Exact recovery of Dirac ensembles from the projection onto spaces of spherical harmonics

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Abstract

In this work we consider the problem of recovering an ensemble of Diracs on the sphere from its projection onto spaces of spherical harmonics. We show that under an appropriate separation condition on the unknown locations of the Diracs, the ensemble can be recovered through Total Variation norm minimization. The proof of the uniqueness of the solution uses the method of 'dual' interpolating polynomials and is based on [8], where the theory was developed for trigonometric polynomials. We also show that in the special case of non-negative ensembles, a sparsity condition is sufficient for exact recovery.

1 Introduction

In many cases, images and signals are observed on spherical manifolds. Typical examples are astrophysics (e.g. [13]), topography [4] and gravity fields sensing [12]. Further example is spherical microphone arrays, used for spatial beam forming [18] and sound recording [19].

A key tool for the analysis of signals on the sphere is spherical harmonics analysis, discussed in detail later on. For instance, the spherical microphone array was analyzed in terms of spherical harmonics in [22]. Additionally, spherical harmonics have been extensively used for various applications in computer graphics, such as modeling of volumetric scattering effects, bidirectional reflectance distribution function, and atmospheric scattering (for more graphical applications, see [24] and the references therein). Spherical harmonics are also used in medical imaging [25], optical tomography [3], several applications in physics such as solving potential problem in electrostatics [15] and the central potential Schrödinger equation in quantum mechanics [9]. Additional applications of spherical harmonics are sampling on the sphere [16, 5] and more recently, compressed sensing [1] and sparse recovery [21, 17]. In some sense, our work relates to these latter fields.

Let $\mathcal{H}_n(\mathbb{S}^{d-1})$ denote the space of homogeneous spherical harmonics of degree n, which is the restriction to the unit sphere of the homogeneous harmonic polynomials of degree n in $\mathbb{R}^d$ [2]. Each subspace $\mathcal{H}_n(\mathbb{S}^{d-1})$ is of dimension

$$a_{n,d} := \frac{(2n+d-2)(n+d-3)!}{n!(d-2)!}, \quad n \in \mathbb{N}, d \geq 2.$$

Also, recall that $L_2(\mathbb{S}^{d-1}) = \oplus_{n=0}^{\infty}\mathcal{H}_n(\mathbb{S}^{d-1})$. Thus, if $\{Y_{n,j}\}$, $j = 1, ..., a_{n,d}$, is an orthonormal basis of $\mathcal{H}_n(\mathbb{S}^{d-1})$, then $f \in L_2(\mathbb{S}^{d-1})$ can be presented as $f = \sum_{n=0}^{\infty} f_n$, where

$$f_n = \sum_{j=1}^{a_{n,d}} \langle f, Y_{n,j}\rangle Y_{n,j}.$$

Using the *Addition Formula* [2], one can write the kernel of the projection onto $\mathcal{H}_n(\mathbb{S}^{d-1})$ as

$$\mathcal{P}_{n,d}(\zeta\cdot\eta) = \sum_{j=1}^{a_{n,d}} Y_{n,j}(\zeta)\overline{Y_{n,j}(\eta)} = \frac{a_{n,d}}{|\mathbb{S}^{d-1}|}P_{n,d}(\zeta\cdot\eta), \qquad \zeta,\eta\in\mathbb{S}^{d-1}, \tag{1.1}$$

where $P_{n,d}$ is univariate ultraspherical Gegenbauer polynomial of order d and degree n. Thus, the projection kernel onto the space $V_N := \oplus_{n=0}^{N}\mathcal{H}_n(\mathbb{S}^{d-1})$ is given by

$$K_N(\zeta\cdot\eta) := \sum_{n=0}^{N}\mathcal{P}_{n,d}(\zeta\cdot\eta). \tag{1.2}$$

Consider the Dirac ensemble

$$f = \sum_m c_m\delta_{\xi_m}, \tag{1.3}$$

where δ_x is a Dirac measure, $c_m \in \mathbb{R}$ are real weights, and $\xi_m \in \Xi \subset \mathbb{S}^{d-1}$, are distinct locations on the sphere. We recall the following definition

Definition 1.1. *Let $\mathcal{B}(A)$ be the Borel σ-Algebra on a compact space A, and denote by $\mathcal{M}(A)$ the associated space of real Borel measures. The Total Variation of a real Borel measure $v \in \mathcal{M}(A)$ over a set $B \in \mathcal{B}(A)$ is defined by*

$$|v|(B) = \sup\sum_k |v(B_k)|,$$

where the supremum is taken over all partitions of B into a finite number of disjoint measurable subsets. The total variation $|v|$ is a non-negative measure on $\mathcal{B}(A)$, and the Total Variation (TV) norm of v is defined as

$$\|v\|_{TV} = |v|(A).$$

For a measure of the form of (1.3), it is easy to see that

$$\|f\|_{TV} = \sum_m |c_m|. \tag{1.4}$$

In this paper we assume that the only information we have on the signal f is its 'orthogonal projection' onto V_N, i.e,

$$y_{n,j} := \langle f, Y_{n,j}\rangle = \sum_m c_m Y_{n,j}(\xi_m), \quad 0\le n\le N, \quad 1\le j\le a_{n,d}. \tag{1.5}$$

To ensure exact recovery of the Dirac ensemble from its projection onto V_N, we impose a separation condition as in [8] for the case of trigonometric polynomials and [6] for the case of algebraic polynomials over $[-1,1]$. To this end, recall that the distance on the sphere between any two points $\xi_1, \xi_2 \in \mathbb{S}^{d-1}$ is given by

$$d(\xi_1,\xi_2) = \arccos(\xi_1\cdot\xi_2). \tag{1.6}$$

Definition 1.2. *A set of points $\Xi \subset \mathbb{S}^{d-1}$ is said to satisfy the minimal separation condition for (sufficiently large) N if*

$$\Delta := \min_{\xi_i, \xi_j \in \Xi, \xi_i \neq \xi_j} d\left(\xi_i, \xi_j\right) \geq \frac{\nu}{N}, \tag{1.7}$$

where ν is a fixed constant that does not depend on N.

The main theorem of this paper concerns exact recovery in the case $d = 3$, i.e. the sphere $\mathbb{S}^2$.

Theorem 1.3. *Let $\Xi = \{\xi_m\}$ be the support of a signed measure of the form (1.3). Let $\{Y_{n,j}\}_{n=0}^{N}$ be any spherical harmonics basis for $V_N(\mathbb{S}^2)$ and let $y_{n,j} = \langle f, Y_{n,j} \rangle$, $0 \leq n \leq N$, $1 \leq j \leq a_{n,3}$. If Ξ satisfies the separation condition of Definition 1.2, then f is the unique solution of*

$$\begin{gathered} \min_{g \in \mathcal{M}(\mathbb{S}^2)} \|g\|_{TV} \quad \textit{subject to} \quad \langle g, Y_{n,j} \rangle = y_{n,j}, \\ n = 0, ..., N, \quad j = 1, ..., a_{n,3}, \end{gathered} \tag{1.8}$$

where $\mathcal{M}(\mathbb{S}^2)$ is the space of signed Borel measures on $\mathbb{S}^2$.

Observe that for applications, Theorem 1.3 is stronger than needed. Indeed, since the form of (the unknown) f is known, one may perform TV minimization over the smaller subspace of Dirac superpositions over the sphere. Designing practical numerical algorithms that leverage on this result is a subject of on going research [7]. Also, we strongly believe that this result holds in higher dimensions and indeed significant parts of the proof can be easily generalized to any dimension. However, there are certain technical challenges (see Section 4.2) which we hope to overcome in future work.

The outline of the paper is as follows. In Section 2 we recall the dual problem of interpolating polynomials. In Section 3 we provide details on the essential ingredient of the dual polynomial construction, which is a well-localized polynomial kernel. In Section 4 we carry out the actual construction of the interpolating polynomial. In Section 5 we review the simpler case of signals with non-negative coefficients, where the separation condition can be replaced by a significantly weaker assumption of sparsity. i.e. that the number of Diracs is $\leq N$.

Finally, we point out that the main result of the paper is of qualitative nature in the following sense. Throughout the proofs we will have for some $k \geq 3$, elements of the type c_k/ν^{k-1}, where c_k are absolute constants that depend only on k, but change from estimate to estimate and ν is the constant from Definition 1.2. Once all estimates are done, ν is selected to be sufficiently large so that c_k/ν^{k-1} and similar quantities are sufficiently small. In this paper, we do not deal with the problem of the sharpness of the constant ν.

2 The dual problem of polynomial interpolation

The proof of Theorem 1.3 can be reduced to a problem in polynomial interpolation. Here we state the real version of a general theorem given in [6] and give its proof for completeness (see also [8, 10]):

Theorem 2.1. *Let $f = \sum_m c_m \delta_{\xi_m}, c_m \in \mathbb{R}$, where $\Xi := \{\xi_m\} \subseteq A$, and A is a compact manifold in $\mathbb{R}^n$. Let Π_D be a linear space of continuous functions of dimension D in A. For any basis $\{P_k\}_{k=1}^D$ of Π_D, let $y_k = \langle f, P_k \rangle$ for all $1 \le k \le D$. If for any set $\{u_m\}$, $u_m \in \mathbb{R}$, with $|u_m| = 1$, there exists $q \in \Pi_D$ such that*

$$q(\xi_m) = u_m \,,\ \forall \xi_m \in \Xi,$$
$$|q(\xi)| < 1 \,,\ \forall \xi \in A \backslash \Xi,$$

then f is the unique real Borel measure satisfying

$$\min_{g \in \mathcal{M}(A)} \|g\|_{TV} \quad \text{subject to} \quad y_k = \langle g, P_k \rangle \,,\ 1 \le k \le D. \tag{2.1}$$

Proof. Let g be a solution of (2.1), and define $g = f + h$. The difference measure h can be decomposed relative to $|f|$ as

$$h = h_\Xi + h_{\Xi^C},$$

where h_Ξ is concentrated in Ξ, and h_{Ξ^C} is concentrated in Ξ^C (the complementary of Ξ). Performing a polar decomposition of h_Ξ yields

$$h_\Xi = |h_\Xi| sgn(h_\Xi)(\xi),$$

where $sgn(h_\Xi)$ is a function on A with values $\{-1, 1\}$ (see e.g. [23]). By assumption, there exists $q \in \Pi_D$ obeying

$$q(\xi_m) = sgn(h_\Xi)(\xi_m) \,,\ \forall \xi_m \in \Xi, \tag{2.2}$$
$$|q(\xi)| < 1 \,,\ \forall \xi \in A \backslash \Xi. \tag{2.3}$$

Also by assumption $\langle g, P_k \rangle = \langle f, P_k \rangle$, for $1 \le k \le D$, and so

$$\langle q, h \rangle = 0. \tag{2.4}$$

Equation (2.4), with the polar decomposition of h_Ξ and (2.2) imply

$$0 = \langle q, h_\Xi \rangle + \langle q, h_{\Xi^C} \rangle = \|h_\Xi\|_{TV} + \langle q, h_{\Xi^C} \rangle.$$

If $h_{\Xi^C} = 0$, then $\|h_\Xi\|_{TV} = 0$, and $h = 0$. Alternatively, if $h_{\Xi^C} \neq 0$,we conclude by property (2.3) that

$$|\langle q, h_{\Xi^C} \rangle| < \|h_{\Xi^C}\|_{TV}.$$

Thus,

$$\|h_{\Xi^C}\|_{TV} > \|h_\Xi\|_{TV}. \tag{2.5}$$

As a result of (2.5), we get

$$\begin{aligned} \|f\|_{TV} &\ge \|f + h\|_{TV} = \|f + h_\Xi\|_{TV} + \|h_{\Xi^C}\|_{TV} \\ &\ge \|f\|_{TV} - \|h_\Xi\|_{TV} + \|h_{\Xi^C}\|_{TV} > \|f\|_{TV}, \end{aligned}$$

which is a contradiction. Therefore, $h = 0$, which implies that f is the unique solution of (2.1). □

In the Figure below, we see an example of an interpolating spherical harmonic polynomial $q : \mathbb{S}^2 \to [0, 1]$ where $N = 50$. The heat map shows dark red at points $\xi_m \in \Xi$, where $q(\xi) = 1$ and blue in regions where q is close to zero.

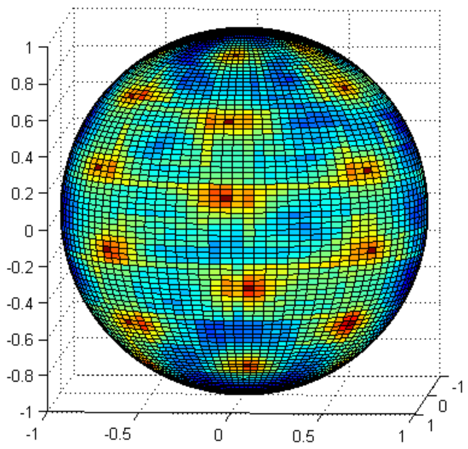

Figure 1: An interpolating polynomial on the sphere.

3 Spherical Harmonics localization

It is well known that the orthogonal projection kernel K_N given by (1.2) does not have good localization. Instead, we follow [20] and for $d = 3$ define the kernel

$$\widetilde{F}_N(\zeta \cdot \eta) := \sum_{n=0}^{\infty} \rho(n/N) \mathcal{P}_{n,3}(\zeta \cdot \eta), \tag{3.1}$$

where $\rho \in C^{\infty}[0, \infty)$ is a smooth non-negative univariate function, satisfying

$$\rho(t) = \begin{cases} 1, & t \in [0, 1/2], \\ \leq 1, & t \in [1/2, 1], \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

We emphasize that $\widetilde{F}_N(\cdot)$ can be regarded as a superposition of Gegenbauer polynomials of degree $\leq N$ and hence also a univariate algebraic polynomial of degree N. Let us impose the following normalization

$$F_N(\zeta \cdot \eta) := \tilde{C}(N) \tilde{F}_N(\zeta \cdot \eta),$$

with $\widetilde{C}(N) > 0$, chosen such that

$$F_N(1) = 1, \tag{3.3}$$

and

$$F_N'(1) \geq \tilde{c} N^2, \tag{3.4}$$

where $\tilde{c} > 0$ is a constant independent of N. Indeed, $\tilde{c}$ can be bounded from below by $1/64$ as follows. Since

$$F_N\left(t\right) = \tilde{C}\left(N\right) \sum_{n=0}^{N} \rho\left(\frac{n}{N}\right) \frac{2n+1}{4\pi} P_{n,3}\left(t\right),$$

and $P_{n,3}(1) = 1$, $\forall n \geq 0$, the normalization $F_N(1) = 1$, gives

$$\tilde{C}(N) = \frac{1}{\sum\limits_{n=0}^{N} \rho\left(\frac{n}{N}\right) \frac{2n+1}{4\pi}}.$$

The derivative formula (see e.g. [2])

$$P'_{n,d}(t) = \frac{n(n+d-2)}{d-1} P_{n-1,d+2}(t), \qquad n \geq 1, d \geq 2,$$

implies

$$F'_N(t) = \tilde{C}(N) \sum_{n=1}^{N} \rho\left(\frac{n}{N}\right) \frac{2n+1}{4\pi} \frac{n(n+1)}{2} P_{n-1,5}(t).$$

Hence, by the properties of ρ (see (3.2))

$$\begin{aligned} F'_N(1) &= \frac{\sum\limits_{n=1}^{N} \rho\left(\frac{n}{N}\right) \frac{2n+1}{4\pi} \frac{n(n+1)}{2}}{\sum\limits_{n=0}^{N} \rho\left(\frac{n}{N}\right) \frac{2n+1}{4\pi}} \\ &\geq \frac{\sum\limits_{n=1}^{N/2} n(n+1)(2n+1)}{2 \sum\limits_{n=0}^{N} (2n+1)} \\ &= \frac{\frac{1}{2}\frac{N}{2}\left(\frac{N}{2}+1\right)\left(\frac{1}{4}N^2 + \frac{3}{2}N + 2\right)}{2N^2 + 4N + 2} \geq \frac{N^2}{64}. \end{aligned}$$

Our construction requires the right form of differentiation. To this end we employ the Lie-Algebra structure on the sphere (see Section 4.2.2 in [2] for more details). For any $\xi_0 \in \mathbb{S}^2$, let $D_{\xi_0,1}, D_{\xi_0,2}$, be the two Lie Algebra matrices associated with the directions of the vectors spanning the tangent plane at $\xi_0 \in \mathbb{S}^2$. The two tangents and hence the matrices, can be determined uniquely (and continuously) to form a right-hand system with ξ_0. These matrices generate parametric families of rotation at angles t in the corresponding directions by the rotation matrices

$$D_{\xi_0,1}(t) := e^{-tD_{\xi_0,1}}, D_{\xi_0,2}(t) := e^{-tD_{\xi_0,2}},$$

where for any matrix B, $e^B := \sum_0^\infty \frac{B^k}{k!}$. We may define the rotational derivatives (if exist) of a function $F : \mathbb{S}^2 \to \mathbb{R}$, at a point $\xi \in \mathbb{S}^2$, by

$$D_{\xi_0,r}F(\xi) := \lim_{t \to 0} \frac{F(D_{\xi_0,r}(t)\xi) - F(\xi)}{t}, \qquad r = 1, 2.$$

Thus, for any point $\xi_1 \in \mathbb{S}^2$, we define the rotational derivatives associated with ξ_0, of the function $F_N(\xi \cdot \xi_1)$, localized at ξ_1, by

$$\begin{aligned} D_{\xi_0,1}F_N(\xi, \xi_1) &:= \lim_{t \to 0} \frac{F_N(D_{\xi_0,1}(t)\xi \cdot \xi_1) - F_N(\xi \cdot \xi_1)}{t}, \\ D_{\xi_0,2}F_N(\xi, \xi_1) &:= \lim_{t \to 0} \frac{F_N(D_{\xi_0,2}(t)\xi \cdot \xi_1) - F_N(\xi \cdot \xi_1)}{t}. \end{aligned}$$

Denoting briefly $\mathcal{P}$ as the orthogonal projector onto V_N, we know by Lemma 4.7 of [2] that for any polynomial $Q \in V_N$,

$$D_{\xi_0,r}Q = D_{\xi_0,r}\mathcal{P}Q = \mathcal{P}D_{\xi_0,r}Q, \qquad r = 1, 2,$$

which implies that $D_{\xi_0,r}F_N(\xi \cdot \xi_1) \in V_N$, $r = 1, 2$, i.e. are spherical harmonics. This is crucial for the construction of the interpolating polynomial (4.3).

First, we investigate the properties of the spherical harmonic $G(\xi, \xi_0) := \xi \cdot \xi_0$, fixed $\xi_0 \in \mathbb{S}^{d-1}$

Lemma 3.1. *For any* $\xi_0, \eta, \eta_1, \eta_2 \in \mathbb{S}^{d-1}$

$$|G(\eta_1, \xi_0) - G(\eta_2, \xi_0)| \leq d(\eta_1, \eta_2) \left[d(\eta, \xi_0) + \max_{j=1,2} d(\eta, \eta_j) \right].$$

Proof. Denote $d_1 := d(\eta_1, \xi_0), d_2 := d(\eta_2, \xi_0)$. Then

$$\begin{aligned} |\eta_1 \cdot \xi_0 - \eta_2 \cdot \xi_0| &= |\cos d_1 - \cos d_2| = 2 |\sin((d_1 - d_2)/2)| |\sin((d_1 + d_2)/2)| \\ &\leq 1/2 |d_1 - d_2| |d_1 + d_2|. \end{aligned}$$

Hence

$$\begin{aligned} |\eta_1 \cdot \xi_0 - \eta_2 \cdot \xi_0| &\leq d(\eta_1, \eta_2) \max\{d(\eta_1, \xi_0), d(\eta_2, \xi_0)\} \\ &\leq d(\eta_1, \eta_2) \left(d(\eta, \xi_0) + \max_{j=1,2} d(\eta, \eta_j) \right). \end{aligned}$$

□

Let $\xi_0, \xi_1, \eta \in \mathbb{S}^2$, $r = 1, 2$ and $0 < t \leq \pi$. If $D_{\xi_1,r}(t)\eta = \eta$, then obviously $D_{\xi_1,r}G(\eta, \xi_0) = 0$. Else, observe that for any rotation matrix A, at an angle t, applied to η, we have $d(A\eta, \eta) \leq t$. Applying this observation and Lemma 3.1, gives

$$\begin{aligned} |D_{\xi_1,r}G(\eta, \xi_0)| &= \lim_{t \to 0} \frac{|D_{\xi_1,r}(t)\eta \cdot \xi_0 - \eta \cdot \xi_0|}{t} \\ &\leq \lim_{t \to 0} \frac{d(D_{\xi_1,r}(t)\eta, \eta)(d(\eta, \xi_0) + d(D_{\xi_1,r}(t)\eta, \eta))}{d(D_{\xi_1,r}(t)\eta, \eta)} \\ &\leq d(\eta, \xi_0). \end{aligned} \tag{3.5}$$

Next, we have the Lipschitz-type estimate

$$\begin{aligned} |D_{\xi_1,r}G(\eta_1, \xi_0) - D_{\xi_1,r}G(\eta_2, \xi_0)| &= \lim_{t \to 0} \frac{|(D_{\xi_1,r}(t) - I)(\eta_1 - \eta_2) \cdot \xi_0|}{t} \\ &\leq \lim_{t \to 0} \frac{||D_{\xi_1,r}(t) - I|||\eta_1 - \eta_2||\xi_0|}{t} \\ &\leq |\eta_1 - \eta_2| \\ &\leq d(\eta_1, \eta_2). \end{aligned} \tag{3.6}$$

This gives for any $\xi_0, \xi_1, \xi_2, \eta \in \mathbb{S}^2$

$$|D_{\xi_1,r_1}D_{\xi_2,r_2}G(\eta, \xi_0)| \leq 1. \tag{3.7}$$

We now recall the following estimate for every $k \geq 1$, $\ell \geq 0$ and $\zeta, \eta \in \mathbb{S}^{d-1}$ [20],

$$\left| F_N^{(\ell)}(\zeta \cdot \eta) \right| \leq \frac{c_{k,\ell} N^{2\ell}}{(1 + Nd(\zeta, \eta))^k}, \tag{3.8}$$

where $c_{k,\ell}$ is a positive constant depending only on k, ℓ. This already gives the good localization of $F_N(\xi \cdot \xi_0)$ at $\xi_0 \in \mathbb{S}^2$, for any $k \geq 1$

$$|F_N(\xi \cdot \xi_0)| \leq \frac{c_k}{(1 + Nd(\xi, \xi_0))^k}. \tag{3.9}$$

Let us proceed with localization of derivatives. For any $\xi_0, \xi_1 \in \mathbb{S}^2$ and $r = 1, 2$ we have the following chain rule

$$\begin{aligned} D_{\xi_1,r} F_N(\xi, \xi_0) &= \lim_{t \to 0} \frac{F_N(D_{\xi_1,r}(t)\xi \cdot \xi_0) - F_N(\xi \cdot \xi_0)}{t} \\ &= \lim_{t \to 0} \frac{(F_N(D_{\xi_1,r}(t)\xi \cdot \xi_0) - F_N(\xi \cdot \xi_0))}{D_{\xi_1,r}(t)\xi \cdot \xi_0 - \xi \cdot \xi_0} \frac{D_{\xi_1,r}(t)\xi \cdot \xi_0 - \xi \cdot \xi_0}{t} \\ &= F_N'(\xi \cdot \xi_0) D_{\xi_1,r} G(\xi, \xi_0). \end{aligned}$$

We note that the above representation of the derivative also shows that it is a spherical polynomial of degree $\leq N$. Furthermore, in the special case where $\xi = \xi_0 = \xi_1$, we get

$$\begin{aligned} D_{\xi_0,r} F_N(\xi_0, \xi_0) &= F_N'(1) \lim_{t \to 0} \frac{D_{\xi_0,r}(t)\xi_0 \cdot \xi_0 - 1}{t} \\ &= F_N'(1) \lim_{t \to 0} \frac{\cos t - 1}{t} = 0. \end{aligned} \tag{3.10}$$

We require the following result that generalizes a lemma from [20]

Lemma 3.2. *Let $\xi_0, \eta, \eta_1, \eta_2 \in \mathbb{S}^2$ with $d(\eta_j, \eta) \leq N^{-1}$, $j = 1, 2$. Then, for any $k \geq 1, \ell \geq 0$,*

$$\left| F_N^{(\ell)}(\eta_1 \cdot \xi_0) - F_N^{(\ell)}(\eta_2 \cdot \xi_0) \right| \leq \frac{c_{k,\ell} d(\eta_1, \eta_2) N^{2\ell+1}}{(1 + Nd(\eta, \xi_0))^k}, \tag{3.11}$$

Proof. First observe that by the triangle inequality for any $\tilde{\eta}$ such that $d(\tilde{\eta}, \eta) \leq N^{-1}$

$$\begin{aligned} Nd(\eta, \xi_0) &\leq N(d(\eta, \tilde{\eta}) + d(\tilde{\eta}, \xi_0)) \\ &\leq N\left(N^{-1} + d(\tilde{\eta}, \xi_0)\right) \\ &\leq 1 + Nd(\tilde{\eta}, \xi_0). \end{aligned}$$

Applying this, (3.8) and Lemma 3.1 yields

$$\begin{aligned} \left| F_N^{(\ell)}(\eta_1 \cdot \xi_0) - F_N^{(\ell)}(\eta_2 \cdot \xi_0) \right| &\leq \max_{d(\tilde{\eta}, \eta) \leq N^{-1}} \left| F_N^{(\ell+1)}(\tilde{\eta} \cdot \xi_0) \right| |\eta_1 \cdot \xi_0 - \eta_2 \cdot \xi_0| \\ &\leq \frac{c_{k+1,\ell+1} N^{2(\ell+1)}}{(1 + Nd(\tilde{\eta}, \xi_0))^{k+1}} d(\eta_1, \eta_2) \left[d(\eta, \xi_0) + N^{-1} \right] \\ &\leq \frac{cN^{2\ell+1} d(\eta_1, \eta_2)}{(1 + Nd(\eta, \xi_0))^k} + \frac{cN^{2\ell+1} d(\eta_1, \eta_2)}{(1 + Nd(\eta, \xi_0))^{k+1}} \\ &\leq \frac{cN^{2\ell+1} d(\eta_1, \eta_2)}{(1 + Nd(\eta, \xi_0))^k}. \end{aligned}$$

□

As a conclusion from Lemma 3.2, we obtain the localization of the derivatives, i.e. for any $\xi_0, \xi_1 \in \mathbb{S}^2$ and $r = 1, 2$

$$\begin{aligned} |D_{\xi_1,r} F_N(\xi,\xi_0)| &= \lim_{t\to 0} \frac{|F_N(D_{\xi_1,r}(t)\xi\cdot\xi_0) - F_N(\xi\cdot\xi_0)|}{t} \\ &\le \lim_{t\to 0} \frac{|F_N(D_{\xi_1,r}(t)\xi\cdot\xi_0) - F_N(\xi\cdot\xi_0)|}{d(D_{\xi_1,r}(t)\xi,\xi)} \\ &\le \frac{c_k N}{(1+Nd(\xi,\xi_0))^k}. \end{aligned} \tag{3.12}$$

Next, we analyze second order derivatives. By the rotation invariance of functions of the type $F_N(\xi\cdot\xi_0)$, we may compute certain values of partial derivatives at the point $\xi_0 = (-1,0,0)$. The rotations at the angle t associated with the partial derivatives at ξ_0 are

$$D_{\xi_0,1}(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}, \quad D_{\xi_0,2}(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denoting $\eta = (\eta_1, \eta_2, \eta_3)$, we get

$$\begin{aligned} D_{\xi_0,1} F_N(\eta,\xi_0) &= \lim_{t\to 0} \frac{F_N\left(\left(\begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}\right)\cdot(-1,0,0)\right) - F_N(-\eta_1)}{t} \\ &= \lim_{t\to 0} \frac{F_N(-\cos t\eta_1 - \sin t\eta_3) - F_N(-\eta_1)}{t} \\ &= \frac{d}{dt} F_N((-\eta_1)\cos t + (-\eta_3)\sin t)\Big|_{t=0} \\ &= (\eta_1 \sin 0 + (-\eta_3)\cos 0)\, F_N'(-\eta_1) \\ &= -\eta_3 F_N'(\eta\cdot\xi_0). \end{aligned}$$

Similarly

$$\begin{aligned} D_{\xi_0,2} F_N(\eta,\xi_0) &= \lim_{t\to 0} \frac{F_N\left(\left(\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}\right)\cdot(-1,0,0)\right) - F_N(-\eta_1)}{t} \\ &= \lim_{t\to 0} \frac{F_N(-\cos t\eta_1 - \sin t\eta_2) - F_N(-\eta_1)}{t} \\ &= \frac{d}{dt} F_N((-\eta_1)\cos t + (-\eta_2)\sin t)\Big|_{t=0} \\ &= (\eta_1 \sin 0 + (-\eta_2)\cos 0)\, F_N'(-\eta_1) \\ &= -\eta_2 F_N'(\eta\cdot\xi_0). \end{aligned}$$

As already observed (see (3.10)),

$$D_{\xi_0,1} F_N(\xi_0,\xi_0) = D_{\xi_0,2} F_N(\xi_0,\xi_0) = 0. \tag{3.13}$$

We now compute mixed partial derivatives of the function $F_N(\xi\cdot\xi_0)$ at the point ξ_0. Again, without the loss of generality we may compute at $\xi_0=(-1,0,0)$

$$\begin{aligned}
D_{\xi_0,2}D_{\xi_0,1}F_N(\eta,\xi_0) &= \lim_{t\to 0}\frac{-\eta_3F_N'\left(\begin{pmatrix}\cos t & \sin t & 0\\ -\sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}\cdot(-1,0,0)\right)+\eta_3F_N'(-\eta_1)}{t}\\
&= \lim_{t\to 0}\frac{-\eta_3F_N'(-\cos t\eta_1-\sin t\eta_2)+\eta_3F_N'(-\eta_1)}{t}\\
&= -\eta_3\frac{d}{dt}F_N'\left((-\eta_1)\cos t+(-\eta_2)\sin t\right)\Big|_{t=0}\\
&= -\eta_3\left(\eta_1\sin 0+(-\eta_2)\cos 0\right)F_N''(-\eta_1)\\
&= \eta_2\eta_3F_N''(\eta\cdot\xi_0).
\end{aligned}$$

This immediately implies

$$D_{\xi_0,2}D_{\xi_0,1}F_N(\xi_0,\xi_0)=D_{\xi_0,1}D_{\xi_0,2}F_N(\xi_0,\xi_0)=0. \tag{3.14}$$

Next, we compute

$$\begin{aligned}
&D_{\xi_0,2}D_{\xi_0,2}F_N(\eta,\xi_0)\\
&= \lim_{t\to 0}\frac{-(-\sin t\eta_1+\cos t\eta_2)F_N'\left(\begin{pmatrix}\cos t & \sin t & 0\\ -\sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}\cdot(-1,0,0)\right)+\eta_2F_N'(-\eta_1)}{t}\\
&= \lim_{t\to 0}\frac{-(-\sin t\eta_1+\cos t\eta_2)F_N'(-\cos t\eta_1-\sin t\eta_2)+\eta_2F_N'(-\eta_1)}{t}\\
&= -\tfrac{d}{dt}(-\sin t\eta_1+\cos t\eta_2)F_N'\left((-\eta_1)\cos t+(-\eta_2)\sin t\right)\Big|_{t=0}\\
&= \eta_1F_N'(\eta\cdot\xi_0)+\eta_2F_N''(\eta\cdot\xi_0).
\end{aligned}$$

With similar computations for $D_{\xi_0,1}D_{\xi_0,1}F_N$, we have,

$$D_{\xi_0,1}D_{\xi_0,1}F_N(\xi_0,\xi_0)=D_{\xi_0,2}D_{\xi_0,2}F_N(\xi_0,\xi_0)=-F_N'(1). \tag{3.15}$$

Proceeding to the next higher order Lipschitz estimate for $\eta,\eta_1,\eta_2\in\mathbb{S}^2$, satisfying $d(\eta_1,\eta),d(\eta_2,\eta)\le N^{-1}$, we have

$$\begin{aligned}
&D_{\xi_1,r}F_N(\eta_1,\xi_0)-D_{\xi_1,r}F_N(\eta_2,\xi_0)\\
&\quad= F_N'(\eta_1\cdot\xi_0)\,D_{\xi_1,r}G(\eta_1,\xi_0)-F_N'(\eta_2\cdot\xi_0)\,D_{\xi_1,r}G(\eta_2,\xi_0)\\
&\quad= (F_N'(\eta_1\cdot\xi_0)-F_N'(\eta_2\cdot\xi_0))\,D_{\xi_1,r}G(\eta_1,\xi_0)+F_N'(\eta_2\cdot\xi_0)\,(D_{\xi_1,r}G(\eta_1,\xi_0)-D_{\xi_1,r}G(\eta_2,\xi_0)).
\end{aligned}$$

Consequently, using (3.5),(3.6), (3.8) and (3.11) for $\ell=1$ yields

$$\begin{aligned}
&|D_{\xi_1,r}F_N(\eta_1,\xi_0)-D_{\xi_1,r}F_N(\eta_2,\xi_0)|\\
&\quad\le |F_N'(\eta_1\cdot\xi_0)-F_N'(\eta_2\cdot\xi_0)|\,d(\eta_1,\xi_0)+|F_N'(\eta_2\cdot\xi_0)|\,d(\eta_1,\eta_2)\\
&\quad\le \tfrac{c_{k+1}N^3}{(1+Nd(\eta\cdot\xi_0))^{k+1}}d(\eta_1,\eta_2)\left(d(\eta,\xi_0)+N^{-1}\right)+\tfrac{c_kN^2d(\eta_1,\eta_2)}{(1+Nd(\eta,\xi_0))^k}\\
&\quad\le \tfrac{c_kN^2d(\eta_1,\eta_2)}{(1+Nd(\eta,\xi_0))^k}.
\end{aligned} \tag{3.16}$$

This implies for any $\xi_0,\xi_1,\xi_2\in\mathbb{S}^2$, $r_1,r_2=1,2$,

$$|D_{\xi_2,r_2}D_{\xi_1,r_1}F_N(\xi,\xi_0)|\le\frac{c_kN^2}{(1+Nd(\xi,\xi_0))^k}. \tag{3.17}$$

Similar calculations give

$$\left|D_{\xi_1,r_1}D_{\xi_2,r_2}F_N\left(\eta_1,\xi_0\right)-D_{\xi_1,r_1}D_{\xi_2,r_2}F_N\left(\eta_2,\xi_0\right)\right|\leq\frac{c_kN^3d\left(\eta_1,\eta_2\right)}{\left(1+Nd\left(\eta,\xi_0\right)\right)^k},\tag{3.18}$$

which in turn yields for any $\xi_0,\xi_1,\xi_2,\xi_3\in\mathbb{S}^2$, $r_1,r_2,r_3=1,2$,

$$\left|D_{\xi_1,r_1}D_{\xi_2,r_2}D_{\xi_3,r_3}F_N\left(\xi\cdot\xi_0\right)\right|\leq\frac{c_kN^3}{\left(1+Nd\left(\xi,\xi_0\right)\right)^k}.\tag{3.19}$$

4 The construction of the interpolating polynomial on $\mathbb{S}^2$

According to Theorem 2.1, a sufficient condition for the recovery of f from its 'orthogonal projection' onto $V_N(\mathbb{S}^2)$ is the existence of $q\in V_N$, satisfying

$$q(\xi_m)=u_m,\quad\forall\xi_m\in\Xi,\tag{4.1}$$
$$|q(\xi)|<1,\quad\forall\xi\notin\Xi,\tag{4.2}$$

for any signed sequence $\{u_m\}$ with unit norm. Following the construction of [8] for $d=2$, we propose that the appropriate form for $d=3$ is

$$q\left(\xi\right):=\sum_{\xi_m\in\Xi}\alpha_mF_N\left(\xi\cdot\xi_m\right)+\beta_mD_{\xi_m,1}F_N\left(\xi,\xi_m\right)+\gamma_mD_{\xi_m,2}F_N\left(\xi,\xi_m\right),\tag{4.3}$$

where $\{\alpha_m\}$,$\{\beta_m\}$, and $\{\gamma_m\}$ are sequences of real coefficients, to be selected later. We point out that, as explained in Section 3, the partial derivatives in (4.3) are spherical harmonics polynomials of degree $\leq N$, and thus $q\in V_N(\mathbb{S}^2)$.

Thus, this section is devoted to the proof of the following proposition:

Proposition 4.1. *If $\Xi\subset\mathbb{S}^2$ satisfies the separation condition of Definition 1.2, then there exist coefficients $\{\alpha_m\}$,$\{\beta_m\}$, and $\{\gamma_m\}$, such that q of the form (4.3) obeys (4.1) and (4.2).*

According to Theorem 2.1, Proposition 4.1 immediately implies Theorem 1.3. The proof of Proposition 4.1 follows the outline of [8] and is given by a series of lemmas, as follows:

Lemma 4.2. *If the separation condition of Definition 1.2 holds, then for any sequence $\{u_m\}$, with $u_m=\{-1,1\}$, there exist coefficients $\{\alpha_m\}$,$\{\beta_m\}$, and $\{\gamma_m\}$, such that*

$$q(\xi_m)=u_m,\tag{4.4}$$
$$D_{\xi_m,1}q(\xi_m)=D_{\xi_m,2}q(\xi_m)=0,\tag{4.5}$$

for all $\xi_m\in\Xi$. Additionally, for any $k\geq3$, there exists a constant c_k, such that

$$\|\alpha\|_\infty\leq1+\frac{c_k}{\nu^{k-1}},\tag{4.6}$$
$$\|\beta\|_\infty\leq\frac{c_k}{N\nu^{k-1}},\tag{4.7}$$
$$\|\gamma\|_\infty\leq\frac{c_k}{N\nu^{k-1}},\tag{4.8}$$

with $\nu > 0$, the constant from the separation condition. Moreover, if $u_1 = 1$, then

$$\alpha_1 \geq 1 - \frac{c_k}{\nu^{k-1}}. \tag{4.9}$$

Lemma 4.3. *If the separation condition in Definition 1.2 holds, then the polynomial (4.3) as constructed in Lemma 4.2 satisfies $|q(\xi)| < 1$ for any $\xi \in \mathbb{S}^2$, obeying*

$$d(\xi, \xi_m) \leq \frac{\sigma}{N},$$

for some $\xi_m \in \Xi$ and sufficiently small $\sigma > 0$.

Lemma 4.4. *If the separation condition in Definition 1.2 holds, then the polynomial (4.3) as constructed in Lemma 4.2 satisfies $|q(\xi)| < 1$ for any $\xi \in \mathbb{S}^2$, obeying*

$$d(\xi, \xi_m) \geq \frac{\sigma}{N}, \quad \forall \xi_m \in \Xi,$$

where σ is the constant of Lemma 4.3.

4.1 Proof of Lemma 4.2

The gradient of any q of the form (4.3), at a point $\xi_k \in \Xi$, is given by

$$\begin{aligned} D_{\xi_k,r} q(\xi_k) = \sum_{\xi_m \in \Xi} & \alpha_m D_{\xi_k,r} F_N(\xi_k, \xi_m) + \beta_m D_{\xi_k,r} D_{\xi_m,1} F_N(\xi_k, \xi_m) \\ & + \gamma_m D_{\xi_k,r} D_{\xi_m,2} F_N(\xi_k, \xi_m), \qquad r = 1, 2. \end{aligned}$$

Conditions (4.4) and (4.5) may be written in matrix notation as

$$\begin{bmatrix} F_0 & \tilde{F}_1^1 & \tilde{F}_1^2 \\ F_1^1 & F_2^{1,1} & F_2^{1,2} \\ F_1^2 & F_2^{2,1} & F_2^{2,2} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}, \tag{4.10}$$

where

$$\begin{aligned} F_0 &:= \{F_N(\xi_k \cdot \xi_m)\}_{k,m}, \\ F_1^r &:= \{D_{\xi_k,r} F_N(\xi_k, \xi_m)\}_{k,m}, \quad r = 1, 2, \\ \tilde{F}_1^r &:= \{D_{\xi_m,r} F_N(\xi_k, \xi_m)\}_{k,m}, \quad r = 1, 2, \\ F_2^{r_1,r_2} &:= \{D_{\xi_k,r_1} D_{\xi_m,r_2} F_N(\xi_k, \xi_m)\}_{k,m}, \quad r_1, r_2 = 1, 2, \end{aligned}$$

and $u = \{u_m\}_m, \alpha = \{\alpha_m\}_m, \beta = \{\beta_m\}_m, \gamma = \{\gamma_m\}_m$. For convenience, we occasionally write (4.10) as

$$\mathcal{F} = \begin{bmatrix} F_0 & \tilde{\mathcal{F}}_1 \\ \mathcal{F}_1 & \mathcal{F}_2 \end{bmatrix}.$$

Our goal is to show that $\mathcal{F}$ is invertible and to estimate the coefficients α, β, γ. To this end, we require the following

Lemma 4.5. *Let $\xi_0 \in \Xi$, where Ξ satisfies the separation condition and let $\xi \in \mathbb{S}^2$, such that $d(\xi,\xi_0) \leq \Delta/2$. Then, for any $k \geq 3$ there exists $c_k > 0$, such that for any $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3 \in \mathbb{S}^2$ and $r_1, r_2, r_3 = 1, 2$,*

$$\sum_{\xi_m \in \Xi \backslash \xi_0} |F_N(\xi \cdot \xi_m)| \leq \frac{c_k}{\nu^{k-1}}, \tag{4.11}$$

$$\sum_{\xi_m \in \Xi \backslash \xi_0} \left| D_{\tilde{\xi}_1, r_1} F_N(\xi, \xi_m) \right|, \sum_{\xi_m \in \Xi \backslash \xi_0} |D_{\xi_m, r_1} F_N(\xi, \xi_m)| \leq \frac{c_k N}{\nu^{k-1}}, \tag{4.12}$$

$$\sum_{\xi_m \in \Xi \backslash \xi_0} \left| D_{\tilde{\xi}_1, r_1} D_{\tilde{\xi}_2, r_2} F_N(\xi, \xi_m) \right| \leq \frac{c_k N^2}{\nu^{k-1}}, \tag{4.13}$$

$$\sum_{\xi_m \in \Xi \backslash \xi_0} \left| D_{\tilde{\xi}_1, r_1} D_{\tilde{\xi}_2, r_2} D_{\tilde{\xi}_3, r_3} F_N(\xi, \xi_m) \right| \leq \frac{c_k N^3}{\nu^{k-1}}. \tag{4.14}$$

Proof. Fix $\xi_0 \in \Xi$. Let Ω_m be the 'ring' about ξ_0 such that

$$\Omega_m := \left\{ \xi \in \mathbb{S}^2 : \frac{\nu m}{N} < d(\xi, \xi_0) \leq \frac{\nu(m+1)}{N} \right\}, 0 \leq m \leq \left\lfloor \frac{\pi N}{\nu} - 1 \right\rfloor.$$

The surface area of the ring is given by [2]

$$|\Omega_m| = 2\pi \left(\cos\left(\frac{\nu}{N} m\right) - \cos\left(\frac{\nu}{N}(m+1)\right) \right).$$

By assumption the set Ξ satisfies the separation condition in Definition 1.2. Hence, the points are the center of pairwise disjoint caps of area $2\pi\left(1 - \cos\frac{\nu}{2N}\right)$. Observe that the cap of any $\xi_k \in \Omega_m$ is contained in the ring

$$\widetilde{\Omega}_m := \left\{ \xi \in \mathbb{S}^2 : \max\left\{ \frac{\nu(m-1/2)}{N}, 0 \right\} < d(\xi, \xi_0) \leq \min\left\{ \frac{\nu(m+3/2)}{N}, \pi \right\} \right\}.$$

Therefore, we can bound the number of points in the ring Ω_m, by

$$\begin{aligned} \#\{\xi_k \in \Omega_m\} &\leq \frac{\left|\tilde{\Omega}_m\right|}{2\pi\left(1 - \cos\frac{\nu}{2N}\right)} = \frac{2\pi\left(\cos\left(\frac{\nu}{N}(m-1/2)\right) - \cos\left(\frac{\nu}{N}(m+3/2)\right)\right)}{2\pi\left(1 - \cos\frac{\nu}{2N}\right)} \\ &= \frac{\sin\left(\frac{\nu}{2N}(2m+1)\right)\sin\left(\frac{\nu}{N}\right)}{\sin^2\left(\frac{\nu}{4N}\right)} \leq \frac{\sin\left(\frac{\nu}{2N}(2m+1)\right) 4 \sin\left(\frac{\nu}{4N}\right)}{\sin^2\left(\frac{\nu}{4N}\right)} \\ &\leq 4\left| \frac{\sin\left(\frac{\nu}{2N}(2m+1)\right)}{\sin\left(\frac{\nu}{4N}\right)} \right| \leq cm, \end{aligned} \tag{4.15}$$

where the constant does not depend on N or ν. Since $d(\xi,\xi_0) \leq \Delta/2$, the point ξ is well-separated from the points $\xi_m \in \Xi\backslash\xi_0$. Therefore, using (3.9) and (4.15) we get for $k \geq 3$

$$\begin{aligned} \sum_{\xi_m \in \Xi \backslash \xi_0} |F_N(\xi \cdot \xi_m)| &\leq c_k \sum_{m=1}^{\infty} \frac{m}{(1+m\nu)^k} \\ &\leq \frac{c_k}{\nu^{k-1}} \sum_{m=1}^{\infty} \frac{1}{m^{k-1}} \leq \frac{c_k}{\nu^{k-1}}. \end{aligned}$$

This proves (4.11). Using (3.12), similar calculations prove (4.12) by

$$\sum_{\xi_m \in \Xi \setminus \xi_0} \left| D_{\tilde{\xi}_1, r_1} F_N(\xi, \xi_m) \right| \leq c_k N \sum_{m=1}^{\infty} \frac{m}{(1+m\nu)^k} \leq \frac{c_k N}{\nu^{k-1}}.$$

The estimates (4.13) and (4.14) are proved in similar manner.

□

We successively use the fact that a sufficient condition for the invertibility of a matrix M is

$$\|I - M\|_\infty < 1, \tag{4.16}$$

where $\|M\|_\infty := max_i \sum_j |m_{i,j}|$. Furthermore (see e.g [14], Corollary 5.6.16),

$$\|M^{-1}\|_\infty \leq \frac{1}{1 - \|I - M\|_\infty}. \tag{4.17}$$

The proof of Lemma 4.2 also requires the following

Lemma 4.6. *If the separation condition holds, then*

$$\|I - F_0\|_\infty \leq \frac{c_k}{\nu^{k-1}}, \tag{4.18}$$

$$\|F_1^r\|_\infty, \|\tilde{F}_1^r\|_\infty \leq N \frac{c_k}{\nu^{k-1}}, r = 1, 2, \tag{4.19}$$

$$\|F_2^{1,2}\|_\infty, \|F_2^{2,1}\|_\infty \leq N^2 \frac{c_k}{\nu^{k-1}}, \tag{4.20}$$

$$\|-F_N'(1)I - F_2^{r,r}\|_\infty \leq N^2 \frac{c_k}{\nu^{k-1}}, \tag{4.21}$$

$$\|(F_2^{r,r})^{-1}\|_\infty \leq \frac{1}{N^2\left(\tilde{c} - \frac{c_k}{\nu^{k-1}}\right)} \quad r = 1, 2, \tag{4.22}$$

where the constant $\tilde{c}$ is given by (3.4).

Proof. Observe that by (3.3), $F_0(k, k) = F_N(1) = 1$. Applying (4.11) to any row in the matrix F_0, yields (4.18)

$$\|I - F_0\|_\infty = \max_{\xi_j \in \Xi} \sum_{\xi_i \in \Xi, \xi_i \neq \xi_j} |F_N(\xi_j \cdot \xi_i)| \leq \frac{c_k}{\nu^{k-1}}.$$

According to (3.13), the diagonals of F_1^r and $\tilde{F}_1^r$, $r = 1, 2$ are zero. Applying (4.12) gives

$$\|F_1^r\|_\infty = \max_{\xi_j \in \Xi} \sum_{\xi_i \in \Xi, \xi_i \neq \xi_j} |D_{\xi_j, r} F_N(\xi_j, \xi_i)| \leq \frac{N c_k}{\nu^{k-1}}.$$

In similar manner, observing from (3.14) that the diagonals of $F_2^{1,2}$ and $F_2^{2,1}$ are zero, (4.13) gives (4.20). Next, we derive from (3.15) and (4.13) that

$$\|-F_N'(1)I - F_2^{r,r}\|_\infty \leq \frac{N^2 c_k}{\nu^{k-1}}.$$

Ultimately, (4.17), (4.21) and (3.4) imply (4.22). □

We may now proceed with the proof of Lemma 4.2. To show that $\mathcal{F}$ is invertible for sufficiently large ν, we show that both $\mathcal{F}_2$ and its Schur complement are invertible [26]. From (4.21), we know that $F_2^{2,2}$ is an invertible matrix for sufficiently large ν. So, $\mathcal{F}_2$ is invertible if the Schur complement of $F_2^{2,2}$ in $\mathcal{F}_2$, given by

$$\mathcal{F}_{s,2} := (\mathcal{F}_2/F_2^{2,2}) = F_2^{1,1} - F_2^{1,2}\left(F_2^{2,2}\right)^{-1}F_2^{2,1},$$

is invertible as well. Using the estimates of Lemma 4.6, (3.4) and assuming $\nu^{k-1} \geq (1+\tilde{c}c_k)/\tilde{c}^2$, we get

$$\begin{aligned}\left\|I - \frac{\mathcal{F}_{s,2}}{-F_N'(1)}\right\|_\infty &\leq \left\|I - \frac{F_2^{1,1}}{-F_N'(1)}\right\|_\infty + \frac{1}{|F_N'(1)|}\left\|F_2^{1,2}\right\|_\infty \left\|F_2^{2,1}\right\|_\infty \left\|\left(F_2^{2,2}\right)^{-1}\right\|_\infty \\ &\leq \frac{c_k}{\nu^{k-1}}.\end{aligned}$$

This implies that

$$\|\mathcal{F}_{s,2}^{-1}\|_\infty \leq \frac{1}{F_N'(1)}\frac{1}{1-\frac{c_k}{\nu^{k-1}}} \leq \frac{1}{\tilde{c}N^2}\left(1+\frac{c_k}{\nu^{k-1}-c_k}\right). \tag{4.23}$$

Since $\mathcal{F}_2$ is invertible for sufficiently large ν, $\mathcal{F}$ is invertible if the Schur complement $\mathcal{F}_s := \mathcal{F}/\mathcal{F}_2$ is invertible as well. Note that

$$\begin{aligned}(\mathcal{F}/F_2^{2,2}) &= \begin{bmatrix} F_0 & \tilde{F}_1^1 \\ F_1^1 & F_2^{1,1}\end{bmatrix} - \begin{bmatrix}\tilde{F}_1^2 \\ F_2^{1,2}\end{bmatrix}\left(F_2^{2,2}\right)^{-1}\begin{bmatrix}F_1^2 & F_2^{2,1}\end{bmatrix} \\ &= \begin{bmatrix} F_0 - \tilde{F}_1^2\left(F_2^{2,2}\right)^{-1}F_1^2 & \tilde{\mathcal{F}}_{s,1} \\ \mathcal{F}_{s,1} & \mathcal{F}_{s,2}\end{bmatrix},\end{aligned}$$

where

$$\mathcal{F}_{s,1} := F_1^1 - F_2^{1,2}(F_2^{2,2})^{-1}F_1^2, \tag{4.24}$$

$$\tilde{\mathcal{F}}_{s,1} := \tilde{F}_1^1 - \tilde{F}_1^2(F_2^{2,2})^{-1}F_2^{2,1}. \tag{4.25}$$

According to Theorem 1.4 in [26],

$$\mathcal{F}_s = \left(\mathcal{F}/F_2^{2,2}\right) / \left(\mathcal{F}_2/F_2^{2,2}\right),$$

and thus, the Schur complement of $\mathcal{F}_2$ is given by

$$\mathcal{F}_s = F_0 - \tilde{\mathcal{F}}_{s,1}\mathcal{F}_{s,2}^{-1}\mathcal{F}_{s,1} - \tilde{F}_1^2(F_2^{2,2})^{-1}F_1^2.$$

Using Lemma 4.6, and assuming $\nu^{k-1} \geq (1+c_k)/\tilde{c}$, we get

$$\|\mathcal{F}_{s,1}\|_\infty \leq \|F_1^1\|_\infty + \|F_2^{1,2}\|_\infty\|(F_2^{2,2})^{-1}\|_\infty\|F_1^2\|_\infty \leq \frac{c_k N}{\nu^{k-1}}. \tag{4.26}$$

A similar estimate holds for $\|\tilde{\mathcal{F}}_{s,1}\|_\infty$. Hence, under similar assumptions on ν

$$\begin{aligned}\|I - \mathcal{F}_s\| &\leq \|I - F_0\|_\infty + \|\mathcal{F}_{s,1}\|_\infty\|\tilde{\mathcal{F}}_{s,1}\|_\infty\|\mathcal{F}_{s,2}^{-1}\|_\infty + \|F_1^2\|_\infty\|\tilde{F}_1^2\|_\infty\|(F_2^{2,2})^{-1}\|_\infty \\ &\leq \frac{c_k}{\nu^{k-1}} + \frac{c_k}{\nu^{2(k-1)}}\frac{1}{\tilde{c}}\left(1+\frac{c_k}{\nu^{k-1}-c_k}\right) + \frac{c_k}{\nu^{2(k-1)}}\frac{1}{\tilde{c}-\frac{c_k}{\nu^{k-1}}} \\ &\leq \frac{c_k}{\nu^{k-1}}.\end{aligned} \tag{4.27}$$

Moreover,

$$\|\mathcal{F}_s^{-1}\|_\infty \leq \frac{1}{1-\frac{c_k}{\nu^{k-1}}} = 1 + \frac{c_k}{\nu^{k-1}-c_k}. \tag{4.28}$$

Therefore, for sufficiently large ν, (4.10) is an invertible matrix. Hence, we can calculate the coefficient sequences by

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix} I \\ -\mathcal{F}_{s,2}^{-1}\mathcal{F}_{s,1} \\ (F_2^{2,2})^{-1}(F_2^{2,1}\mathcal{F}_{s,2}^{-1}\mathcal{F}_{s,1} - F_1^2)\end{bmatrix}\mathcal{F}_s^{-1}u. \tag{4.29}$$

We now proceed to estimate the coefficients. We begin with the observation that

$$\|\alpha\|_\infty \leq \|\mathcal{F}_s^{-1}\|_\infty \leq 1 + \frac{c_k}{\nu^{k-1}-c_k}.$$

In addition, using (4.23), (4.26) and (4.28), for sufficiently large ν, get

$$\begin{aligned}\|\beta\|_\infty \leq & \|\mathcal{F}_{s,2}^{-1}\|_\infty \|\mathcal{F}_{s,1}\|_\infty \|\mathcal{F}_s^{-1}\|_\infty \\ & \leq \frac{c_k}{N\nu^{k-1}}.\end{aligned}$$

Using the same estimates with additional estimates from Lemma 4.6 give

$$\begin{aligned}\|\gamma\|_\infty &\leq \|(F_2^{2,2})^{-1}\|\|F_2^{1,2}\|_\infty \|\mathcal{F}_{s,2}^{-1}\|_\infty \|\mathcal{F}_{s,1}\|_\infty \|\mathcal{F}_s^{-1}\|_\infty \\ &\leq \frac{c_k}{N\nu^{k-1}}.\end{aligned}$$

Finally, if $u_1 = 1$, we can apply (4.27), (4.28) and the assumption that $|u_m| = 1$, for each m, to obtain

$$\begin{aligned}\alpha_1 &= \left(\left(I - (I - \mathcal{F}_s^{-1})\right)u\right)_1 \\ &= u_1 - \left((I - \mathcal{F}_s^{-1})u\right)_1 \\ &\geq 1 - \|\mathcal{F}_s^{-1}\|_\infty \|I - \mathcal{F}_s\|_\infty \\ &\geq 1 - \frac{c_k}{\nu^{k-1}}.\end{aligned}$$

This completes the proof of Lemma 4.2.

4.2 Proof of Lemma 4.3

Without loss of generality, assume that at $\xi_1 \in \Xi$, the interpolation condition is $q(\xi_1) = 1$. Let $\xi \in \mathbb{S}^2$ such that $d(\xi_1, \xi) \leq \sigma/N$ for sufficiently small $0 < \sigma < 1$ (to be chosen later). The Hessian of $q(\xi)$ at ξ is

$$H(q)(\xi) = \begin{bmatrix} (D_{\xi,1})^2 q(\xi) & D_{\xi,1}D_{\xi,2}q(\xi) \\ D_{\xi,1}D_{\xi,2}q(\xi) & (D_{\xi,2})^2 q(\xi)\end{bmatrix}.$$

We wish to show that for sufficiently small $\sigma > 0$ and large enough ν, $\det(H(\xi)) > 0$ and $Tr(H(\xi)) < 0$, which implies that both eigenvalues are strictly negative and therefore q is concave at ξ. For $r=1,2$

$$\begin{aligned}(D_{\xi,r})^2 q(\xi) \le{}& \alpha_1 (D_{\xi,r})^2 F_N(\xi,\xi_1) + \|\beta\|_\infty \left|(D_{\xi,r})^2 D_{\xi_1,1}F_N(\xi,\xi_1)\right| \\ &+ \|\gamma\|_\infty \left|(D_{\xi,r})^2 D_{\xi_1,2}F_N(\xi,\xi_1)\right| \\ &+ \|\alpha\|_\infty \sum_{\xi_m\in\Xi\setminus\xi_1} \left|(D_{\xi,r})^2 F_N(\xi,\xi_m)\right| \\ &+ \|\beta\|_\infty \left(\sum_{\xi_m\in\Xi\setminus\xi_1} \left|(D_{\xi,r})^2 D_{\xi_m,1}F_N(\xi,\xi_m)\right|\right) \\ &+ \|\gamma\|_\infty \left(\sum_{\xi_m\in\Xi\setminus\xi_1} \left|(D_{\xi,r})^2 D_{\xi_m,2}F_N(\xi,\xi_m)\right|\right).\end{aligned}$$

We estimate the first left hand term using (4.9), (3.15), (4.6) and then (3.18)

$$\begin{aligned}\alpha_1 (D_{\xi,r})^2 F_N(\xi,\xi_1) &= \alpha_1 (D_{\xi,r})^2 F_N(\xi,\xi) + \alpha_1 \left((D_{\xi,r})^2 F_N(\xi,\xi_1) - (D_{\xi,r})^2 F_N(\xi,\xi)\right) \\ &\le -\left(1-\frac{c_k}{\nu^{k-1}}\right) F_N'(1) + \left(1+\frac{c_k}{\nu^{k-1}-c_k}\right) c_k N^3 d(\xi,\xi_1) \\ &\le -N^2\left(\tilde{c}\left(1-\frac{c_k}{\nu^{k-1}}\right) - \left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)c_k\sigma\right).\end{aligned}$$

The next two terms are estimated using the bounds on α,β (4.7), (4.8) and (3.19)

$$\|\beta\|_\infty \left|(D_{\xi,r})^2 D_{\xi_1,1}F_N(\xi,\xi_1)\right|, \|\gamma\|_\infty \left|(D_{\xi,r})^2 D_{\xi_1,2}F_N(\xi,\xi_1)\right| \le \frac{c_k}{\nu^{k-1}}N^2.$$

Estimates (4.6) and (4.13) give

$$\|\alpha\|_\infty \sum_{\xi_m\in\Xi\setminus\xi_1} \left|(D_{\xi,r})^2 F_N(\xi,\xi_m)\right| \le \left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)\frac{c_k}{\nu^{k-1}}N^2.$$

Using (4.7), (4.8) and (4.14)

$$\|\beta\|_\infty \left(\sum_{\xi_m\in\Xi\setminus\xi_1} \left|(D_{\xi,r})^2 D_{\xi_1,1}F_N(\xi,\xi_1)\right|\right), \|\gamma\|_\infty \left(\sum_{\xi_m\in\Xi\setminus\xi_1} \left|(D_{\xi,r})^2 D_{\xi_1,2}F_N(\xi,\xi_1)\right|\right) \le \frac{c_k}{\nu^{k-1}}N^2.$$

Thus, for sufficiently small σ and large ν

$$(D_{\xi,r})^2 q(\xi) \le -N^2\left(\tilde{c}\left(1-\frac{c_k}{\nu^{k-1}}\right) - \left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)c_k\sigma + \frac{c_k}{\nu^{k-1}}\right) < 0.$$

We proceed with the estimate of the two other entries of the Hessian

$$\begin{aligned}&|D_{\xi,1}D_{\xi,2}q(\xi)| \le \alpha_1 |D_{\xi,1}D_{\xi,2}F_N(\xi,\xi_1)| + \|\beta\|_\infty |D_{\xi,1}D_{\xi,2}D_{\xi_1,1}F_N(\xi,\xi_1)| \\ &+ \|\gamma\|_\infty D_{\xi,1}D_{\xi,2}D_{\xi_1,2}F_N(\xi,\xi_1) \\ &+ \|\alpha\|_\infty \sum_{\xi_m\in\Xi\setminus\xi_1} |D_{\xi,1}D_{\xi,2}F_N(\xi,\xi_m)| \\ &+ \|\beta\|_\infty \left(\sum_{\xi_m\in\Xi\setminus\xi_1} |D_{\xi,1}D_{\xi,2}D_{\xi_1,1}F_N(\xi,\xi_1)|\right) \\ &+ \|\gamma\|_\infty \left(\sum_{\xi_m\in\Xi\setminus\xi_1} |D_{\xi,1}D_{\xi,2}D_{\xi_1,2}F_N(\xi,\xi_1)|\right).\end{aligned}$$

Using first (4.6), (3.14) and then (3.18) yields

$$\begin{aligned}\alpha_1 \left|D_{\xi,1}D_{\xi,2}F_N(\xi,\xi_1)\right| &\le \left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)\left|D_{\xi,1}D_{\xi,2}F_N(\xi,\xi_1)-D_{\xi,1}D_{\xi,2}F_N(\xi,\xi)\right| \\ &\le \left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)c_k N^3 d(\xi,\xi_1) \\ &\le \left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)c_k\sigma N^2.\end{aligned}$$

Combining with similar estimates as in the previous case results in

$$\left|D_{\xi,1}D_{\xi,2}q(\xi)\right| \le N^2\left(\left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)c_k\sigma+\frac{c_k}{\nu^{k-1}}+\left(1+\frac{c_k}{\nu^{k-1}-c_k}\right)\frac{c_k}{\nu^{k-1}}\right).$$

It is now clear, that we can chose sufficiently small σ and large enough ν such that $\left|D_{\xi,1}D_{\xi,2}q(\xi)\right| < \left|(D_{\xi,r})^2 q(\xi)\right|$ and $(D_{\xi,r})^2 q(\xi) < 0$, $r = 1, 2$. This gives that $\det(H(\xi)) > 0$ and $Tr(H(\xi)) < 0$. To finish the proof, we have to show that $q(\xi) > -1$

$$\begin{aligned}q(\xi) &\ge \alpha_0 F_N(\xi\cdot\xi_1) - \|\alpha\|_\infty \sum_{\xi_m\in\Xi\setminus\xi_1}\left|F_N(\xi\cdot\xi_m)\right| \\ &\quad - \|\beta\|_\infty\sum_{\xi_m\in\Xi}\left|D_{\xi,1}F_N(\xi,\xi_m)\right| - \|\gamma\|_\infty\sum_{\xi_m\in\Xi}\left|D_{\xi,2}F_N(\xi,\xi_m)\right| \\ &\ge \left(1-\frac{c}{\nu^{k-1}}\right)\left(1+F_N(\xi\cdot\xi_1)-F_N(\xi\cdot\xi)\right) - \left(1+\frac{c_k}{\nu^{k-1}}\right)\frac{c_k}{\nu^{k-1}} - \frac{2c_k}{\nu^{2(k-1)}} \\ &\ge \left(1-\frac{c}{\nu^{k-1}}\right)(1-c_k\sigma) - \frac{2c_k}{\nu^{2(k-1)}}.\end{aligned}$$

Clearly, for large ν and small σ, $q(\xi) > -1$. For the case where $q(\xi_1) = -1$, the proof is almost identical except for the fact that we show that q is convex in the neighborhood of ξ_1 and $q(\xi) < 1$, for $d(\xi,\xi_1) < \sigma/N$.

4.3 Proof of Lemma 4.4

Let $\xi \in \mathbb{S}^2$ and $\xi_1 \in \Xi$, such that $\sigma/N \le d(\xi,\xi_1) \le \Delta/2$. We need to show that for sufficiently large ν, $|q(\xi)| < 1$. First observe that using only the first order estimate for $F_N(\xi\cdot\xi_1)$, with the normalization $F_N(\xi_1,\xi_1) = 1$

$$|\alpha_1|\left|F_N(\xi\cdot\xi_1)\right| \le \left(1+\frac{c_k}{\nu^{k-1}}\right)\frac{1}{1+\sigma}.$$

Consequently, for sufficiently large ν, using also the estimates of Lemmas 4.2, 4.5 and (3.12) gives

$$
\begin{aligned}
|q(\xi)| \leq{} & \|\alpha\|_\infty |F_N(\xi\cdot\xi_1)| + \|\beta\|_\infty |D_{\xi_1,1}F_N(\xi,\xi_1)| + \|\gamma\|_\infty |D_{\xi_1,2}F_N(\xi,\xi_1)| \\
& + \|\alpha\|_\infty \sum_{\xi_m\in\Xi\backslash\xi_1} |F_N(\xi\cdot\xi_m)| + \|\beta\|_\infty \sum_{\xi_m\in\Xi\backslash\xi_1} |D_{\xi_1,1}F_N(\xi,\xi_m)| \\
& + \|\gamma\|_\infty \sum_{\xi_m\in\Xi\backslash\xi_1} |D_{\xi_1,2}F_N(\xi,\xi_m)| \\
\leq{} & \left(1+\frac{c_k}{\nu^{k-1}}\right)\frac{1}{1+\sigma} + \frac{2c_k}{\nu^{k-1}}\frac{c_k}{(1+\sigma)^k} + \left(1+\frac{c_k}{\nu^{k-1}}\right)\frac{c_k}{\nu^{k-1}} + \frac{2c_k}{\nu^{2(k-1)}} \\
<{} & 1.
\end{aligned}
$$

The case where $d(\xi,\xi_m) > \Delta/2$, for each $\xi_m \in \Xi$ is easier. In this case, where ξ is well separated from all the points of Ξ, we can use estimates similar to the those of Lemma 4.5, to get

$$
\begin{aligned}
|q(\xi)| \leq{} & \|\alpha\|_\infty \sum_{\xi_m\in\Xi} |F_N(\xi\cdot\xi_m)| + \|\beta\|_\infty \sum_{\xi_m\in\Xi} |D_{\xi_m,1}F_N(\xi\cdot\xi_m)| \\
& + \|\gamma\|_\infty \sum_{\xi_m\in\Xi} |D_{\xi_m,2}F(\xi\cdot\xi_m)| \\
\leq{} & \left(1+\frac{c_k}{\nu^{k-1}}\right)\frac{c_k}{\nu^{k-1}} + \frac{c_k}{\nu^{2(k-1)}}.
\end{aligned}
$$

This completes the proof.

5 Non-Negative Signals

In this section, we show that for the special case of non-negative Dirac ensembles

$$
f = \sum_m c_m \delta_{\xi_m}\,,\ c_m > 0\,,\ \xi_m \in \Xi, . \tag{5.1}
$$

a sparsity condition is sufficient for exact recovery (compare with the discrete case [11]). We start by presenting a sufficient condition for the reconstruction of the signal from its projection onto V_N. Here we give a general version of the theorem as follows:

Theorem 5.1. *Let $f = \sum_m c_m\delta_{\xi_m}$, where $\Xi = \{\xi_m\} \subset A$, with A a compact manifold in $\mathbb{R}^d$ and $c_m > 0$. Let Π_D be a linear space of continuous functions of dimension D in A. For any basis $\{P_k\}_{k=1}^D$ of Π_D, let $y_k = \langle f, P_k\rangle$ for all $1 \leq k \leq D$. If there exists $q \in \Pi_D$ such that*

$$
\begin{aligned}
q(\xi_m) = 1 \quad & \xi_m \in \Xi, && (5.2)\\
|q(\xi)| < 1 \quad & \xi \notin \Xi, && (5.3)
\end{aligned}
$$

then, f is the unique minimizer over all non-negative measures of the following

$$
min_{g\in\mathcal{M}(\mathcal{A})}\|g\|_{TV} \quad s.t. \quad y_k = \langle g, P_k\rangle, k = 1,\dots,D. \tag{5.4}
$$

Proof. Let g be the solution of (5.4), and set $g = f + h, h \neq 0$. Let $h = h_\Xi + h_{\Xi^C}$ be the Lebesgue decomposition of h relative to $|f|$, so that h_Ξ is supported on Ξ.

Additionally, $h_\Xi = \sum d_m \delta_{\xi_m}$ for some real $\{d_m\}$. Also, since g is a non-negative measure, $f + h_\Xi$ is also non-negative, implying $c_m + d_m \geq 0$ for all $\xi_m \in \Xi$. Thus, $\|f + h_\Xi\|_{TV} = \sum_m (c_m + d_m)$.

We observe that

$$0 = \langle q, h \rangle = \langle q, h_\Xi \rangle + \langle q, h_{\Xi^C} \rangle = \sum_m d_m + \langle q, h_{\Xi^C} \rangle. \tag{5.5}$$

Plainly, if $h_{\Xi^C} = 0$, then $h_\Xi = 0$, and consequently $h = 0$. Else, if $h_{\Xi^C} \neq 0$, we obtain

$$\left| \sum_m d_m \right| = \left| \int q dh_{\Xi^C} \right| < \|h_{\Xi^C}\|_{TV}. \tag{5.6}$$

This leads to the following contradiction

$$\begin{aligned} \|f\|_{TV} &\geq \|f + h\|_{TV} = \|f + h_\Xi\|_{TV} + \|h_{\Xi^C}\|_{TV} \\ &> \sum_m (c_m + d_m) + \left| \sum_m d_m \right| \\ &= \|f\|_{TV} + \left| \sum_m d_m \right| + \sum_m d_m \geq \|f\|_{TV} \end{aligned} \tag{5.7}$$

Therefore, $f = g$. □

We now show that a polynomial $q \in V_N(\mathbb{S}^{d-1})$, $d \geq 2$, obeying (5.2) and (5.3) can be constructed with a sparsity condition replacing the separation condition. Assuming that $|\Xi| = s \leq N$, we construct the following polynomial

$$q(\xi) := 1 - 2^{-(s+1)} \prod_{m=1}^{s} (1 - \xi \cdot \xi_m). \tag{5.8}$$

As already noted function $G(\xi) = \xi \cdot \xi_0$, is a spherical harmonic and thus also $1 - G(\xi)$. The fact that a product of spherical harmonics of degrees N_1, N_2 is a spherical harmonic of degree $N_1 + N_2$ and the computation of the corresponding representation is known as Clebsch - Gordan. Plainly, as long as $s \leq N$, $q \in V_N$. Moreover, $q(\xi_m) = 1$, and $0 \leq q(\xi) < 1$ for any $\xi \notin \Xi$.

As a result of the above construction, we may apply Theorem 5.1 to obtain exact recovery for non-negative Dirac ensembles whenever the sparsity condition $|\Xi| \leq N$ holds.

Observe that the case of univariate non-negative Dirac trains and spaces of trigonometric polynomials is a special case of the above, with $d = 2$. Therefore, a sparsity condition can replace the separation condition of [8]. For $d = 2$, the construction of the interpolating polynomial over knots $\{t_m\} \subset [-\pi, \pi]$, takes the form

$$q(t) = 1 - 2^{-(s+1)} \prod_{j=1}^{s} (1 - \cos(t - t_m)), \qquad t \in [-\pi, \pi]. \tag{5.9}$$

Similarly, in [6] the authors showed that the separation condition is a sufficient condition for the reconstruction of signals of the form (1.3) from their projection onto the space of algebraic polynomials of degree N over $[-1, 1]$. If the signal is known to be non-negative, a sufficient condition for reconstruction is $|\Xi| \leq N/2$, by the construction of the following algebraic polynomial (see also [10])

$$q(\xi) = 1 - 4^{-(s+1)} \prod_{i=1}^{s} (\xi - \xi_m)^2. \tag{5.10}$$

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