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Brief paper

On the use of one bit quantizers in networked control[☆]

Graham C. Goodwin[a], Mauricio Esteban Cea Garrido[a,1], Arie Feuer[b], David Q. Mayne[c]

[a] School of Electrical Engineering and Computer Science, University of Newcastle, 2308, Australia
[b] Electrical Engineering Department, Technion-Israel Institute of Technology, Haifa, Israel
[c] Department of Electrical and Electronic Engineering, Imperial College, London, UK

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ABSTRACT

This paper addresses the following problem in networked control: "If a control law is implemented over a channel that supports a certain fixed bit rate what is the best choice for the control update rate and, consequently, the number of bits carried in each sample?" A restricted architecture in which linear filters are used for the encoder/decoder is considered and a quantizer with linear feedback is deployed. Subject to these restrictions, a procedure for designing the controller and associated filters is presented. These filters are then deployed to choose the best number of bits per control update. It is shown, subject to the above restrictions, that it is generally best to use one bit per sample, in which case, the control update rate is equal to the bit rate. Our analysis has two points of departure from contemporary literature in this area. Firstly, we focus on bits per unit time, as opposed to bits per sample. Secondly we use a fixed number of bits in every time period as opposed to an average bit rate.

1. Introduction

Control theory has traditionally ignored communication constraints, but the recent developments in networked control systems, and the problems arising there from, have inspired considerable interest in the interplay between communication and control (Antsaklis & Baillieul, 2004; Wong & Brockett, 1997). A major focus in this literature has been on the effect of network constraints on performance and stability; typical constraints are limits on (average) data rate, random delays and lost packets. There has been important progress in several areas, (see Braslavsky, Middleton, and Freudenberg (2007), Goodwin, Silva, and Quevedo (2010), Lian, Moyne, and Tilbury (2003), Ling and Lemmon (2004), Nair and Evans (2004), Nair, Fagnani, Zampieri, and Evans (2007), Nilsson (1998), Savkin (2006), Schenato, Sinopoli, Franceschetti, Poolla, and Sastry (2007), Seiler and Sengupta (2005) and Tatikonda and Mitter (2004)). The current paper adopts an alternative point of view and assumes that the constraint is on the bit rate in the communication channel between control law and plant rather than the sampling rate. There are several ways that the question considered here could be formulated, for example where the constraint on bit rate lies between controller and plant, between plant and controller or both. Here, we explore the first of these options. This choice was motivated, in part, by the practical problem of inner loop power control in WCDMA mobile communications (Cea & Goodwin, 2011; Dahlman, Parkvall, Skold, & Beming, 2007). In this problem, the input update period is set to $\Delta = 0.667$ ms and, in traditional implementations, 1 bit per sample is used. However, it would be possible to maintain the same input bit rate whilst changing the input update period to $p\Delta$ and to use p bits/sample. This change of paradigm raises the question as to whether, or not, the choice $p = 1$, used in practice, is the best choice. Preliminary simulation studies conducted by the present authors suggest that 1 bit/update is actually the best choice. The bit rate is the product of the number of bits per sample and the control update frequency. Since p bits per sample permits 2^p quantization levels, a higher value for p reduces the quantization error but also increases the period over which the input must be held, therefore, the ability of the controller to reduce the effect of disturbances. This decomposition of the bit rate into the product of bits per sample and input update frequency results in an obvious trade-off and leads to the question: "What is the best allocation of a given bit rate into the

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E-mail addresses: Graham.Goodwin@newcastle.edu.au (G.C. Goodwin), Mauricio.Cea@uon.edu.au (M.E. Cea Garrido), feuer@ee.technion.ac.il (A. Feuer), d.mayne@imperial.ac.uk (D.Q. Mayne).

[1] Tel.: +61 2 4921 7378; fax: +61 2 4960 1712.

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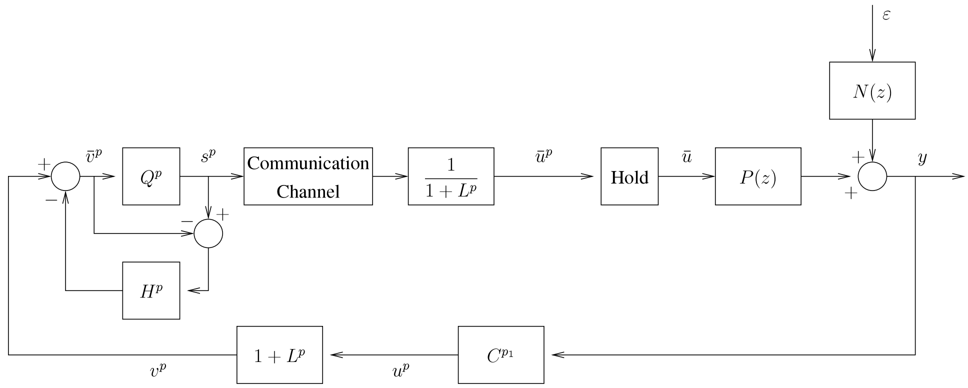

Fig. 1. Proposed four degree of freedom architecture (C^{p_1}, L^p, H^p and Q^p).

number of bits per sample and the number of control updates per second?" This paper addresses this question.

A restrictive (pragmatic) view of network control in which linear filters are utilized for the encoder/decoder pair is adopted. The single input single output case is considered, and a quantizer with linear feedback is deployed to assign the signal of interest to the available bits. The analysis is restricted to open loop stable systems. It would be interesting to relax these restrictions. However, these restrictions are used here to simplify the analysis. We leave it for future research to consider other scenarios. Subject to the above restrictions, a design procedure in which, for each choice of the number of bits in the quantizer, the optimal controller, encoder/decoder and quantizer feedback are chosen. These designs are then used to choose the optimal number of bits/control update. It is shown, surprisingly in our view, that one bit per control update is typically the best choice. Consequently a control update rate equal to the available bit rate is best. This choice corresponds to implementing the control law using a scaled sign function.

2. A class of models

Consider a single input single output linear continuous time system and assume the following:

A.1 The bit rate of the input channel (between controller and plant) is restricted to B_r bits/s so that $\Delta_1 = 1/B_r$ s is the smallest possible control update period.

The output is always sampled at period Δ_1 and an appropriate anti-aliasing filter is deployed at this sample period. Filtering at the lower sample period is implicit in the form of the controller. The input is held constant for p samples to allow a p bit quantizer to be used i.e. p bits are used to code the input signal. When the input is up-sampled to period Δ_1, then the resulting system can be described, without loss of generality, in innovations form (Anderson & Moore, 1979; Goodwin & Sin, 1984) as follows:

$$x_{k+1} = Ax_k + B\bar{u}_k + K\varepsilon_k \tag{1}$$

$$y_k = Cx_k + \varepsilon_k \tag{2}$$

where $x_k \in \mathbb{R}^n$, $\bar{u}_k \in \mathbb{R}^1$, $y_k \in \mathbb{R}^1$, $\varepsilon_k \in \mathbb{R}^1$ are the state, plant input, plant output and innovations sequence having variance σ_ε respectively. Furthermore, assume the transfer function $C(zI - A)^{-1}B$ to have relative degree $d + 1 < n$. Hence, for $d = 0$, $CB \neq 0$ and for $d \geq 1$,

$$CA^iB = 0 \quad \forall\, i = 0, 1, \ldots, d-1; \qquad CA^dB \neq 0. \tag{3}$$

Additional assumptions are introduced as follows:

A.2 The discrete time transfer function from $\bar{u}$ to y is stable and minimum phase.

A.3 A uniform-interval-nearest-neighbour quantizer with 2^p levels is deployed.

A.4 All bits used in the quantizer are communicated between controller and plant over a serial link supporting b_r bits/s.

A.5 The communication channel is error free.

Note that, a quantizer which allocates p bits/sample introduces a transmission delay of period $\Delta_p = p\Delta_1$.

3. Feedback architecture and quantizer

The proposed architecture for the feedback system is shown in Fig. 1. In this figure, a superscript p denotes either a downsampled signal with period $p\Delta_1$ or a system that operates at period $p\Delta_1$. The architecture shown in Fig. 1 has the following degrees of freedom: C^{p1} (a controller which is driven by a signal with sample period Δ_1, which outputs a control every pth sample), $(1 + L_p)$ and $(1 + L_p)^{-1}$ (the channel coder and decoder), $Q^p(\cdot)$ a 2^p level quantizer having step size λ^p and H^p (providing feedback around the quantizer). For realizability L^p and H^p are constrained to be strictly proper. We define $s^p = \bar{v}^p + q^p$ where q^p denotes the quantization error sequence. At sample time $k = \ell p$, $\ell \in \mathbb{Z}^+$, the controller has knowledge of all past outputs (sampled at period Δ_1), that is $y_{\ell p}, y_{\ell p-1}, y_{\ell p-2} \ldots$. The controller then generates an input signal u^p_ℓ. This input signal is held for p samples. The up-sampled input signal is denoted as u_k. Note that $u_{\ell p+i} = u^p_\ell$, $i = 0, 1, \ldots, p-1$. After filtering by $(1 + L^p)$ the input signal is quantized to 2^p levels which leads to a p bit representation. It takes $p\Delta_1$ s to transmit these p bits over the communication channel to the plant input thus satisfying the bit rate constraint. The signal is passed through $(1 + L^p)^{-1}$, then a series to parallel conversion is applied followed by D/A conversion. This process produces a piecewise constant control signal constrained to the same 2^p levels. This signal is then passed through a p sample hold so that the plant input, $\bar{u}_k$, is constant for p successive samples. During this period, the next p bits are received, allowing the next plant input to be reconstructed, and so on. The total delay between sample time, $k = \ell p$, (the time that a sample of the output is taken) and the first time that the resultant control, $u^p_{\ell p}$, effects the output of the plant is $(d + p + 1)$ samples. Note that, due to the p sample hold nature of the input, the output at sample period Δ_1 will be cyclostationary with period p.

We assume a uniform quantizer, which is characterized by the step size between quantization levels. $Q^p[\cdot]$ denotes the quantization operation. The quantization error is defined as

$$q^p = \bar{v}^p - Q[\bar{v}^p] \tag{4}$$

$$= F^p_\lambda[\bar{v}^p] \tag{5}$$

where

$$F_\lambda^p[x] = \begin{cases} x + (2p+1)\lambda & \forall x \in (-\infty, -2(2^{p-1}-1)] \\ x + (2i-1)\lambda & \forall x \in (-2i, -2(i-1)] \\ \quad i = 1, 2, \ldots, 2^{p-1}-1 & \\ x - (2i-1)\lambda & \forall x \in (2(i-1), 2i] \\ \quad i = 1, 2, \ldots, 2^{p-1}-1 & \\ x - (2p+1)\lambda & \forall x \in [2(2^{p-1}-1), \infty). \end{cases} \tag{6}$$

4. Performance criteria and design guidelines

It is important that the performance for different values of p be compared on a fair basis. Since the output is cyclo-stationary with period p, the average performance over the cyclo-stationary period used to compare designs via the cost function:

$$J_A(p) = \frac{1}{p}\sum_{i=1}^{p} E\left\{y(\ell p + p + d + i)^2\right\}. \tag{7}$$

As remarked earlier, for each p, there are four degrees of freedom in the architecture of Fig. 1, namely C^{p1}, L^p, H^p and the choice of quantization levels in Q^p. The design of each of these elements is discussed below.

- C^{p1}: This controller is designed to minimize $J_A(p)$ considering $q^p = 0$.
- L^p: This filter is designed to minimize the variance of $\bar{v}^p$ (the input to the quantizer) considering $q^p = 0$.
- H^p: This filter is designed to minimize $J_A(p)$ when q^p is modelled as a white sequence[2] and ε is considered to be zero.
- Q^p: The levels in the quantizer are chosen to minimize $J_A(p)$ when both ε and q^p are present.

Although restrictive, the above framework underlies contemporary work on the signal to noise ratio approach to network control (Braslavsky et al., 2007; Goodwin et al., 2010). Moreover, it is known that this strategy leads to similar insights to those obtained by more general formulations (Goodwin et al., 2010).

Recall that, the control signal is subject to the constraint

$$\bar{u}_k = \bar{u}_{p\ell} \quad \text{for } p\ell \le k < p(\ell+1), \ \ell \in \mathbb{Z}. \tag{8}$$

This constraint leads to the result that the system, when viewed at period Δ_1, is p-periodic and that the output is cyclo-stationary. A lifted form of the model is used to derive the key results. Hence, for $k = p\ell$ and subject to the control constraint (8):

$$x_{p(\ell+1)} = x^p_{p(\ell+1)} = A_P x^p_{p\ell} + B_P \bar{u}_{p(\ell-1)} + K_P \varepsilon^p_{p\ell} \tag{9}$$

$$y^p_{p\ell} = C_P x^p_{p\ell} + D_{P,u}\bar{u}_{p(\ell-1)} + D_{P,\varepsilon}\varepsilon^p_{p\ell} \tag{10}$$

where $\varepsilon^p_{p\ell} = [\varepsilon_{p\ell} \ \ \varepsilon_{p\ell+1} \ \ \cdots \ \ \varepsilon_{p(\ell+1)-1}]^T$, $y^p_{p\ell} = [y_{p\ell} \ \ y_{p\ell+1} \ \ \cdots \ \ y_{p(\ell+1)-1}]^T$ and

$$A_P = A^p; \qquad B_P = \sum_{i=1}^{p} A^{p-i}B \tag{11}$$

$$C_P = \left[(C)^T \quad (CA)^T \quad \ldots \quad (CA^{p-1})^T\right]^T; \qquad K_P = \left[A^{p-1}K \quad A^{p-2}K \quad \cdots \quad K\right] \tag{12}$$

$$D_{P,u} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{p-2}B & CA^{p-3}B & \cdots & 0 \end{bmatrix}\mathbf{1}; \qquad D_{P,\varepsilon} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ CK & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{p-2}K & CA^{p-3}K & \cdots & 1 \end{bmatrix} \tag{13}$$

where $\mathbf{1} \in \mathbb{R}^{p\times 1}$ is a column vector of ones.

Lemma 1 (*Design of C^{p1}*). *The following controller minimizes (7):*

$$x^C_{p(\ell+1)} = A_C x^C_{p\ell} + B_C u^p_{p(\ell-1)} + K_C y^p_{p\ell} \tag{14}$$

$$u^p_{p\ell} = C_C x^C_{p\ell} + D_C y_{p\ell} + a u^p_{p(\ell-1)} \tag{15}$$

where

$$A_C = (A-KC)^p; \qquad B_C = \sum_{i=1}^{p}(A-KC)^{i-1}B \tag{16}$$

$$K_C = \left[(A-KC)^{p-1}K \quad (A-KC)^{p-2}K \quad \cdots \quad K\right] \tag{17}$$

$$C_C = -\frac{G_2^T}{G_2^TG_2}\begin{bmatrix} CA^{p+d} \\ CA^{p+d+1} \\ \vdots \\ CA^{2p+d-1} \end{bmatrix}(A-KC); \qquad D_C = -\frac{G_2^T}{G_2^TG_2}\begin{bmatrix} CA^{p+d} \\ CA^{p+d+1} \\ \vdots \\ CA^{2p+d-1} \end{bmatrix}K \tag{18}$$

$$a = -\frac{G_2^TG_1}{G_2^TG_2} \tag{19}$$

$$G_1 = \begin{bmatrix} CA^{p+d}B & \cdots & CA^{d+2}B & CA^{d+1}B \\ CA^{p+d+1}B & \cdots & CA^{d+3}B & CA^{d+2}B \\ \vdots & \ddots & \vdots & \vdots \\ CA^{2p+d-1}B & \cdots & CA^{d+p+1}B & CA^{d+p}B \end{bmatrix}\mathbf{1}; \qquad G_2 = \begin{bmatrix} CA^dB & 0 & \cdots & 0 \\ CA^{d+1}B & CA^dB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{d+p-1}B & CA^{d+p-2}B & \cdots & CA^dB \end{bmatrix}\mathbf{1}. \tag{20}$$

Proof. From (1), (2) and (3) it is seen that the predicted output given data up to time $k = p\ell$ satisfies

$$\begin{bmatrix} \widehat{y}_{p(\ell+1)+d+1} \\ \widehat{y}_{p(\ell+1)+d+2} \\ \vdots \\ \widehat{y}_{p(\ell+2)+d} \end{bmatrix} = \begin{bmatrix} CA^{p+d} \\ CA^{p+d+1} \\ \vdots \\ CA^{2p+d-1} \end{bmatrix}(A-KC)\,x_{p\ell} + \begin{bmatrix} CA^{p+d} \\ CA^{p+d+1} \\ \vdots \\ CA^{2p+d-1} \end{bmatrix}Ky_{p\ell}$$

[2] Note that this assumption is a working hypothesis and it is not used when evaluating the final performance.

$$+\begin{bmatrix} CA^{p+d}B & \cdots & CA^{d+2}B & CA^{d+1}B \\ CA^{p+d+1}B & \cdots & CA^{d+3}B & CA^{d+2}B \\ \vdots & \ddots & \vdots & \vdots \\ CA^{2p+d-1}B & \cdots & CA^{d+p+1}B & CA^{d+p}B \end{bmatrix}\begin{bmatrix} u^p_{p(\ell-1)} \\ \vdots \\ u^p_{p\ell-2} \\ u^p_{p\ell-1} \end{bmatrix}$$

$$+\begin{bmatrix} CA^{d}B & 0 & \cdots & 0 \\ CA^{d+1}B & CA^dB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{d+p-1}B & CA^{d+p-2}B & \cdots & CA^dB \end{bmatrix}\begin{bmatrix} u^p_{p\ell} \\ u^p_{p\ell+1} \\ \vdots \\ u^p_{p(\ell+1)-1} \end{bmatrix} \tag{21}$$

where (3) has been used. Then, applying (8)

$$\begin{bmatrix} \widehat{y}_{p(\ell+1)+d+1} \\ \widehat{y}_{p(\ell+1)+d+2} \\ \vdots \\ \widehat{y}_{p(\ell+2)+d} \end{bmatrix} = \begin{bmatrix} CA^{p+d} \\ CA^{p+d+1} \\ \vdots \\ CA^{2p+d-1} \end{bmatrix}(A-KC)\,x_{p\ell} + \begin{bmatrix} CA^{p+d} \\ CA^{p+d+1} \\ \vdots \\ CA^{2p+d-1} \end{bmatrix}Ky_{p\ell}$$

$$+\begin{bmatrix} CA^{p+d}B & \cdots & CA^{d+2}B & CA^{d+1}B \\ CA^{p+d+1}B & \cdots & CA^{d+3}B & CA^{d+2}B \\ \vdots & \ddots & \vdots & \vdots \\ CA^{2p+d-1}B & \cdots & CA^{d+p+1}B & CA^{d+p}B \end{bmatrix}\mathbf{1}u^p_{p(\ell-1)}$$

$$+\begin{bmatrix} CA^{d}B & 0 & \cdots & 0 \\ CA^{d+1}B & CA^dB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{d+p-1}B & CA^{d+p-2}B & \cdots & CA^dB \end{bmatrix}\mathbf{1}u^p_{p\ell}. \tag{22}$$

Since the norm of the vector on the RHS is a simple square function of $u_{p\ell}$, the optimal control can be derived as

$$u^p_{p\ell} = C_C x^p_{p\ell} + D_C y_{p\ell} + a u^p_{p(\ell-1)} \tag{23}$$

where C_C, D_C and a are as given in (18)–(19). We note however, that the plant state $x^p_{p\ell}$ used in the above derivation is not available, so an observer is needed.

$$x^C_{p\ell+1} = Ax^C_{p\ell} + Bu^p_{p(\ell-1)-d} + K\left(y_{p\ell} - Cx^C_{p\ell}\right). \tag{24}$$

Replacing $x^p_{p\ell}$ by $x^C_{p\ell}$, (15) is obtained. □

Lemma 2 (*Design of* L^p). (i) *The optimal filter* $1+L^p$ *is defined by the following equations:*

$$x^L_{p(\ell+1)} = A_L x^L_{p\ell} + B_P u^p_{p(\ell-1)} + K_L\left(u^p_{p\ell} - au^p_{p(\ell-1)}\right) \tag{25}$$

$$v^p_{p\ell} = C_L x^L_{p\ell} + u^p_{p\ell} - au^p_{p(\ell-1)} \tag{26}$$

$$A_L = A_P + K_L C_L \tag{27}$$

$$C_L = -\left(C_C + D_C C\right) \tag{28}$$

where K_L *is the Kalman filter gain designed for the plant:*

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{\omega}_k \tag{29}$$

$$\tilde{y}_k = \tilde{C}\tilde{x}_k + \tilde{\nu}_k \tag{30}$$

where $\tilde{A} = A_P$, $\tilde{C} = C_L$ *and*

$$E\left\{\begin{pmatrix}\tilde{\omega}_k \\ \tilde{\nu}_k\end{pmatrix}\begin{pmatrix}\tilde{\omega}_k^T & \tilde{\nu}_k^T\end{pmatrix}\right\} = \begin{bmatrix} K_P K_P^T & A^{p-1}KD_C^T \\ (A^{p-1}KD_C^T)^T & (D_C)^2 \end{bmatrix}\sigma_\varepsilon^2. \tag{31}$$

(ii) *The inverse filter,* $(1+L^p)^{-1}$*, is given by*

$$x^{IL}_{p(\ell+1)} = A_P x^{IL}_{p\ell} + B_P \bar{u}^p_{p(\ell-1)} + K_L v^p_{p\ell} \tag{32}$$

$$\bar{u}^p_{\ell p} = -C_L x^{IL}_{\ell p} + a\bar{u}^p_{p(\ell-1)} + v^p_{p\ell}. \tag{33}$$

Proof. Recall the controller given in (15). For the purpose of the current derivation, note that the *controller state* is (asymptotically) equal to the plant state. Hence one can equivalently write (15) as:

$$u^p_{\ell p} = C_C x^p_{\ell p} + D_C y_{\ell p} + au^p_{(\ell-1)p}. \tag{34}$$

Using the plant model (1) and recalling the plant equation in its equivalent lifted form (9)

$$u^p_{\ell p} = C_C x^p_{\ell p} + D_C\left(Cx^p_{\ell p} + \varepsilon_{\ell p}\right) + au^p_{(\ell-1)p} \tag{35}$$

where A_P, B_P and K_P are given in (11) and (12). Eqs. (9) and (35) can be seen as a system having process noise $K_P\varepsilon^p_{p\ell} \sim N\left(0, \sigma_\varepsilon^2 K_P K_P^T\right)$ and measurement noise $D_C\varepsilon_{\ell p} \sim N\left(0, \sigma_\varepsilon^2 (D_C)^2\right)$. The process noise and measurement noise have cross-covariance given by $\sigma_\varepsilon^2 A^{p-1}KD_C^T$. This system can be rewritten in innovations form:

$$x'^p_{p(\ell+1)} = A_P x'^p_{p\ell} + B_P\bar{u}^p_{p(\ell-1)} + K_L v^p_{p\ell} \tag{36}$$

$$\bar{u}^p_{\ell p} = \left(C_C + D_C C\right)x'^p_{\ell p} + a\bar{u}^p_{(\ell-1)p} + v^p_{p\ell} \tag{37}$$

where K_L is the corresponding Kalman gain and $v^p_{p\ell}$ is the new innovation sequence. Eqs. (36), (37) are the inverse filter $\frac{1}{1+L^p}$ with input $v^p_{p\ell}$ and output $\bar{u}^p_{\ell p}$. The corresponding inverse, that is the filter $1+L^p$, is

$$x^L_{p(\ell+1)} = A_P x^L_{p\ell} + B_P u^p_{p(\ell-1)} + K_L\left(u^p_{\ell p} - \left(C_C + D_C C\right)x^L_{\ell p} - au^p_{(\ell-1)p}\right) \tag{38}$$

$$v^p_{p\ell} = u^p_{\ell p} - \left(C_C + D_C C\right)x^L_{\ell p} - au^p_{(\ell-1)p}. \tag{39}$$

This result establishes (25) and (26). Also, since $v^p_{p\ell}$ is an innovations sequence, it is i.i.d with minimal variance. This concludes the proof of (i). The inverse filter, as stated in (36), (37) is given by (32), (33). □

To design H^p, ε_k is temporarily set to zero and q^p is modelled as a white noise sequence. Recall that the goal of H^p is to minimize the impact of q^p on the output. From Fig. 1, the input $s_{p\ell}$ to the filter $\frac{1}{1+L^p}$ is

$$s_{p\ell} = \bar{v}^p_{p\ell} + q_{p\ell} - m_{p\ell} \tag{40}$$

where $m_{p\ell}$ is the output of H^p. Also, $q_{p\ell}$ is assumed to be an i.i.d process. The resulting system can be viewed as an LTI system at rate $p\ell$ with input $\left(q_{p\ell} - m_{p\ell}\right)$ and output

$$y^p_{p\ell} = \left[(y_{p\ell})^T \quad (y_{p\ell+1})^T \quad \ldots \quad (y_{p(\ell+1)-1})^T\right]^T. \tag{41}$$

We then have:

Lemma 3. *The model linking inputs* $q_{p\ell} - m_{p\ell}$ *to output* $y^p_{p\ell}$ *can be written as:*

$$x^S_{p(\ell+1)} = A_S x^S_{p\ell} + B_S\left(q_{p\ell} - m_{p\ell}\right) \tag{42}$$

$$y^p_{p\ell} = C_S x^S_{p\ell} \tag{43}$$

where

$$A_S = \begin{bmatrix} a + D_C e_1^T D_P & D_C e_1^T C_P & \mathbf{0} & C_C & C_L & -C_L \\ B_P & A_P & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ D_C e_1^T D_P & D_C e_1^T C_P & a & C_C & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_C & A_C & \mathbf{0} & \mathbf{0} \\ K_L D_C e_1^T D_P & K_L D_C e_1^T C_P & B_P & K_L C_C & A_L & \mathbf{0} \\ B_P + K_L D_C e_1^T D_P & K_L D_C e_1^T C_P & 0 & K_L C_C & K_L C_L & A_P \end{bmatrix} \tag{44}$$

$$B_S = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & K_L^T \end{bmatrix}^T \tag{45}$$

$$C_S = \begin{bmatrix} D_P & C_P & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{46}$$

where $\mathbf{0}$ *are zero matrices of appropriate dimensions.*

Proof. The closed loop system is composed of several subsystems as outlined below. In the following equations the various matrices are computed using the equations numbers in brackets A_C (16), B_C (16), K_C (17), C_C (18), D_C (18), a (19), A_L (27), C_L (28), A_P (11), B_P (11), C_P (12), $D_{P,u}$ (13), $D_{P,\varepsilon}$ (13) and K_P (12).

(i) Delay system 1 (input u_ℓ and output x_ℓ^0):

$$x_{\ell+1}^0 = u_\ell. \tag{47}$$

(ii) The plant in lifted form (input x_ℓ^0 and output y_ℓ):

$$x_{\ell+1}^p = A_P x_\ell^p + B_P x_\ell^0 \tag{48}$$

$$y_\ell^p = C_P x_\ell^p + D_P x_\ell^0. \tag{49}$$

(iii) The controller (with delay system 2p), (input y_ℓ and output $\widetilde{u}_\ell$):

$$\widetilde{x}_{\ell+1}^0 = \widetilde{u}_\ell \tag{50}$$

$$x_{\ell+1}^C = A_C x_\ell^C + B_C \widetilde{x}_\ell^0 \tag{51}$$

$$\widetilde{u}_\ell = C_C x_\ell^C + a\widetilde{x}_\ell^0 + D_c e_1^T y_\ell. \tag{52}$$

(iv) The $1 + L^p$ filter (inputs $\widetilde{u}_\ell, \widetilde{x}_\ell^0$ and output v_ℓ^p):

$$x_{\ell+1}^L = A_L x_\ell^L + B_P \widetilde{x}_\ell^0 + K_L \left(\widetilde{u}_\ell - a\widetilde{x}_\ell^0\right) \tag{53}$$

$$v_\ell^p = C_L x_\ell^L + \widetilde{u}_\ell - a\widetilde{x}_\ell^0. \tag{54}$$

(v) The filter $\frac{1}{1+L^p}$ (inputs $v_\ell^p + q_\ell, x_\ell^0$ and output u_ℓ):

$$x_{\ell+1}^{IL} = A_P x_\ell^{IL} + B_P x_\ell^0 + K_L \left(v_\ell^p + q_\ell\right) \tag{55}$$

$$u_\ell = -C_L x_\ell^{IL} + a x_\ell^0 + v_\ell^p + q_\ell. \tag{56}$$

Combining the equations in (i) to (v) into a system (x_ℓ^S, A_S, B_S, C_S) with input q_ℓ and output y_ℓ one obtains (42) to (46) where

$$x_\ell^S = \left[(x_\ell^0)^T \quad (x_\ell^p)^T \quad (\widetilde{x}_\ell^0)^T \quad (x_\ell^C)^T \quad (x_\ell^L)^T \quad (x_\ell^{IL})^T\right]^T \in \mathbb{R}^{2(2n+1)}. \quad \square \tag{57}$$

Lemma 4 (*Design of H^p*). *Under the working hypothesis that $q_{p\ell}$ is an i.i.d. sequence, then the variance of $y_{p(\ell+1+\bar{n})}$ is minimized by choosing $m_{p\ell}$ via the following equations:*

$$\bar{x}_{p(\ell+1)}^S = A_S \bar{x}_{p\ell}^S + B_S (q_{p\ell} - m_{p\ell}) \tag{58}$$

$$m_{p\ell} = \frac{1}{t_0} e_p C_S A^{\bar{n}+1} \bar{x}_{p\ell}^S \tag{59}$$

where

$$e_p = \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix} \quad e_p \in \mathbb{R}^{1\times p}; \qquad t_0 = e_p C_S A_S^{\bar{n}} B_S \tag{60}$$

and where $\bar{n}$ is the first integer such that $e_p C_S A_S^{\bar{n}} B_S \neq 0$.

Proof. Comparing (58) and (42), it is seen that $\bar{x}_{p\ell}^S$ converges asymptotically to $x_{p\ell}^S$. Hence the second equation in (15) can (asymptotically) be expressed as

$$y_{p(\ell+1)-1} = e_p C^S \bar{x}_{p\ell}^S. \tag{61}$$

Iterating this equation using (58) one obtains

$$y_{p(\ell+1)+\bar{n}} = e_p p C_S A^{\bar{n}+1} \bar{x}_{p\ell}^S + e_p C_S A_S^{\bar{n}} B_S [q_{p\ell} - m_{p\ell}] \tag{62}$$

$$= e_p C_S A^{\bar{n}+1} \bar{x}_{p\ell}^S + t_0 [q_{p\ell} - m_{p\ell}]. \tag{63}$$

Hence, using (59),

$$y_{p(\ell+1)+\bar{n}} = t_0 q_{p\ell} \tag{64}$$

which is an i.i.d. sequence having minimal variance. $\square$

Table 1
Simulation results.

p	1	2	3	4
Best choice for λ_p	0.95	0.25	0.14	0.09
Average output variance achieved with λ^p	2.3268	2.9363	3.1056	3.1346

Having designed C^{p_1}, H^p and L^p, we now fix these quantities and proceed to the design of λ^p (the quantizer step size). The design of λ^p is carried out using a full Monte Carlo simulation, and by then conducting a line search on λ^p, for each p, to choose the best value of λ^p that minimizes (7). The resulting (best) performance, for each value of p, are compared. Note that the earlier working hypothesis that q^p was a white sequence is no longer used in this phase. Indeed, this is why a Monte Carlo approach has been used since an analytic solution is intractable due to the nonlinear behaviour of the quantizer.

5. Example

Many cases have been simulated and, in all cases, $p = 1$ turned out to be the best choice. An illustrative case is given below.

The plant (at sample period Δ_1) including anti-aliasing filter is modelled[3] by (1).

$$A = \begin{bmatrix}0.9 & -0.1\\ 0.7 & 0.8\end{bmatrix}; \qquad B = \begin{bmatrix}0.8\\ 1\end{bmatrix}; \qquad K = \begin{bmatrix}1\\ 1\end{bmatrix}; \qquad C = \begin{bmatrix}1 & 0\end{bmatrix}. \tag{65}$$

Note that this state space system is a stable, minimum phase, LTI system of relative degree 1.

C^{p_1}, L^p, H^p are designed as in Section 4. Then, a full Monte Carlo simulation based on the circuit shown in Fig. 1 is used. (Note that q^p is not modelled as an i.i.d. sequence in the simulations.) λ^p is chosen by a line search to minimize (7) for each p. The results are shown in Table 1. From Table 1 it is clear that $p = 1$ is best. In the tested example it gives, approximately, 25% improvement on the output variance over any other p.

6. Conclusions

This paper has studied the control problem that arises when the control signal is implemented over a bit rate constrained communication channel and when the control update rate is, otherwise, unconstrained. Several simplifying assumptions have been made, so as to simplify the analysis tractable. The conclusion, based on the restrictive assumptions made, is that it is best to use 1 bit per sample and hence to choose the control update rate equal to the inverse of the bit rate. More complex scenarios, for example nonminimum phase and or unstable plants are the subject of on-going research. Note that, in the latter case, one can only expect local stability since the range of a fixed bit quantizer is necessarily finite. Also, more sophisticated designs for the coder/decoder and quantization could be considered. We (boldly) conjecture that $p = 1$ is the best choice under very general conditions. In this context, we hope that this paper may inspire other researchers to examine more general scenarios.

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[3] The innovations variance is taken as 1 without loss of generality since all results are simply scaled by the innovations variance.

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Graham C. Goodwin graduated from the University of New South Wales with B.Sc. (Physics) 1965, B.E. Honours I (Electrical Engineering) 1967 and Ph.D. 1971. In 2010 he was awarded the IEEE Control Systems Field Award and in 2013 he received the Rufus T. Oldenburger Medal. Other international awards include the 1999 IEEE Control Systems Society Hendrik Bode Lecture Prize, a Best Paper award by IEEE Transactions on Automatic Control, a Best Paper award by Asian Journal of Control, and two Best Engineering Text Book awards from the International Federation of Automatic Control in (1984 and 2005). He received the 2008 Quazza Medal from the International Federation of Automatic Control, the 2010 Nordic Process Control Award, and the 2011 Asian Control Association Wook Hyun Kwon Education Award. He is a Fellow of IEEE; an Honorary Fellow of Institute of Engineers, Australia; a Fellow of the International Federation of Automatic Control, a Fellow of the Australian Academy of Science; a Fellow of the Australian Academy of Technology, Science and Engineering; a Member of the International Statistical Institute; a Fellow of the Royal Society, London and a Foreign Member of the Royal Swedish Academy of Sciences. He holds Honorary Doctorates from Lund Institute of Technology, Sweden and the Technion, Israel. He is the co-author of nine books, four edited books, 218 international journal papers and 322 refereed international conference papers. He has successfully supervised 38 Ph.D. students. These hold senior positions in major international universities and industry. He has presented 60 Keynote Addresses at major international conferences. He is a Distinguished Professor at the Harbin Institute of Technology (China), Northwestern University (China), Zhengzhou University (China) and Universidad Técnica Federico Santa María (Chile). He holds several research grants covering diverse areas including Power Electronics, 3G and 4G Mobile Communications, Ambulance Scheduling, and Artificial Pancreas development. He holds 16 International Patents covering rolling mill technology, telecommunications, mine planning and mineral exploration.

Mauricio Esteban Cea Garrido was born in Vina del Mar, Chile. He obtained his professional title of Ingeniero Civil Electrónico and a Master of Science of Engineering from the Universidad Técnica Federico Santa María (Chile) in 2008, obtaining the Best Electronic Engineering Student Award. He completed his Ph.D. at The Centre for Complex Dynamic Systems and Control at The University of Newcastle in temporal and spatial quantization issues in nonlinear filtering. His research interest include Sampled-Data systems, Nonlinear Filtering, Modelling and Optimization. He holds 2 international Patents in Telecommunications and has recently moved to Finance Modelling and Optimization.

Arie Feuer has been with the Electrical Engineering at the Technion—Israel Institute of Technology, since 1983 where he is currently a Professor Emeritus. He received his B.Sc. and M.Sc. from the Technion in 1967 and 1973 respectively and his Ph.D. from Yale University in 1978. From 1967 to 1970 he was in an industry working on automation design and between 1978 and 1983 with Bell Labs in Holmdel, N.J. Between the years 1994 and 2002 he served as the President of the Israel Association of Automatic Control and was a Member of the IFAC Council during the years 2002–2005. He is a Fellow of the IEEE. In the last 22 years he has been regularly visiting the EE&CS department at the University of Newcastle. In 2009 he received an honorary doctorate from the University of Newcastle. His current research interests include:

1. Medical imaging—in particular, ultrasound and CT.
2. Resolution enhancement of digital images and videos.
3. 3D video and multi-view data.
4. Sampling and combined representations of signals and images.
5. Adaptive systems in signal processing and control.
6. Applications of vector quantization.

David Q. Mayne received the Ph.D. and D.Sc. degrees from the University of London, and the degree of Doctor of Technology, honoris causa, from the University of Lund, Sweden. He has held posts at the University of the Witwatersrand, the British Thomson Houston Company, University of California, Davis and Imperial College London where he is now a Senior Research Investigator. His research interests include optimization, optimization based design, nonlinear control and model predictive control.