(t),\cdots,\eta_l(t),\xi_0(t),\cdots,\xi_h(t)\}. \tag{5}$$

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