Analysis of pulse dropout in harmonically mode-locked fiber lasers by use of the Lyapunov method

Moshe Horowitz

Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel, and Department of Computer Science and Electrical Engineering, University of Maryland, Baltimore County, 1000 Hilltop Circle, Baltimore, Maryland 21250

Curtis R. Menyuk

Department of Computer Science and Electrical Engineering, University of Maryland, Baltimore County, 1000 Hilltop Circle, Baltimore, Maryland 21250, and Laboratory for Telecommunications Sciences, U.S. Army Research Laboratory, Room 131, Building 601, 2800 Powder Mill Road, Adelphi, Maryland 20783-1197

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Using a stability analysis based on the Lyapunov method, we study pulse dropout in an actively mode-locked fiber laser. The analysis gives a limit on the maximum pulse duration and the minimum laser power that are needed for stable operation without pulse dropout. The stability of pulse trains was studied analytically and validated numerically for different pulse shapes.

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Actively mode-locked erbium-doped fiber lasers that are used in high-data-rate optical communication systems are harmonically mode locked for high repetition rates.[1,2] In Ref. 3 these lasers were studied with an innovative numerical technique based on modeling the propagation of a small number of individual pulses and a superpulse that represents the pulses that are not simulated individually. It was shown that the combination of the Kerr effect, the slow response time of the laser amplifier, and the mode locker can cause pulse dropout when the laser is insufficiently pumped. Our numerical approach allows us to model the propagation of several pulses with different energies and shapes inside the laser cavity, and therefore it can be used to study the dynamics of the laser and to accurately find the limits on the stable operating regime; however, this approach can be computationally time consuming, so it is useful to approximate explicitly the dependence of pulse dropout on the laser parameters as a starting point for a more-comprehensive numerical analysis. In this Letter we present a simplified stability analysis based on the Lyapunov method that allows us to determine the stability threshold and the minimum power required for stable operation with fair accuracy. The pulse is found to be stable only when its duration is shorter than a threshold value. Therefore, when the laser is insufficiently pumped, it generates a limited number of short and intense pulses with the same duration, while other pulses are dropped so that the average power is kept approximately constant.

Prior study of the stability of actively mode-locked fiber lasers or storage rings[4,5] has been largely based on using soliton perturbation theory for solving the Ginzburg–Landau equation, modified to include amplifier filtering and active mode locking (this equation is also refered to as the master equation of mode locking). This work, which is based on earlier pioneering work by Haus[6] and by others,[7] includes a number of simplifying assumptions. The most important of these assumptions is this: (1) Every pulse in the laser cavity is the same. This assumption, which is certainly false in general, makes it impossible to study the laser dynamics or to find the precise limits on the stable operating regime; however, it is a reasonable assumption when one is studying the stability of an established pulse train. Other assumptions are as follows: (2) The pulse change is small at any fixed point in the laser from one round trip to the next. (3) The bandwidth of the pulse is small compared with the bandwidth of the gain medium and (or) the optical filtering. (4) The time duration of the pulse is much shorter than the period of the mode locking. Additionally, to use soliton perturbation theory, one must assume that (5) the pulse shape remains nearly hyperbolic secant during its round trip through the laser.

The simplified theory presented here differs from prior study of the stability of actively mode-locked fiber lasers in three major respects. Instead of using soliton perturbation theory, we use the simpler yet more general Lyapunov method, which is widely used in many areas of science and engineering for study of the stability of nonlinear systems.[8,9] Using this approach, we do not need to assume that the pulse has a hyperbolic-secant profile as in assumption (5). We need only assume that the FWHM, τ, is a decreasing function of the pulse energy, W. From a practical standpoint, the dispersion management that is used in modern fiber lasers leads to pulse shapes that are not hyperbolic secant.[1,10] We have found numerically that, even in a laser with a uniform-dispersion map, the pulse shape is different from hyperbolic secant when the power is low or when the cavity length is relatively short. The second aspect in which our work differs from prior work is that we validated our results with

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full numerical simulations that include many pulses, as is appropriate in fiber lasers. Finally, we do not assume that the pulse energy is nearly constant inside the amplifier, so we can model the filtering behavior of the amplifier more accurately.

We start the analysis by calculating the transmissivity of the cavity for the pulse energy. Assuming that $A(T, t, z)$ is the slowly varying envelope of the pulse at the central frequency ω_c, where t is the time variable, T is a slow time variable on the scale of the cavity round-trip time T_R, and z is the location in the cavity, we can write the change in the pulse energy, $W = \int_{-\infty}^{\infty} |A(T, t, z)|^2 \mathrm{d}t$, in a single round trip as

$$\frac{\Delta W}{\Delta T} = \frac{1}{T_R}\left(\prod_{i=1}^{n} T_i - 1\right)W \equiv \frac{1}{T_R} F(W)W\,, \qquad (1)$$

where T_i is the transmissivity of the ith element in the cavity, $F(W)$ is the cavity loss in a round trip, and $\Delta T = T_R$ is the time change. Using the Lyapunov linearization method,[8,9] we find that the pulse is asymptotically stable at an equilibrium point $W = W_{\mathrm{eq}}$ if $-1 < \mathrm{d}F/\mathrm{d}W|_{W_{\mathrm{eq}}} < 0$. We note that the first condition, $-1 < \mathrm{d}F/\mathrm{d}W|_{W_{\mathrm{eq}}}$, is always obeyed if assumption (2) holds. Indeed, it is a mathematical way of specifying the required smallness of the change in one round trip. Hence we focus on the second condition.

To use the Lyapunov method we need to know only the loss contributions that depend on the energy. In a cavity that contains an amplifier, a mode locker, and fibers with both chromatic dispersion and Kerr nonlinearity, the energy loss that depends on the energy is due to the mode locker and the effective amplifier filtering. We assume that the cavity fibers determine the pulse shape $f(t/\tau)$, where τ is the FWHM at the mode locker and the amplifier. We also assume that the pulse shape does not change significantly during one pass through the mode locker and the amplifier. We note that we need to know the pulse shape only in the amplifier and the mode locker. The pulse shape can change significantly in other locations of the cavity, as occurs in dispersion-managed lasers.[1] Assuming that the amplitude transmission of the modelocker equals $T(t) = 1 - M[1 - \cos(\omega_m t)]$, where M is the modulation depth and ω_m is the modulation frequency, we find, using assumption (4), that the change in the pulse energy that is due to the mode locker ΔW_m equals $-W(M\omega_m{}^2\tau^2)\langle x^2\rangle$, where $\langle x^2\rangle = \int_{-\infty}^{\infty} |f(x)|^2 x^2 \mathrm{d}x / \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x$. Note that $\langle x^2\rangle$ depends only on the pulse shape and not on τ.

Using assumptions (1) and (3), and assuming that the pulse is generated at the frequency where the gain is maximum, i.e., $\omega_c = \omega_0$, where ω_0 is the resonant frequency of the amplifier, we can approximate the gain coefficient of the amplifier in the frequency domain as[6] $g(z) = g_0[1 - (\omega - \omega_c)^2/\omega_g{}^2]/[1 + W(z)/TP_s]$, where T is the time interval between adjacent pulses, g_0 is the small gain coefficient, ω_g is the amplifier bandwidth, P_s is the saturation power of the amplifier, and z is the position inside the amplifier. In practice, the bandwidth in fiber lasers is often limited by optical filters rather than by the gain medium, but we can take this point into account by appropriately choosing ω_g.[11] Assuming that the pulse amplitude $A(T, t, z) = A_g(z) f(t/\tau)$, we obtain

$$\frac{\mathrm{d}W(z)}{\mathrm{d}z} = \frac{2g_0}{1 + W(z)/(TP_s)}\left(1 - \frac{\langle\omega^2\rangle}{\omega_g{}^2\tau^2}\right)W(z)\,. \qquad (2)$$

Integrating Eq. (2) and calculating the energy loss ΔW_g that is due to the effective amplifier filtering, we obtain $\Delta W_g = (2G\langle\omega^2\rangle W)/(\omega_g{}^2\tau^2)$, where $\langle\omega^2\rangle = \int_{-\infty}^{\infty} |\mathrm{d}f(x)/\mathrm{d}x|^2 \mathrm{d}x / \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x$. The quantity $G = g_0 l_g/(1 + \ln T_0 + 2g_0 l_g)$ is the average gain coefficient, where T_0 is the energy transmissivity of the laser cavity that does not depend on the pulse and l_g is the amplifier length. In deriving the amplifier loss we assumed that the energy-constant loss, $1 - T_0$, is much higher than the loss that depends on the pulse duration.

The total loss $F(W)$ equals $F_\tau + T_0\overline{G}$, where $\overline{G}$ is the constant gain of the amplifier and $F_\tau(W) = (\Delta W_g + \Delta W_m)/W$ is the loss that is due to the amplifier and the mode locker, which we calculated above. The constant gain actually can vary on a slow time scale of milliseconds, owing to the slow response of the erbium-doped fiber amplifier, but this variation is too slow to affect the stability discussed in the Letter. Figure 1 shows the dependence of $F_\tau(W)$ on the pulse duration for a hyperbolic-secant pulse and a Gaussian pulse and the results obtained from our numerical analysis for a single pulse and from soliton perturbation theory.[4] The results were obtained for a laser with a length of $L = 200$ m, cavity transmissivity $T_0 = 0.1$, dispersion coefficient $D = -3$ ps/(nm km), nonlinear coefficient $\gamma_{\mathrm{nl}} = 2\ \mathrm{W}^{-1}\ \mathrm{km}^{-1}$, $f_m = \omega_m/2\pi = 10$ GHz, $M = 0.5$, and $\Delta\Lambda_g = 20$ nm, where $\Delta\Lambda_g$ is the FWHM of the gain coefficient. Close to the boundary between the stable and the unstable operating regimes, the pulse in this laser has a hyperbolic-secant profile, and we obtain good quantitative agreement between the numerical and the analytical models for a hyperbolic-secant pulse. On the other hand, there is a significant

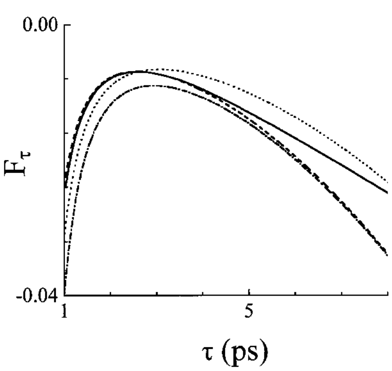

Fig. 1. Cavity loss depending on the pulse duration, F_τ, versus the FWHM of pulse duration, τ, obtained from the numerical model (solid curve), from soliton perturbation theory (dashed–dotted curve), and from Eq. (3) for a hyperbolic-secant pulse (dashed curve) and a Gaussian pulse (dotted curve).

discrepancy between the numerical model and the result obtained with soliton perturbation theory, since in fiber lasers the loss is high, and therefore the power and the effective amplifier bandwidth change significantly along the amplifier. Mathematically, the difference between the two analytical models is due to the term G in the amplifier loss, which becomes $-\ln(T_0)/2$ in conventional laser models.

Noting that $\tau = h(W)$ depends only on the energy and that $\mathrm{d}h/\mathrm{d}W < 0$ by assumption, we show that the Lyapunov stability criterion ($\mathrm{d}F/\mathrm{d}W < 0$) gives the maximum value of τ that is required for stable operation:

$$\tau < \tau_{\max} = \left(\frac{2G\langle\omega^2\rangle}{\langle x^2\rangle M\omega_m{}^2\omega_g{}^2}\right)^{1/4}. \quad (3)$$

For hyperbolic-secant pulses the coefficient $\langle\omega^2\rangle/\langle x^2\rangle = 4\ [2\ \mathrm{sech}^{-1}(2^{-1/2})]^4/\pi^2$. When the constant loss is small, $T_0 \approx 1$, and the result is equal to that found with soliton perturbation theory for solitons that propagate inside a soliton storage ring.[4] For a Gaussian pulse the coefficient $\langle\omega^2\rangle/\langle x^2\rangle = [4\ \ln(2)]^2$.

Using the soliton relation[12,13] $W = 4\ \mathrm{sech}^{-1}(2^{-1/2}) \times ED/\tau\gamma_{\mathrm{nl}}$, where D is the average dispersion and E is the energy-enhancement factor,[12,13] we can approximately calculate the energy of a single pulse. Multiplying the single-pulse energy by the number of cavity pulses, $N = \omega_m/2\pi$, we obtain the minimum average intracavity power needed for stable operation,

$$P_{\min} = \frac{\mathrm{sech}^{-1}(2^{-1/2})ED}{\pi\gamma_{\mathrm{nl}}}\,\omega_m{}^{3/2}\left(\frac{8M\omega_g{}^2\langle x^2\rangle}{G\langle\omega^2\rangle}\right)^{1/4}. \quad (4)$$

When the intracavity power is less than $P_{\min}$, the laser generates a limited number of short and intense pulses, each with a duration shorter than $\tau_{\max}$, while other pulses are dropped. We note that Eq. (4) predicts a very rapid increase in $P_{\min}$ when the repetition rate ω_m increases, owing to the increase of the pulse number (proportional to ω_m), the increase in the mode-locker effect (proportional to $\omega_m{}^{1/2}$), and the increase in the energy-enhancement factor, which strongly depends on $\tau_{\max}$. We also note that the minimum power calculated in this Letter is only a lower limit on the laser power, since practical lasers must be able to recover from a large deterioration in pulses owing to changes in environmental conditions and not only from small perturbations. This issue will be discussed elsewhere.[3]

We have compared the results of our analytical and numerical models for the laser that was analyzed in Fig. 1. Our reduced model for a hyperbolic-secant pulse yields the results that the maximum pulse duration $\tau_{\max} = 2.6$ ps and $P_{\min} = 25.7$ mW. These results are in quantitative agreement with those obtained from our full numerical simulation, $\tau_{\max} = 2.85$ ps and $P_{\min} = 22.7$ mW, where $\tau_{\max}$ is the duration of the pulses that remain in the cavity when the laser power is insufficient and $P_{\min}$ is the minimum average power needed to avoid pulse dropout. When the repetition rate $f_m = \omega_m/2\pi$ increases to 20 GHz, the reduced model yields a maximum pulse duration equal to 1.82 ps and a minimum power needed for stable operation that is larger by a factor of $2^{1.5} = 2.82$ relative to what is needed at 10 GHz. When the repetition rate $f_m = 30$ GHz, the power increases by a factor of $3^{1.5} = 5.2$, and $\tau_{max} = 1.48$ ps. These results are in good agreement with the results obtained from our full numerical model: The minimum power increases by a factor of 2.8 (5.3) and the maximum pulse duration decreases to 2.05 ps (1.65 ps) when the repetition rate increases to 20 GHz (30 GHz). When the repetition rate is increased to 100 GHz, our reduced model indicates that $P_{\min} = 800$ mW. In practical lasers it is both difficult and expensive to obtain such high power, and therefore one might put a narrow filter inside the cavity to increase the maximum pulse duration. We have also analyzed a laser with a dispersion map similar to that used in the study reported in Ref. 1. The average dispersion D was 0.1 ps/nm, the cavity length L was 190 m, and the dispersion-map strength factor γ (Refs. 12 and 13) was 4.8 for pulses with a 1.4-ps duration. Using Eq. (4) and the connection between E and γ,[12,13] we obtain for a Gaussian pulse $\tau_{\max} = 3.1$ ps and $P_{\min} = 1.25$ mW, in good agreement with the numerical results, $\tau_{\max} = 3.5$ ps and $P_{\min} = 1.0$ mW.

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