Generalization of Results on Vector Sampling Expansion

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***Abstract*— This paper addresses the problem of parsimonious sampling of filtered signals. A vector of bandlimited signals is passed through a MIMO filter and then sampled. Conditions on the sampling rates are presented that allow for perfect reconstruction of the original signals. We extend previous results in two aspects. First, we allow for different bandwidths in signal vector entries while previous results were restricted to the same bandwidth in all entries. Second, we emphasize the parsimonious aspect of the sampling. Namely, in each case, including those previously treated, we highlight the gains which can be made via downsampling.**

***Index Terms*— Sampling, generalized sampling, reconstruction**

I. INTRODUCTION

It is well known that a bandlimited signal can be completely reconstructed from its samples if the sampling is sufficiently "dense". In many applications, a signal is measured ('viewed') through a number of channels and then sampled. An obvious question of interest is, given the configuration of the channels (which could be viewed as filter bank), what would be the most beneficial sampling policy while maintaining the ability to reconstruct the observed signal. By beneficial we mean the least possible sampling rates.

A cornerstone result in this area is the, so-called, generalized sampling expansion (GSE) due to Papoulis [6]. This result has been further generalized in several directions. For example, there has been work on efficient ways of implementing the Papoulis reconstruction [1], results on multidimensional signals, i.e. functions of several variables, [2], [3] and results on vector signals, i.e. vector functions of a scalar variable, [7], [8]. Our focus in the current paper is on the latter class of problems which are referred to as vector sampling expansion (VSE) problems. The investigation of vector sampling expansions (VSE) is motivated by many practical scenarios - multiaccess wireless communication systems, radar or sonar systems with multiple transmitters and multiple receivers, RGB color acquisition systems, to name a few.

Our contribution consists of two parts. In the first part we revisit the cases treated in [7], [8] in a more general setting using a more convenient notation. These cases are limited by the requirement that all input vector signal entries have the same bandwidth. In the second part of our contribution we present novel results for cases where this constraint is removed. Thus, we allow for a different bandwidth in each entry of the measured vector signal and discuss potential resulting gains.

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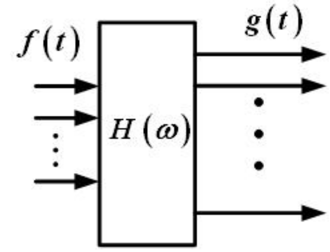

Fig. 1. The vector sampling expansion setup.

II. PROBLEM SET UP

We consider the situation where a vector of signals is passed through a multi-input multi-output (MIMO) filter and then sampled. The core question of interest is: "What can be gained in sampling rates if the number of outputs is larger then the number of inputs ?". A number of applications where this problem arises has been discussed earlier and in [7].

More specifically, consider the setup in Figure 1. The signal $\mathbf{f}(t) \in \mathbb{R}^N$, $t \in \mathbb{R}$, passes through a MIMO filter denoted by $H(\omega) \in \mathbb{C}^{M\times N}$ with $\mathbf{g}(t) \in \mathbb{R}^M$ as the output. We then have, in the frequency domain, that

$$\widehat{\mathbf{g}}(\omega) = H(\omega)\widehat{\mathbf{f}}(\omega) \tag{1}$$

Here, and elsewhere in the paper, a '$\widehat{\ }$' on a variable denotes its Fourier transform.

We assume that each of the input vector entries $f_n(t)$, is band limited with the bandwidth W_n, $n = 1, 2, ..., N$. We will consider both the cases where $W_n = W$ (a constant) and where W_n depends on n. Earlier work, as described in [7], [8], deals only with cases where all entries of the *input* vector $\mathbf{f}(t)$, have the *same* bandwidth.

We wish to emphasize here that our concern in each scenario is, whether a perfect reconstruction is possible for *any* MIMO filter $H(\omega)$. Namely, we assume that one has the freedom to choose this filter as desired. All the conditions presented here are such that, if satisfied, a filter exists for which perfect reconstruction is possible.

The layout of the remainder of the paper is as follows: In Section III, we treat the cases where all input entries have the same bandwidth. We treat separately the two possibilities, all output entries sampled at the same rate in III-A and at different rates in III-B. In Section IV, we treat the cases where input entries may have different bandwidths, with the same two possibilities for the output sampling rates in IV-A and IV-B. Through some examples we highlight the value of recognizing the different bandwidths at the input. Finally, in Section V some concluding remarks are provided.

III. Equal bandwidths of input entries

Here we revisit the cases treated in [7], [8]. Specifically, we assume all input entries have the *same bandwidth*, namely, $W_n = W$ for $n = 1, 2, ..., N$. When the ratio M/N is an integer, it has been shown in [8] that $\mathbf{g}(t)$ can be sampled at a M/N slower rate whilst ensuring that perfect reconstruction (PR) is possible. This is repeated in [7] with the additional claim that for uniform sampling this is a necessary condition as well. Here we rederive these results using an alternative argument based on straightforward algebra. In fact, we make a more general statement covering the case when the ratio M/N is not an integer and show that a gain of $\lfloor M/N \rfloor$ can be made in sampling rate while PR is still possible.

A. Output entries sampled at the same (uniform) rate

Suppose all entries of the output vector $\mathbf{g}(t)$, are sampled uniformly at the interval

$$T_o = \alpha \frac{\pi}{W} \tag{2}$$

Under these conditions, we can establish:

Theorem 1: Let $\mathbf{f}(t)$, $\mathbf{g}(t)$, and T_o be as above. Then, perfect reconstruction (PR) of $\mathbf{f}(t)$ from the data $\{\mathbf{g}(kT_o)\}_{k\in\mathbb{Z}}$ is possible if and only if $\alpha \leq \left\lfloor \frac{M}{N} \right\rfloor$.

Proof: The proof is provided in Appendix A. ■

Remark 2: The result in [7] for this case applies to the choice $\alpha = \frac{M}{N}$.

Remark 3: For discrete time signals a statement similar to Theorem 1 can be made. PR of the input vector sequence, in this case, is possible if the decimation (down sampling) of the output sequence is by an *integer* factor $\leq \left\lfloor \frac{M}{N} \right\rfloor$.

B. Output entries sampled at different (uniform) rates

In this subsection we again consider the case where all input entries are of the same bandwidth W, but each entry of the output vector $\mathbf{g}(t)$ may be sampled at a different sampling rate. Specifically, we assume that the mth output entry is sampled at intervals

$$T_m = \frac{\widetilde{Q}_m}{\widetilde{R}_m} \frac{\pi}{W} \tag{3}$$

where $\widetilde{Q}_m, \widetilde{R}_m$ are coprime integers. Let $Q = 1\,\mathrm{cm}\left(\widetilde{Q}_m\right)$ then (3) can be rewritten as

$$\begin{aligned} T_m R_m &= Q \frac{\pi}{W} \\ &= T_o \end{aligned} \tag{4}$$

Clearly, a necessary condition for PR is that the rates satisfy the following inequality

$$2NW \leq \sum_{m=1}^{M} \frac{2\pi}{T_m}$$

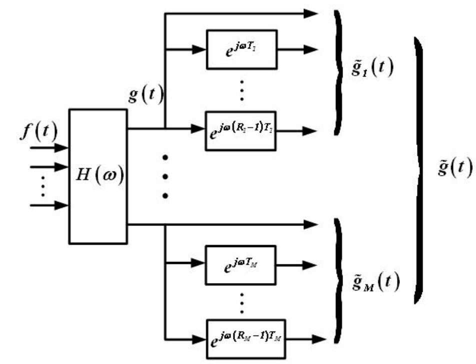

Fig. 2. Equivalent configuration with equal output sampling.

Using (4), this implies

$$NQ \leq \sum_{m=1}^{M} R_m \tag{5}$$

We next convert this problem into an equivalent one with *equal* sampling rates at the output. To do this, we employ the polyphase representation (see e.g. [9]). We generate the vectors

$$\widetilde{\mathbf{g}}_m(t) = \begin{bmatrix} g_m(t) \\ g_m(t+T_m) \\ \vdots \\ g_m(t+(R_m-1)T_m) \end{bmatrix} \in \mathbb{R}^{R_m} \tag{6}$$

and concatenate them to obtain

$$\widetilde{\mathbf{g}}(t) = \begin{bmatrix} \widetilde{\mathbf{g}}_1(t) \\ \widetilde{\mathbf{g}}_2(t) \\ \vdots \\ \widetilde{\mathbf{g}}_M(t) \end{bmatrix} \in \mathbb{R}^{\sum_{m=1}^{M} R_m} \tag{7}$$

where g_m is the mth entry of $\mathbf{g}$. It is readily seen that the original data, i.e. $\left\{\{g_m(k_m T_m)\}_{k_m\in\mathbb{Z}}\right\}_{m=1}^{M}$, is equivalent to the data $\left\{\{\widetilde{\mathbf{g}}_m(kT_o)\}_{m=1}^{M}\right\}_{k\in\mathbb{Z}}$. Thus, we have converted the problem of non equal output sampling rates to a problem with uniform sampling. We want PR of $\mathbf{f}(t)$ from $\widetilde{\mathbf{g}}(t)$ sampled at T_o. By (6) we observe that the transfer matrix from $\mathbf{f}(t)$ to $\widetilde{\mathbf{g}}(t)$ is given by

$$H_Q(\omega) = C(\omega) H(\omega) \in \mathbb{C}^{\left(\sum_{m=1}^{M} R_m\right)\times N} \tag{8}$$

where

$$C(\omega) = \operatorname{diag}\left\{ \begin{bmatrix} 1 \\ e^{j\omega T_m} \\ \vdots \\ e^{j\omega(R_m-1)T_m} \end{bmatrix} \right\} \in \mathbb{C}^{\left(\sum_{m=1}^{M} R_m\right)\times M} \tag{9}$$

This is illustrated in Figure 2. The problem now is a familiar one.

We know that PR is possible if the matrix $\widetilde{H}(\omega) \in \mathbb{C}^{\left(\sum_{m=1}^{M} R_m\right)\times QN}$ as defined in the proof of Theorem 1 (see (39) in Appendix A, with M_o replaced by Q and $H_Q(\omega)$ replacing $H(\omega)$) has a full column rank for all

$\omega \in \left[-W, \frac{2\pi}{T_o} - W\right)$. We thus investigate whether or not this matrix can be made to have full column rank by a choice of $H(\omega)$. From (8) and (9) we observe that the matrix

$$\widetilde{H}(\omega) = \left[H_Q(\omega), H_Q\left(\omega + \frac{2\pi}{T_o}\right), \cdots, H_Q\left(\omega + \frac{2\pi(Q-1)}{T_o}\right)\right] \quad (10)$$

can be rewritten as

$$\widetilde{H}(\omega) = \mathrm{diag}\{G_m(\omega)\} \cdot F \cdot \mathrm{diag}\left\{H\left(\omega + \frac{2\pi q}{T_o}\right)\right\} \in \mathbb{C}^{\left(\sum_{m=1}^{M} R_m\right) \times NQ} \quad (11)$$

where

$$G_m(\omega) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{j\omega T_m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{j\omega(R_m-1)T_m} \end{bmatrix} W_{R_m} \in \mathbb{C}^{R_m \times R_m} \quad (12)$$

$$F = \begin{bmatrix} F_0 & \cdots & F_{Q-1} \end{bmatrix} \in \mathbb{R}^{\left(\sum_{m=1}^{M} R_m\right) \times QM} \quad (13)$$

W_{R_m} is the R_m dimensional DFT matrix and

$$F_q = \begin{bmatrix} \mathbf{e}^{R_1}_{q \mod R_1}\left(\mathbf{e}^M_0\right)^T \\ \vdots \\ \mathbf{e}^{R_m}_{q \mod R_m}\left(\mathbf{e}^M_{m-1}\right)^T \\ \vdots \\ \mathbf{e}^{R_M}_{q \mod R_M}\left(\mathbf{e}^M_{M-1}\right)^T \end{bmatrix} \in \mathbb{R}^{\left(\sum_{m=1}^{M} R_m\right) \times M} \quad (14)$$

We remind the reader that here $\mathbf{e}^R_r$ denotes the $(r+1)$th column of the R dimensional identity matrix. With this as background, we are now in a position to establish the following:

Theorem 4: Let T_m, R_m, F, Q, M and N be as above. Then, PR of $\mathbf{f}(t)$ from $\left\{\{g_m(k_m T_m)\}_{k_m \in \mathbb{Z}}\right\}_{m=1}^{M}$ is possible if and only if there exists a set subspaces $\{\mathcal{S}_q\}_{q=0}^{Q-1}$ such that

$$\mathcal{S}_q \subseteq \mathrm{span}\{F_q\} \quad (15)$$

$$\dim \mathcal{S}_q = N \quad \text{for every} \quad 0 \leq q \leq Q-1 \quad (16)$$

and

$$\mathcal{S}_0 + \mathcal{S}_1 + \ldots + \mathcal{S}_{Q-1} = \mathbb{R}^{QN} \quad (17)$$

(by $\mathrm{span}\{F_q\}$ we mean the subspace spanned by the columns of the matrix F_q).

Proof: The proof is provided in Appendix B. ■

Remark 5: Going back to the definition of the blocks F_q in (14) we observe that each block consists of M distinct unit vectors $\mathbf{e}_r^{\left(\sum_{m=1}^{M} R_m\right)}$. Hence, using Theorem 4 the question of PR translates to whether, in every block F_q, out of its M columns, one can choose a subset of N columns not chosen in any other block. This restatement of the problem can be recognized as a well known problem in combinatorics, namely, the Hall *marriage problem.* This was observed and stated in [7]. A necessary and sufficient condition for this problem to have a solution has been given by Hall in [4] (or in [5]). This condition, stated in our terms, is as follow: *A solution exists if and only if rank* $\begin{bmatrix} F_{q_1}, & F_{q_2}, & \cdots & , F_{q_J} \end{bmatrix} \geq JN$ *for all subsets of* J *blocks,* $\left\{F_{q_j}\right\}_{j=1}^{J}$, $1 \leq J \leq Q$. This leads to the following sufficient condition for the result of Theorem 4:

Theorem 6: Under the conditions of Theorem 4, PR is possible if (5) holds and

$$\sum_{m=1}^{M} \left\lceil \frac{Q-J}{\left\lfloor \frac{Q}{R_m} \right\rfloor} \right\rceil \leq (Q-J)N \quad \text{for all} \quad 1 \leq J \leq Q \quad (18)$$

Proof: We provide the proof in Appendix C. (See also [7]) ■

Remark 7: The sufficient condition given in Theorem 6 holds when all $\widetilde{R}_m = 1$, since then, all R_m divide Q and

$$\sum_{m=1}^{M} \left\lceil \frac{Q-J}{\left\lfloor \frac{Q}{R_m} \right\rfloor} \right\rceil = \sum_{m=1}^{M} \left\lceil \frac{Q-J}{\frac{Q}{R_m}} \right\rceil \leq \sum_{m=1}^{M} \frac{Q-J}{\frac{Q}{R_m}} = \frac{Q-J}{Q} \sum_{m=1}^{M} R_m \leq \frac{Q-J}{Q} QN$$

The following Theorem provides a special case of the above result which generalizes Papoulis' GSE result [6] to the case of non equal sampling in the output vector.

Theorem 8: Under the conditions of Theorem 4, when $N = 1$, PR is possible if and only if (5) holds.

Proof: We have already argued that this condition is necessary. So, we only need to establish sufficiency. Since $N = 1$, (5) becomes $\sum_{m=1}^{M} R_m \geq Q$. Define for every $0 \leq q < Q$ the integer $1 \leq m_q \leq M$ such that

$$\begin{aligned} m_q &= 1 \quad \text{for} \quad 0 \leq q < R_1 \\ m_q &= 2 \quad \text{for} \quad R_1 \leq q < R_1 + R_2 \\ &\vdots \\ m_q &= M \quad \text{for} \quad \sum_{m=1}^{M-1} R_m \leq q < \sum_{m=1}^{M} R_m \end{aligned}$$

Then, we observe from (14) that the unit vector $\mathbf{e}^{\left(\sum_{m=1}^{M} R_m\right)}_{\left(q \mod R_{m_q} + \sum_{m=1}^{m_q-1} R_m\right)}$ is such that

$$F_p^T \mathbf{e}^{\left(\sum_{m=1}^{M} R_m\right)}_{\left(q \mod R_{m_q} + \sum_{m=1}^{m_q-1} R_m\right)} = \delta_{p-q} \mathbf{e}^M_{m_q-1}$$

where δ_{p-q} is the Kronecker Delta. This means that $\mathbf{e}^{\left(\sum_{m=1}^{M} R_m\right)}_{\left(q \mod R_{m_q} + \sum_{m=1}^{m_q-1} R_m\right)}$ is one of the columns of F_q. If we choose $\mathcal{S}_q = \mathrm{span}\left\{\mathbf{e}^{\left(\sum_{m=1}^{M} R_m\right)}_{\left(q \mod R_{m_q} + \sum_{m=1}^{m_q-1} R_m\right)}\right\}$ the conditions of Theorem 4 are clearly satisfied and PR is possible. This completes the proof of the Theorem. ■

IV. Different bandwidths in each input entry

We next consider the case where each input entry may have a different bandwidth. Specifically, let the nth input entry $f_n(t)$ be of bandwidth W_n. One possible approach to this problem, would be to choose the largest bandwidth of the input entries as the bandwidth of the whole input and thus convert the problem to one dealt with earlier in Section III. However, this is clearly not the best utilization of the fact that $M > N$. In the sequel we propose a more efficient approach.

We again discuss two possibilities for sampling of the output entries.

A. Output entries sampled at the same (uniform) rate

Suppose all output entries are sampled at a uniform rate T_o. Let $\{L_n\}_{n=1}^{N}$ be a set of integers such that

$$M \geq \sum_{n=1}^{N} L_n \tag{19}$$

Then we may establish the following:

Theorem 9: Let $\mathbf{f}(t)$, $\{W_n\}$, $\mathbf{g}(t)$ and T_o be as defined above. Perfect reconstruction (PR) of $\mathbf{f}(t)$ from the data $\{\mathbf{g}(kT_o)\}_{k\in\mathbb{Z}}$ is possible (for appropriate choice of $H(\omega)$) if

$$T_o \leq \frac{\pi L_n}{W_n} \quad \text{for} \quad n = 1, ..., N \tag{20}$$

Proof: First we note that from (19) and (20)

$$\begin{aligned} T_o \sum_{n=1}^{N} W_n &\leq \pi \sum_{n=1}^{N} L_n \\ &\leq \pi M \end{aligned}$$

so that

$$2\sum_{n=1}^{N} W_n \leq M\frac{2\pi}{T_o}$$

Namely, the necessary condition relating output sampling rates to input bandwidths is satisfied. Let us now consider the set of equations

$$\Phi(t,\omega)\widetilde{H}(\omega) = \left[\begin{array}{ccc} E_1(t), & \cdots & , E_N(t) \end{array}\right] \tag{21}$$

for all t and $0 \leq \omega \leq \frac{\pi}{T_o}$ where $\Phi(t,\omega) \in \mathbb{C}^{N\times 2M}$ and

$$\widetilde{H}(\omega) = \left[\begin{array}{cc} I & 0 \\ 0 & e^{j\omega T_o} I \end{array}\right] \left[\begin{array}{ccc} \widetilde{H}_1(\omega), & \cdots & , \widetilde{H}_N(\omega) \end{array}\right] \in \mathbb{C}^{2M\times 2\sum_{n=1}^{N} L_n} \tag{22}$$

where

$$\widetilde{H}_n(\omega) = \left[\begin{array}{c} \widehat{\mathbf{h}}_n\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots \\ e^{-j\pi L_n}\widehat{\mathbf{h}}_n\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots \end{array} \begin{array}{c} \widehat{\mathbf{h}}_n\left(\omega - (L_n-1)\frac{\pi}{T_o}\right), \cdots \\ e^{-j\pi(L_n-1)}\widehat{\mathbf{h}}_n\left(\omega - (L_n-1)\frac{\pi}{T_o}\right), \cdots \end{array} \begin{array}{c} \widehat{\mathbf{h}}_n\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \\ e^{j\pi(L_n-1)}\widehat{\mathbf{h}}_n\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \end{array}\right] \in \mathbb{C}^{2M\times 2L_n} \tag{23}$$

$\widehat{\mathbf{h}}_n(\omega)$ is the nth column of $H(\omega)$ and

$$E_n(t) = \mathbf{e}_{n-1}^{N}\left[\begin{array}{cccc} e^{-j\frac{\pi}{T_o}L_n t}, & e^{-j\frac{\pi}{T_o}(L_n-1)t}, & \cdots & \cdots, e^{j\frac{\pi}{T_o}(L_n-1)t} \end{array}\right] \in \mathbb{C}^{N\times 2L_n} \tag{24}$$

with $\mathbf{e}_{n-1}^{N}$ being the nth column of the N dimensional identity matrix. In Appendix D we establish that, with $\widetilde{H}(\omega)$ as defined above, there exists an $H(\omega)$ such that $\widetilde{H}(\omega)$ is full column rank for all $0 \leq \omega \leq \frac{\pi}{T_o}$.

Since from (19) we have $\sum_{p=1}^{N} L_p \leq M$, having $\widetilde{H}(\omega)$ full rank for all $0 \leq \omega \leq \frac{\pi}{T_o}$ guarantees that (21) has a solution. Let $\Phi(t,\omega)$ be a solution of (21). Define

$$\varphi(t) = \frac{T_o}{\pi}\int_0^{\frac{\pi}{T_o}} \Phi(t,\omega) e^{j\omega t} d\omega \in \mathbb{R}^{N\times 2M} \tag{25}$$

Then, from (21) and (24), we note that $\Phi(t+2T_o,\omega) = \Phi(t,\omega)$ so that

$$\begin{aligned} \varphi(t-2kT_o) &= \frac{T_o}{\pi}\int_0^{\frac{\pi}{T_o}} \Phi(t-2kT_o,\omega) e^{j\omega(t-2kT_o)} d\omega \\ &= \frac{T_o}{\pi}\int_0^{\frac{\pi}{T_o}} \Phi(t,\omega) e^{j\omega t} e^{-j\omega 2kT_o} d\omega \end{aligned}$$

Hence, we see that $\varphi(t-2kT_o)$ are the coefficients of the Fourier series of $\Phi(t,\omega)e^{j\omega t}$ defined on $\left[0, \frac{\pi}{T_o}\right]$. Namely

$$\Phi(t,\omega) e^{j\omega t} = \sum_{k\in\mathbb{Z}} \varphi(t-2kT_o) e^{j\omega 2kT_o} \text{ for all } 0 \leq \omega \leq \frac{\pi}{T_o} \tag{26}$$

Then, from (21)-(24) we have

$$\Phi(t,\omega)\widetilde{H}_n(\omega) = E_n(t) \text{ for } n = 1, 2, ..., N \text{ and } 0 \leq \omega \leq \frac{\pi}{T_o}$$

or, for $l = -L_n, .., L_n - 1$

$$\Phi(t,\omega)\left[\begin{array}{c} \widehat{\mathbf{h}}_n\left(\omega + l\frac{\pi}{T_o}\right) \\ e^{j\left(\omega + l\frac{\pi}{T_o}\right)T_o}\widehat{\mathbf{h}}_n\left(\omega + l\frac{\pi}{T_o}\right) \end{array}\right] = \mathbf{e}_{n-1}^{N} e^{j\frac{\pi}{T_o}lt}$$
$$\text{for } n = 1, 2, ..., N \text{ and } 0 \leq \omega \leq \frac{\pi}{T_o}$$

Multiplying both sides by $e^{j\omega t}$ and substituting (26) we obtain

$$\sum_{k\in\mathbb{Z}} \varphi(t-2kT_o) e^{j\omega 2kT_o}\left[\begin{array}{c} \widehat{\mathbf{h}}_n\left(\omega + l\frac{\pi}{T_o}\right) \\ e^{j\left(\omega + l\frac{\pi}{T_o}\right)T_o}\widehat{\mathbf{h}}_n\left(\omega + l\frac{\pi}{T_o}\right) \end{array}\right] = \mathbf{e}_{n-1}^{N} e^{j\left(\omega + \frac{\pi}{T_o}l\right)t}$$

Since the above result holds for all $0 \leq \omega \leq \frac{\pi}{T_o}$ and $l = -L_n, .., L_n - 1$ we conclude that

$$\sum_{k\in\mathbb{Z}} \varphi(t-2kT_o) e^{j\omega 2kT_o}\left[\begin{array}{c} \widehat{\mathbf{h}}_n(\omega) \\ e^{j\omega T_o}\widehat{\mathbf{h}}_n(\omega) \end{array}\right] = \mathbf{e}_{n-1}^{N} e^{j\omega t} \tag{27}$$

holds for all $|\omega| \leq L_n\frac{\pi}{T_o}$ and $n = 1, ..., N$.

Using the appropriate definitions, we have,

$$\begin{aligned} \widehat{\mathbf{g}}(\omega) &= H(\omega)\widehat{\mathbf{f}}(\omega) \\ &= \sum_{n=1}^{N} \widehat{\mathbf{h}}_n(\omega)\widehat{f}_n(\omega) \\ &= \sum_{n=1}^{N} \widehat{\mathbf{g}}_n(\omega) \end{aligned} \tag{28}$$

and, using (20)

$$\begin{aligned}\mathbf{f}(t) &= \sum_{n=1}^{N} f_n(t)\,\mathbf{e}_{n-1}^{N} \\ &= \sum_{n=1}^{N} \frac{1}{2\pi}\int_{-W_n}^{W_n} \widehat{f}_n(\omega)\,\mathbf{e}_{n-1}^{N} e^{j\omega t} d\omega \\ &= \sum_{n=1}^{N} \frac{1}{2\pi}\int_{-L_n\frac{\pi}{T_o}}^{L_n\frac{\pi}{T_o}} \widehat{f}_n(\omega)\,\mathbf{e}_{n-1}^{N} e^{j\omega t} d\omega\end{aligned}$$

Substituting (27) and (28) we obtain

$$\begin{aligned}\mathbf{f}(t) &= \sum_{k\in\mathbb{Z}} \varphi(t-2kT_o) \sum_{n=1}^{N} \frac{1}{2\pi}\int_{-L_n\frac{\pi}{T_o}}^{L_n\frac{\pi}{T_o}} e^{j\omega 2kT_o} \\ &\qquad \begin{bmatrix} \widehat{\mathbf{h}}_n(\omega) \\ e^{j\omega T_o}\widehat{\mathbf{h}}_n(\omega)\end{bmatrix} \widehat{f}_n(\omega) \\ &= \sum_{k\in\mathbb{Z}} \varphi(t-2kT_o) \sum_{n=1}^{N} \frac{1}{2\pi}\int_{-L_n\frac{\pi}{T_o}}^{L_n\frac{\pi}{T_o}} e^{j\omega 2kT_o} \\ &\qquad \begin{bmatrix} \widehat{\mathbf{g}}_n(\omega) \\ e^{j\omega T_o}\widehat{\mathbf{g}}_n(\omega)\end{bmatrix} d\omega \\ &= \sum_{k\in\mathbb{Z}} \varphi(t-2kT_o) \sum_{n=1}^{N} \begin{bmatrix} \mathbf{g}_n(2kT_o) \\ \mathbf{g}_n((2k+1)T_o)\end{bmatrix} \\ &= \sum_{k\in\mathbb{Z}} \varphi(t-2kT_o) \begin{bmatrix} \mathbf{g}(2kT_o) \\ \mathbf{g}((2k+1)T_o)\end{bmatrix}\end{aligned}$$

which is the required reconstruction of the input vector from the output sampled at T_o. This completes the proof. ■

Example 10: We present a simple example to illustrate the potential benefit in taking into account the different bandwidths of input entries. Let the input vector consist of two entries (i.e. $N = 2$) with bandwidths $W_1 = 3W_o$ and $W_2 = 2W_o$ (for some $W_o > 0$) and the output vector of five entries (i.e. $M = 5$). We assume here that all outputs are sampled at the same sampling rate T_o and are interested in finding the largest T_o possible which still enables reconstruction with an appropriate MIMO filter $H(\omega)$. One approach would be to consider both input entries as if they had the same bandwidth, $\max\{W_n\} = 3W_o$. Then, using Theorem 1 we have that $T_o \le \lfloor \frac{M}{N} \rfloor \frac{\pi}{W} = 2\frac{\pi}{3W_o}$. As an alternative, let us try the approach suggested in Theorem 9. We choose $L_1 = 3$ and $L_2 = 2$ which clearly satisfy (19). Then, by the theorem, we have $T_o \le \frac{\pi L_1}{W_1} = \frac{\pi L_2}{W_2} = \frac{\pi}{W_o}$ - clearly larger than we got with the first approach. We wish, though, to point out that conditions on the MIMO filter may be different in both approaches. The key point being that, in each approach, a filter enabling PR exists for the corresponding T_o.

B. Output entries sampled at different (uniform) rates

Finally, we tackle the most general case for contiguous bandwidths - i.e. with different bandwidths, W_n, at the input entries and different uniform sampling rates, T_m, for the output entries. We assume, as we did in Section III-B that

$$T_o = T_m R_m \quad \text{for} \quad m = 1, 2, .., M$$

for some T_o and integers R_m. Repeating the arguments in Section III-B leads to the same conclusion, namely, that the data $\left\{\{g_m(k_m T_m)\}_{k_m\in\mathbb{Z}}\right\}_{m=1}^{M}$ is equivalent to the data $\{\widetilde{\mathbf{g}}(kT_o)\}_{k\in\mathbb{Z}}$ where $\widetilde{\mathbf{g}}(t)$ is defined in (6) and (7). Hence, the problem of signals passing through the MIMO filter $H(\omega)$ and sampled at different rates T_m, is replaced with the problem of signals passing through the MIMO filter $H_Q(\omega)$ defined in (8) and (9) and sampled at the same rate T_o.

We also use ideas similar to those used in Section.IV-A. Specifically, let $\{L_n\}_{n=1}^{N}$ be a set of integers such that

$$\sum_{m=1}^{M} R_m \ge \sum_{n=1}^{N} L_n \tag{29}$$

Note that $\sum_{m=1}^{M} R_m$ is the number of outputs with $H_Q(\omega)$ as the MIMO filter which makes (29) equivalent to (19). We rewrite equation (21) for all t and $0 \le \omega \le \frac{\pi}{T_o}$

$$\Phi(t,\omega)\,\widetilde{H}(\omega) = \begin{bmatrix} E_1(t), & \cdots & , E_N(t) \end{bmatrix}$$

where $E_n(t)$ is as in (24), $\Phi(t,\omega) \in \mathbb{C}^{N\times 2\left(\sum_{m=1}^{M} R_m\right)}$ and the matrix $\widetilde{H}(\omega)$ is now defined by

$$\widetilde{H}(\omega) = \begin{bmatrix} I & 0 \\ 0 & e^{j\omega T_o} I \end{bmatrix} \begin{bmatrix} \widetilde{H}_1(\omega), & \cdots & , \widetilde{H}_N(\omega) \end{bmatrix} \in \mathbb{C}^{2\left(\sum_{m=1}^{M} R_m\right)\times 2\left(\sum_{n=1}^{N} L_n\right)} \tag{30}$$

where

$$\widetilde{H}_n(\omega) = \left[\begin{matrix} \widehat{\mathbf{h}}_{Q,n}\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots \\ e^{-j\pi L_n}\widehat{\mathbf{h}}_{Q,n}\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots \end{matrix}\right. \\ \begin{matrix} \widehat{\mathbf{h}}_{Q,n}\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots \\ e^{-j\pi(L_n-1)}\widehat{\mathbf{h}}_{Q,n}\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots \end{matrix} \\ \left.\begin{matrix} \widehat{\mathbf{h}}_{Q,n}\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \\ e^{j\pi(L_n-1)}\widehat{\mathbf{h}}_{Q,n}\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \end{matrix}\right] \in \mathbb{C}^{2\left(\sum_{m=1}^{M} R_m\right)\times 2L_n} \tag{31}$$

and $\widehat{\mathbf{h}}_{Q,n}(\omega)$ is the nth column of $H_Q(\omega)$. We may then establish the following:

Theorem 11: Let $\mathbf{f}(t)$, $\{W_n\}$, $\mathbf{g}(t)$, $\{T_m\}$, $\{R_m\}$, $\{L_n\}$ and T_o be as above. Perfect reconstruction (PR) of $\mathbf{f}(t)$ from the data $\left\{\{g_m(k_m T_m)\}_{k_m\in\mathbb{Z}}\right\}_{m=1}^{M}$ (or, equivalently, the data $\{\widetilde{\mathbf{g}}(kT_o)\}_{k\in\mathbb{Z}}$) is possible if

$$T_o \le \frac{\pi L_n}{W_n} \quad \text{for} \quad n = 1, ..., N \tag{32}$$

and the matrix $\widetilde{H}(\omega)$ has full column rank for all $0 \le \omega \le \frac{\pi}{T_o}$.

Proof: The proof is similar to the proof of Theorem 9. ■

The crucial difference between the results in Theorems 9 and 11 is related to the rank of the matrix $\widetilde{H}(\omega)$. While in the case dealt with in Theorem 9 this matrix can be made full column rank by proper choice of the original MIMO filter $H(\omega)$, this is not not necessarily true for the case treated in Theorem 11. We will next investigate the possibilities for achieving this property using arguments akin to those in Section III-B.

Without loss of generality, we assume that $L_1 \geq L_2 \geq ... \geq L_N$. Let us define the integers $\{N_q\}_{q=-L_1}^{L_1-1}$ as

$$\begin{aligned} N_q &= \left\{ \text{number of } L_n \geq \frac{1}{2} + \left| q + \frac{1}{2} \right| \right\} \\ &\leq N \end{aligned} \tag{33}$$

Note that $N_{2p-L_1} = N_{L_1-2p-1}$ and $\sum_{p=0}^{L_1-1} N_{2p-L_1} = \sum_{p=0}^{L_1-1} N_{L_1-2p-1} = \sum_{n=1}^{N} L_n$. We may then establish:

Theorem 12: Let $\{F_p\}_{p=0}^{L_1-1}$ be defined as in (14). Then $\widetilde{H}(\omega)$ can be made full column rank by choice of $H(\omega)$ if and only if there exists a set of subspaces $\{\mathcal{S}_p\}_{p=0}^{L_1-1}$such that

$$\mathcal{S}_p \subseteq \operatorname{span}\{F_p\} \tag{34}$$

$$\dim \mathcal{S}_p = N_{2p-L_1} \quad \text{for every} \quad 0_1 \leq p \leq L_1 - 1 \tag{35}$$

and

$$\mathcal{S}_0 + \mathcal{S}_1 + ... + \mathcal{S}_{L_1-1} = \mathbb{R}^{\sum_{n=1}^{N} L_n} \tag{36}$$

Proof: The proof is given in Appendix E. ∎

Remark 13: As in Section III-B, we note that the blocks F_p consist of unit vectors. Hence, the existence of the subspaces $\mathcal{S}_p$ is equivalent to the ability to choose, in each block F_p, a subset of N_{2p-L_1} columns which are not chosen in any other block. This again, can readily be recognized as a slightly more general but well known problem in combinatorics called the *bi-marriage problem*. Hall's condition [4] for the existence of a solution to this combinatorics problem, stated in our terms, is: For every set of J blocks, $\left\{F_{p_j}\right\}_{j=1}^{J}$, $1 \leq J \leq L_1$, *rank* $\left[\begin{array}{cccc} F_{p_1}, & F_{p_2}, & \cdots, & F_{p_J} \end{array}\right] \geq \sum_{j=1}^{J} N_{2p_j-L_1}$.

Example 14: We again, give an example to demonstrate the potential advantage of Theorem 12 over Theorem 4. Consider the same system and inputs as in Example 10. However, here, not all the outputs are sampled at the same rate. Specifically, let $T_1 = T_2 = T_3 = \frac{7\pi}{12W_o}$ and $T_4 = T_5 = \frac{7\pi}{3W_o}$. We note that, assuming $W = \max\{W_n\} = 3W_o$, with $R_1 = R_2 = R_3 = 4$, $R_4 = R_5 = 1$ and $Q = 7$ the conditions (4) and (5) hold for $T_o = \frac{7\pi}{3W_o}$. *However*, as *rank* $\left[\begin{array}{cccccc} F_0, & F_1, & F_2, & F_4, & F_5, & F_6 \end{array}\right] = 11 < JN = 6 \times 2 = 12$, which implies according to Theorem 4 *that there exist no MIMO filter for which PR is possible*. However, if we use the result in Theorem 12 we can verify that for every set of $1 \leq J \leq 7$ block $\left\{F_{p_j}\right\}_{j=1}^{J}$, *rank* $\left[\begin{array}{cccc} F_{p_1}, & F_{p_2}, & \cdots, & F_{p_J} \end{array}\right] \geq \sum_{j=1}^{J} N_{2p_j-L_1}$. Specifically, for the case Hall's Theorem failed above, we get *rank* $\left[\begin{array}{cccccc} F_0, & F_1, & F_2, & F_4, & F_5, & F_6 \end{array}\right] = 11 \geq N_{-7} + N_{-5} + N_{-3} + N_1 + N_3 + N_5 = 11$ and it is satisfied in its modified form. This clearly demonstrates that ignoring the different bandwidths of the input entries may lead to the wrong conclusions.

V. Conclusion

This paper has addressed the problem of perfect reconstruction of a vector signal from samples of a (vector) filtered version of the signal. The results presented here further generalize previously published results in this area with the aim of finding the most parsimonious data in each case treated, while preserving the ability to get a perfect reconstruction (PR) of the original vector signal. In particular, conditions for PR for the case which allows for different bandwidths in each input vector entry are presented with examples which highlight the potential benefits.

Appendix A

Proof of Theorem 1

Proof: We note first that, as the total rate of the output has to be at least equal to the total rate of the input, we *must have* $\alpha \leq \frac{M}{N}$. The proof will consist of two parts. We will first show that reconstruction is possible for $\alpha \leq \left\lfloor \frac{M}{N} \right\rfloor$. In the second part we will show that reconstruction is impossible for $\left\lfloor \frac{M}{N} \right\rfloor < \alpha$.

Part 1. Let us denote $\alpha \leq \left\lfloor \frac{M}{N} \right\rfloor = M_o$ so that $T_o \leq M_o \frac{\pi}{W}$. We then introduce the *matrix* of functions $\varphi(t) \in \mathbb{R}^{N \times M}$ defined by

$$\varphi(t) = \frac{T_o}{2\pi} \int_{-\frac{M_o \pi}{T_o}}^{\frac{(2-M_o)\pi}{T_o}} \Phi(\omega, t) e^{j\omega t} d\omega \tag{37}$$

where $\Phi(\omega, t) \in \mathbb{C}^{N \times M}$ are the solutions of the following set of linear equations

$$\Phi(\omega, t) \widetilde{H}(\omega) = \left[I_N, e^{jt\frac{2\pi}{T_o}} I_N, \cdots, e^{jt(M_o-1)\frac{2\pi}{T_o}} I_N \right] \tag{38}$$

defined for all t and for $\omega \in [-\frac{M_o \pi}{T_o}, \frac{(2-M_o)\pi}{T_o})$. I_N denotes the N - dimensional identity matrix and $\widetilde{H}(\omega) \in \mathbb{C}^{M \times NM_o}$ is given by

$$\begin{aligned} \widetilde{H}(\omega) = \Bigg[H(\omega), H\left(\omega + \frac{2\pi}{T_o}\right), \cdots, \\ H\left(\omega + \frac{2\pi(M_o - 1)}{T_o}\right) \Bigg] \end{aligned} \tag{39}$$

With proper choice of $H(\omega)$, $\widetilde{H}(\omega)$ can be made full column rank for every $\omega \in [-\frac{M_o \pi}{T_o}, \frac{(2-M_o)\pi}{T_o})$. This guarantees that the set of equations (38) has a solution (not necessarily unique) as $\left\lfloor \frac{M}{N} \right\rfloor \leq \frac{M}{N}$) which is both sufficient (see e.g.[6]) and necessary (see [3]) condition for PR. The reconstruction formula is then given by $\mathbf{f}(t) = \sum_{k \in \mathbb{Z}} \varphi(t - kT_o) \mathbf{g}(kT_o)$

Part 2. Let us assume now that $\left\lfloor \frac{M}{N} \right\rfloor < \alpha = \frac{Q}{R} \leq \frac{M}{N}$. Then, we have $RT_o = Q\frac{\pi}{W}$. Using a polyphase representation ([9]), we observe that the data $\{\mathbf{g}(kT_o)\}_{k \in \mathbb{Z}}$ is equivalent to the data $\left\{\widetilde{\mathbf{g}}\left(lQ\frac{\pi}{W}\right) = \widetilde{\mathbf{g}}(lRT_o)\right\}_{l \in \mathbb{Z}}$ where

$$\widetilde{\mathbf{g}}(t)^T = \left[\mathbf{g}(t)^T, \mathbf{g}(t + T_o)^T, \cdots, \mathbf{g}(t + (R-1)T_o)^T \right] \tag{40}$$

Hence, we have converted the problem of PR of $\mathbf{f}(t)$ from $\mathbf{g}(t)$ sampled at T_o to the PR of $\mathbf{f}(t)$ from $\widetilde{\mathbf{g}}(t)$ sampled at $RT_o = Q\frac{\pi}{W}$. (See Figure 2 with all polyphase blocks identical and output components shuffled). Using (40) we observe that the transfer function from $\mathbf{f}(t)$ to $\widetilde{\mathbf{g}}(t)$ is given by

$$H_R(\omega) = \begin{bmatrix} 1 \\ e^{j\omega T_o} \\ \vdots \\ e^{j\omega(R-1)T_o} \end{bmatrix} \otimes H(\omega) \tag{41}$$

where $\otimes$ denotes the Kronecker product. Then, (38) becomes

$$\Phi\left(\omega,t\right)\widetilde{H}\left(\omega\right)=\left[I_N,e^{jt\frac{2\pi}{RT_o}}I_N,\cdots,e^{jt(Q-1)\frac{2\pi}{RT_o}}I_N\right]$$

with $\Phi\left(\omega,t\right)\in\mathbb{C}^{N\times MR}$ and

$$\widetilde{H}\left(\omega\right)=\left[H_R\left(\omega\right),H_R\left(\omega+\frac{2\pi}{RT_o}\right),\cdots,\right.$$
$$\left.H_R\left(\omega+\frac{2\pi\left(Q-1\right)}{RT_o}\right)\right]\in\mathbb{C}^{MR\times QN}\quad(42)$$

Hence, PR is possible *if (and only if)* the matrix $\widetilde{H}\left(\omega\right)$ is full column rank for all $\omega\in\left[-W,\frac{2\pi}{RT_o}-W\right)$. However, we will show that this matrix can never have full column rank independent of the choice of $H\left(\omega\right)$.From (41) we have for $q=0,1,...,Q-1$

$$\begin{aligned}H_R\left(\omega+\frac{2\pi q}{RT_o}\right)&=\begin{bmatrix}1\\ e^{j\left(\omega+\frac{2\pi q}{RT_o}\right)T_o}\\ \vdots\\ e^{j\left(\omega+\frac{2\pi q}{RT_o}\right)(R-1)T_o}\end{bmatrix}\\&\quad\otimes H\left(\omega+\frac{2\pi q}{RT_o}\right)\\&=\left(\operatorname{diag}\left\{e^{j\omega rT_o}\right\}\begin{bmatrix}1\\ e^{j\frac{2\pi q}{R}}\\ \vdots\\ e^{j\frac{2\pi q}{R}(R-1)}\end{bmatrix}\right)\\&\quad\otimes H\left(\omega+\frac{2\pi q}{RT_o}\right)\\&=\left(\operatorname{diag}\left\{e^{j\omega rT_o}\right\}W_R\mathbf{e}^R_{q\mod R}\right)\\&\quad\otimes H\left(\omega+\frac{2\pi q}{RT_o}\right)\end{aligned}$$

where W_R is the R - dimensional DFT matrix, $m\mod R=m-\left\lfloor\frac{M}{N}\right\rfloor R$ and $\mathbf{e}^R_r$ denotes the $(r+1)$th column of the R dimensional identity matrix, I_R. Then we have from (42)

$$\widetilde{H}\left(\omega\right)=\left(\left(\operatorname{diag}\left\{e^{j\omega rT_o}\right\}W_R\right)\otimes I_M\right)\widetilde{H}_1\left(\omega\right)$$

where

$$\widetilde{H}_1\left(\omega\right)=\left[\mathbf{e}^R_0\otimes H\left(\omega\right),\ \mathbf{e}^R_1\otimes H\left(\omega+\frac{2\pi}{RT_o}\right),\cdots\right.$$
$$\mathbf{e}^R_{(Q-2)\mod R}\otimes H\left(\omega+\frac{2\pi\left(Q-2\right)}{RT_o}\right),$$
$$\left.\mathbf{e}^R_{(Q-1)\mod R}\otimes H\left(\omega+\frac{2\pi\left(Q-1\right)}{RT_o}\right)\right]\quad(43)$$

From (43) and since $\left\lfloor\frac{Q}{R}\right\rfloor=\left\lfloor\frac{M}{N}\right\rfloor<\frac{Q}{R}$ we observe that the first M rows of $\widetilde{H}_1\left(\omega\right)$ contain $\left\lfloor\frac{Q}{R}\right\rfloor+1$ non zero blocks, of dimension $M\times N$ each. Hence, since $\left(\left\lfloor\frac{M}{N}\right\rfloor+1\right)N>M$, these columns must be linearly dependent and the matrix $\widetilde{H}_1\left(\omega\right)$ cannot be full column rank. This completes the proof of the theorem. ■

Appendix B
Proof of Theorem 4

Proof: As seen in the preamble to the theorem, PR is possible if and only if the matrix $\widetilde{H}\left(\omega\right)$ as defined in (10) is full column rank for every $\omega\in\left[-W,\frac{2\pi}{T_o}-W\right)$. Using (11) we note that $\operatorname{diag}\left\{G_m\left(\omega\right)\right\}$ is square and nonsingular for all ω. Hence, the result reduces to the question of whether the matrix $F\cdot\operatorname{diag}\left\{H\left(\omega+\frac{2\pi q}{T_o}\right)\right\}$ can be made full column rank for every $\omega\in\left[-W,\frac{2\pi}{T_o}-W\right)$. We note that this product of matrices consists of the Q blocks $F_qH\left(\omega+\frac{2\pi q}{T_o}\right)$.

(i) Sufficiency. Suppose that the subspaces defined in (15) to (17) exist. Then $H\left(\omega+\frac{2\pi q}{T_o}\right)$ can be chosen so that $\operatorname{span}\left\{F_qH\left(\omega+\frac{2\pi q}{T_o}\right)\right\}=S_q$ for every $\omega\in\left[-W,\frac{2\pi}{T_o}-W\right)$. Hence, by (17) $F\cdot\operatorname{diag}\left\{H\left(\omega+\frac{2\pi q}{T_o}\right)\right\}$ is full column rank for every $\omega\in\left[-W,\frac{2\pi}{T_o}-W\right)$.

(ii) Necessity. Suppose there exists $H\left(\omega\right)$ such that $F\cdot\operatorname{diag}\left\{H\left(\omega+\frac{2\pi q}{T_o}\right)\right\}$ is full column rank for all $\omega\in\left[-W,\frac{2\pi}{T_o}-W\right)$. Then we can define, for any $\omega\in\left[-W,\frac{2\pi}{T_o}-W\right)$ and $q=0,1,...,Q-1$

$$\mathcal{S}_q=\operatorname{span}\left\{F_qH\left(\omega+\frac{2\pi q}{T_o}\right)\right\}$$

which clearly satisfy (15) and $\dim\mathcal{S}_q\leq N$. Furthermore, since $F\cdot\operatorname{diag}\left\{H\left(\omega+\frac{2\pi q}{T_o}\right)\right\}$ is assumed to have full column rank we must have

$$\begin{aligned}&\operatorname{span}\left\{F\cdot\operatorname{diag}\left\{H\left(\omega+\frac{2\pi q}{T_o}\right)\right\}\right\}\\&=\quad\mathcal{S}_0+\mathcal{S}_1+...+\mathcal{S}_{Q-1}=\mathbb{R}^{QN}\end{aligned}$$

hence, $\dim\mathcal{S}_q=N$. This completes the proof. ■

Appendix C
Proof of Theorem 6

Proof: Recall that each block F_q consists of M distinct unit vectors. Let us call the mth one in each block unit vector of type m. From the structure of each block (see (14)) we observe that the unit vector of type m in block F_q $(q=0,1,...,Q-1)$ is of the form

$$\begin{bmatrix}\mathbf{0}_{R_1}\\ \vdots\\ \mathbf{e}^{R_m}_{q\mod R_m}\\ \vdots\\ \mathbf{0}_{R_M}\end{bmatrix}\in\mathbb{R}^{\left(\sum_{m=1}^MR_m\right)}$$

Hence, there are exactly R_m distinct unit vectors of type m. If two blocks contain the same unit vector, it has to be of the same type and each such vector appears in $\left\lfloor\frac{Q}{R_m}\right\rfloor$ blocks. Let us thus consider a set of $1\leq J<Q$ blocks. For a particular unit vector of type m to be excluded from this set, the complementary set must contain all the $\left\lfloor\frac{Q}{R_m}\right\rfloor$ blocks

which contain it. Since the complementary set of $Q-J$ blocks may contain *at most* $\left\lfloor \frac{Q-J}{\lfloor \frac{Q}{R_m} \rfloor} \right\rfloor$ such unit vectors, the set of J blocks has *at least* $R_m - \left\lfloor \frac{Q-J}{\lfloor \frac{Q}{R_m} \rfloor} \right\rfloor$ distinct unit vectors of type m. Hence, the total number of distinct unit vectors must be *larger or equal to* $\sum_{m=1}^{M} \left(R_m - \left\lfloor \frac{Q-J}{\lfloor \frac{Q}{R_m} \rfloor} \right\rfloor \right)$. Thus, from (5) and (18) we obtain

$$\begin{aligned} \mathit{rank}\left[F_{q_1} F_{q_2} \cdots F_{q_J}\right] &\geq \sum_{m=1}^{M} \left(R_m - \left\lfloor \frac{Q-J}{\left\lfloor \frac{Q}{R_m} \right\rfloor} \right\rfloor \right) \\ &\geq QN - (Q-J)\,N \\ &\geq JN \end{aligned}$$

Thus, Hall's necessary and sufficient condition [4] is satisfied and the 'marriage problem' has a solution which completes the proof of the claim. ■

Appendix D
Rank of matrix $\widetilde{H}(\omega)$ in equations (22) (23)

Going back to eqn. (22) and (23) we note that since $e^{j\pi l} = (-1)^l$ we can write

$$\widetilde{H}_n(\omega) = \begin{bmatrix} \widehat{\mathbf{h}}_n\left(\omega - L_n \frac{\pi}{T_o}\right), \cdots & \widehat{\mathbf{h}}_n\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots & \widehat{\mathbf{h}}_n\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \\ (-1)^{-L_n}\,\widehat{\mathbf{h}}_n\left(\omega - L_n \frac{\pi}{T_o}\right), \cdots & (-1)^{(1-L_n)}\,\widehat{\mathbf{h}}_n\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots & (-1)^{(L_n-1)}\,\widehat{\mathbf{h}}_n\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \end{bmatrix} \in \mathbb{C}^{2M \times 2L_n} \quad (44)$$

Namely, we can choose a permutation matrix P so that

$$\left[\widetilde{H}_1(\omega) \cdots \widetilde{H}_N(\omega)\right] P = \begin{bmatrix} \widetilde{\widetilde{H}}_1(\omega) & \widetilde{\widetilde{H}}_2(\omega) \\ -\widetilde{\widetilde{H}}_1(\omega) & \widetilde{\widetilde{H}}_2(\omega) \end{bmatrix} \in \mathbb{C}^{2M \times 2\sum_{n=1}^{N} L_n} \quad (45)$$

where $\widetilde{\widetilde{H}}_1(\omega) \in \mathbb{C}^{M \times \sum_{n=1}^{N} L_n}$ consists of all the columns $\left\{\widehat{\mathbf{h}}_n\left(\omega + q\frac{\pi}{T_o}\right) : q \text{ odd}\right\}_{n=1}^{N}$ and $\widetilde{\widetilde{H}}_2(\omega) \in \mathbb{C}^{M \times \sum_{n=1}^{N} L_n}$ consists of all the columns $\left\{\widehat{\mathbf{h}}_n\left(\omega + q\frac{\pi}{T_o}\right) : q \text{ even}\right\}_{n=1}^{N}$. Hence, by proper choice of $H(\omega)$, $\widetilde{\widetilde{H}}_1(\omega), \widetilde{\widetilde{H}}_2(\omega)$ can each be guaranteed to be full column rank for all $0 \leq \omega \leq \frac{\pi}{T_o}$. This in turn guarantees that $\begin{bmatrix} \widetilde{\widetilde{H}}_1(\omega) & \widetilde{\widetilde{H}}_2(\omega) \\ (-1)^{L_1}\widetilde{\widetilde{H}}_1(\omega) & (-1)^{L_1-1}\widetilde{\widetilde{H}}_2(\omega) \end{bmatrix}$ is full column rank and so are $\begin{bmatrix} \widetilde{H}_1(\omega) & \cdots & \widetilde{H}_N(\omega) \end{bmatrix}$ and $\widetilde{H}(\omega)$ as required for PR.

Appendix E
Proof of Theorem 12

Proof: By substituting from (9)

$$\widehat{\mathbf{h}}_{Q,n}\left(\omega + q\frac{\pi}{T_o}\right) = C\left(\omega + q\frac{\pi}{T_o}\right)\widehat{\mathbf{h}}_n\left(\omega + q\frac{\pi}{T_o}\right)$$

we can rewrite (31), using the fact that $R_m T_m = T_o$,

$$\widetilde{H}_n(\omega) = \begin{bmatrix} C\left(\omega - L_n\frac{\pi}{T_o}\right)\widehat{\mathbf{h}}_n\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots & C\left(\omega - (L_n-1)\frac{\pi}{T_o}\right)\widehat{\mathbf{h}}_n\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots & C\left(\omega + (L_n-1)\frac{\pi}{T_o}\right)\widehat{\mathbf{h}}_n\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \\ e^{-j\pi L_n} C\left(\omega - L_n\frac{\pi}{T_o}\right)\widehat{\mathbf{h}}_n\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots & e^{-j\pi(L_n-1)} C\left(\omega - (L_n-1)\frac{\pi}{T_o}\right)\widehat{\mathbf{h}}_n\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots & e^{j\pi(L_n-1)} C\left(\omega + (L_n-1)\frac{\pi}{T_o}\right)\widehat{\mathbf{h}}_n\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \end{bmatrix} \in \mathbb{C}^{2\left(\sum_{m=1}^{M} R_m\right) \times 2L_n}$$

and, recalling that we assumed (without loss of generality) that $L_1 \geq L_2 \geq \ldots \geq L_N$,

$$\begin{aligned} \widetilde{H}_n(\omega) &= \begin{bmatrix} C\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots & C\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots & C\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \\ e^{-j\pi L_n} C\left(\omega - L_n\frac{\pi}{T_o}\right), \cdots & e^{-j\pi(L_n-1)} C\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) \cdots & e^{j\pi(L_n-1)} C\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \end{bmatrix} A_n(\omega) \\ &= \begin{bmatrix} C\left(\omega - L_1\frac{\pi}{T_o}\right), \cdots & C\left(\omega - (L_1-1)\frac{\pi}{T_o}\right) \cdots & C\left(\omega + (L_1-1)\frac{\pi}{T_o}\right) \\ e^{-j\pi L_1} C\left(\omega - L_1\frac{\pi}{T_o}\right), \cdots & e^{-j\pi(L_1-1)} C\left(\omega - (L_1-1)\frac{\pi}{T_o}\right) \cdots & e^{j\pi(L_1-1)} C\left(\omega + (L_1-1)\frac{\pi}{T_o}\right) \end{bmatrix} \begin{bmatrix} 0 \\ A_n(\omega) \\ 0 \end{bmatrix} \end{aligned} \quad (46)$$

where

$$A_n(\omega) = \begin{bmatrix} \widehat{\mathbf{h}}_n\left(\omega - L_n\frac{\pi}{T_o}\right) & 0 & \cdots & 0 \\ 0 & \widehat{\mathbf{h}}_n\left(\omega - (L_n-1)\frac{\pi}{T_o}\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{\mathbf{h}}_n\left(\omega + (L_n-1)\frac{\pi}{T_o}\right) \end{bmatrix} \in \mathbb{C}^{2ML_n \times 2L_n} \quad (47)$$

Hence, with an appropriate permutation matrix $P \in \mathbb{R}^{\left(2\sum_{n=1}^{N} L_n\right)\times\left(2\sum_{n=1}^{N} L_n\right)}$ we can write

$$
\begin{aligned}
&\left[\widetilde{H}_1(\omega)\,\widetilde{H}_2(\omega)\cdots\widetilde{H}_N(\omega)\right] \\
&\qquad P = \left[\begin{array}{cc} \widetilde{\widetilde{H}}_1(\omega) & \widetilde{\widetilde{H}}_2(\omega) \\ (-1)^{L_1}\,\widetilde{\widetilde{H}}_1(\omega) & (-1)^{(L_1-1)}\,\widetilde{\widetilde{H}}_2(\omega) \end{array}\right]
\end{aligned}
\tag{48}
$$

where

$$
\begin{aligned}
\widetilde{\widetilde{H}}_1(\omega) = \; & \left[C\left(\omega - L_1\tfrac{\pi}{T_o}\right)B_{-L_1}(\omega)\cdots,\right. \\
& C\left(\omega - (L_1-2)\tfrac{\pi}{T_o}\right)B_{2-L_1}(\omega)\cdots \\
& \left. C\left(\omega + (L_1-2)\tfrac{\pi}{T_o}\right)B_{L_1-2}(\omega)\right] \\
& \in \mathbb{C}^{\left(\sum_{m=1}^{M} R_m\right)\times\left(\sum_{n=1}^{N} L_n\right)}
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
\widetilde{\widetilde{H}}_2(\omega) = \; & \left[C\left(\omega + (L_1-1)\tfrac{\pi}{T_o}\right)B_{L_1-1}(\omega),\cdots\right. \\
& C\left(\omega + (L_1-3)\tfrac{\pi}{T_o}\right)B_{L_1-3}(\omega)\cdots \\
& \left. C\left(\omega - (L_1-1)\tfrac{\pi}{T_o}\right)B_{1-L_1}(\omega)\right] \\
& \in \mathbb{C}^{\left(\sum_{m=1}^{M} R_m\right)\times\left(\sum_{n=1}^{N} L_n\right)}
\end{aligned}
\tag{50}
$$

The that each block $B_q(\omega) \in \mathbb{C}^{M\times N_q}$ (N_q as defined in (33) and recall that $\sum_{p=0}^{L_1-1} N_{2p-L_1} = \sum_{p=0}^{L_1-1} N_{L_1-2p-1} = \sum_{n=1}^{N} L_n$) consists of unique entries which are entries of the MIMO filter $H(\omega)$ shifted to different frequency bins. Hence, each such block can be selected at will by selecting $H(\omega)$ appropriately. Furthermore, using arguments similar to those used in Appendix A, to guarantee that $\left[\begin{array}{cccc} \widetilde{H}_1(\omega) & \widetilde{H}_2(\omega) & \cdots & \widetilde{H}_N(\omega) \end{array}\right]$ has full column rank we need to guarantee that both $\widetilde{\widetilde{H}}_1(\omega)$ and $\widetilde{\widetilde{H}}_2(\omega)$ have full column rank.

We next observe from (9) that we can write

$$
\begin{aligned}
\widetilde{\widetilde{H}}_1(\omega) = \operatorname{diag}\left\{\left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & e^{j\left(\omega - L_1\frac{\pi}{T_o}\right)T_m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{j\left(\omega - L_1\frac{\pi}{T_o}\right)(R_m-1)T_m} \end{array}\right] W_{R_m}\right\} \cdot \\
\cdot F \cdot \operatorname{diag}\left\{B_{-L_1}(\omega), B_{2-L_1}(\omega), ..., B_{L_1-2}(\omega)\right\}
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
\widetilde{\widetilde{H}}_2(\omega) = \operatorname{diag}\left\{\left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & e^{j\left(\omega + (L_1-1)\frac{\pi}{T_o}\right)T_m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{j\left(\omega + (L_1-1)\frac{\pi}{T_o}\right)(R_m-1)T_m} \end{array}\right] \overline{W}_{R_m}\right\} \cdot \\
\cdot F \cdot \operatorname{diag}\left\{B_{L_1-1}(\omega), B_{L_1-3}(\omega), ..., B_{1-L_1}(\omega)\right\}
\end{aligned}
\tag{52}
$$

where W_{R_m} is the R_m dimensional DFT matrix,

$$
F = \left[\begin{array}{cccc} F_0, & F_1, & \cdots, & F_{L_1-1} \end{array}\right]
$$

and F_p is as defined in (14). From here, by using arguments identical to those used in the proof of Theorem 4 we can conclude that both $\widetilde{\widetilde{H}}_1(\omega)$ and $\widetilde{\widetilde{H}}_2(\omega)$ can be made full column rank if and only if (34) - (36) hold. This in turn means that so does $\left[\begin{array}{cccc} \widetilde{H}_1(\omega), & \widetilde{H}_2(\omega), & \cdots, & \widetilde{H}_N(\omega) \end{array}\right]$ and, by (30), so is $\widetilde{H}(\omega)$, which completes the proof of the theorem. ■

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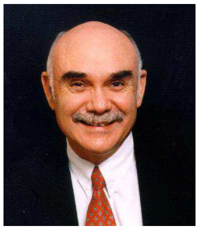

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