A Statistical–Mechanical View on Code Ensembles and Random Coding Exponents

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General Background

Relations between information theory and statistical physics:

- The maximum entropy principle: Jaynes, Shore & Johnson, Burg, ...
- Physics of information: Landauer, Bennet, Maroney, Plenio & Vitelli, ...
- Large deviations theory: Ellis, Oono, McAllester, ...
- Random matrix theory: Wigner, Balian, Foschini, Telatar, Tse, Hanly, Shamai, Verdú, Tulino, ...
- Coding and spin glasses: Sourlas, Kabashima, Saad, Kanter, Mézard, Montanari, Nishimori, Tanaka, ...

Physical insights and analysis tools are ‘imported’ to IT.

In This Talk We:

- Briefly review basic background in statistical physics.
- Describe relationships between coding and spin glasses.
- Relate performance measures in coding to physical quantities.
- Develop an analysis technique inspired by stat–mech.
- Discuss extensions of the basic models.

Background in Statistical Physics

Consider a system with $n >> 1$ particles which can lie in various microstates, $\{\boldsymbol{x} = (x_1, \ldots, x_n)\}$, e.g., a combination of locations, momenta, angular momenta, spins, ...

For every $\boldsymbol{x}$, $\exists$ energy $\mathcal{E}(\boldsymbol{x})$ – Hamiltonian.

Example: For $x_i = (\boldsymbol{p}_i, \boldsymbol{r}_i)$,

$$\mathcal{E}(\boldsymbol{x}) = \sum_{i=1}^{n} \left(\frac{\|\boldsymbol{p}_i\|^2}{2m} + mgh_i \right).$$

Basic Background (Cont'd)

In thermal equilibrium, $\boldsymbol{x} \sim$ Boltzmann–Gibbs distribution:

$$P(\boldsymbol{x}) = \frac{e^{-\beta \mathcal{E}(\boldsymbol{x})}}{Z(\beta)}$$

where $\beta = \frac{1}{kT}$, k – Boltzmann's constant, T – temperature, and

$$Z(\beta) = \sum_{\boldsymbol{x}} e^{-\beta \mathcal{E}(\boldsymbol{x})}, \quad \text{a normalization factor} = \text{partition function}$$

$\phi(\beta) = \ln Z(\beta) \quad \Rightarrow$ many physical quantities:

free energy: $F = -\frac{\phi}{\beta}$; mean internal energy: $E = -\frac{d\phi}{d\beta}$;

entropy: $S = \phi - \beta \frac{d\phi}{d\beta}$; heat capacity: $C = -\beta^2 \frac{d^2\phi}{d\beta^2}$; ...

From now on: $T \leftarrow kT \quad \Rightarrow \quad \beta = \frac{1}{T}$.

Bckgd Cont'd: Stat. Mech. of Magnetic Materials

Example: magnetic material – each particle has a magnetic moment (spin) – a 3D vector which tends to align with the

net magnetic field = external field + effective fields of other particles.

Quantum mechanics: each spin $\in$ discrete set of values, e.g., for spin $\frac{1}{2}$:

$$\text{spin up}: \quad x_i = +\frac{1}{2} \Rightarrow +1$$

$$\text{spin down}: \quad x_i = -\frac{1}{2} \Rightarrow -1$$

Background Cont'd: The Ising Model

$$\mathcal{E}(\boldsymbol{x}) = -H \cdot \sum_{i=1}^{n} x_i - J \cdot \sum_{<i,j>} x_i x_j$$

$J = 0$ – paramagnetic: no interactions $\Rightarrow$ spins are independent:

$$\begin{aligned} \text{magnetization} \quad &\stackrel{\triangle}{=} \quad m = \boldsymbol{E}\left\{\frac{1}{n}\sum_i X_i\right\} \\ &= \quad (+1) \cdot \frac{e^{\beta H}}{2\cosh(\beta H)} + (-1) \cdot \frac{e^{-\beta H}}{2\cosh(\beta H)} = \tanh(\beta H) \end{aligned}$$

$J > 0$ – ferromagnetic; $J < 0$ – antiferromagnetic.

The Ising Model (Cont'd)

Strong interaction $\Rightarrow$ two conflicting effects:

- 2nd law $\Rightarrow$ entropy $\uparrow$ $\Rightarrow$ disorder $\uparrow$
- Interaction energy $\downarrow$ $\Rightarrow$ order $\uparrow$.

Q: Who wins?

A: Depends on temperature:

$$Z = \sum_{\boldsymbol{x}} e^{-\beta \mathcal{E}(\boldsymbol{x})} = \sum_{E} N(E) e^{-\beta E} = \sum_{E} \exp\{S(E) - \beta E\}$$

- High temperature – disorder (paramagnetism).
- Low temperature – order: magnetization (sometimes spontaneous).

Abrupt passage $\Rightarrow$ phase transition.

Background Cont'd: Other Models

Interactions between remote pairs:

$$\mathcal{E}_I(\boldsymbol{x}) = -\sum_{i,j} J_{ij} x_i x_j$$

$\{J_{ij}\}$ with mixed signs $\Rightarrow$ spin glass.

Disorder: $\{J_{ij}\} =$ quenched random variables.

- Edwards–Anderson (EA): $J_{ij} \sim$ i.i.d. Gaussian; neighbors only.
- Sherrington–Kirkpatrick (SK): J_{ij} same, but all pairs.
- p–spin glass model: Like SK, but products of p spins.
- Random Energy model (REM): $p \to \infty \Rightarrow \{\mathcal{E}_I(\boldsymbol{x})\} =$ i.i.d. Gaussian.

Background Cont'd: The REM (Derrida, 1980,81)

Very simple, but rich enough for phase transitions.

$$Z(\beta) = \sum_{x=1}^{2^n} e^{-\beta \mathcal{E}_I(x)} = \int \mathrm{d}E \cdot N(E) e^{-\beta E} \quad \mathcal{E}_I(x) \sim \mathcal{N}(0, nJ^2/2)$$

$$\overline{N(E)} \approx 2^n \cdot \Pr\{\mathcal{E}_I(x) \approx E\} \doteq 2^n \cdot e^{-E^2/(nJ^2)} = \exp\{n[\ln 2 - (E/nJ)^2]\}.$$

- $\overline{N(E)}$ with a negative exponent $\iff |E| \geq E_0 \stackrel{\Delta}{=} nJ\sqrt{\ln 2} \Rightarrow N(E) \sim 0$.
- $|E| < E_0 \Rightarrow N(E)$ concentrates rapidly around $\overline{N(E)}$.

Typical realization:

$$Z(\beta) \approx \int_{-E_0}^{E_0} \mathrm{d}E \cdot \overline{N(E)} \cdot e^{-\beta E}.$$

The REM (Cont'd)

$$Z(\beta) \approx \int_{-E_0}^{E_0} \mathrm{d}E \cdot \exp\left\{n\left[\ln 2 - \left(\frac{E}{nJ}\right)^2\right]\right\} \cdot e^{-\beta E}$$

$$\doteq \exp\left\{n \cdot \max_{|E|\leq E_0}\left[\ln 2 - \left(\frac{E}{nJ}\right)^2 - \beta E\right]\right\} \stackrel{\triangle}{=} \exp\{n\phi(\beta)\}$$

$$\phi(\beta) = -\beta F(\beta) = \begin{cases} \ln 2 + \frac{\beta^2 J^2}{4} & \beta < \frac{2}{J}\sqrt{\ln 2} \\ \beta J\sqrt{\ln 2} & \beta \geq \frac{2}{J}\sqrt{\ln 2} \end{cases}$$

Phase transition at $\beta = \beta_0 \stackrel{\triangle}{=} \frac{2}{J}\sqrt{\ln 2}$:

- High temp. ($\beta < \beta_0$) – paramagnetic phase: entropy> 0; $Z(\beta)$ dominated by exponentially many $\boldsymbol{x}$'s at $E = -n\beta J^2/2$.
- Low temp. ($\beta \geq \beta_0$) – spin–glass phase: $\phi =$ linear, entropy $= 0$, frozen at ground–state $E = -E_0$ with sub–exponentially few dominant $\boldsymbol{x}$'s.

REM & Random Coding (Mézard & Montanari, 2008)

BSC(p) $+$ a random code $\mathcal{C} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{M-1}\}$, $M = e^{nR}$, (fair coin tossing).

Posterior:

$$P(\boldsymbol{x}|\boldsymbol{y}) = \frac{P(\boldsymbol{y}|\boldsymbol{x})}{\sum_{\boldsymbol{x}' \in \mathcal{C}} P(\boldsymbol{y}|\boldsymbol{x}')} = \frac{e^{-\ln[1/P(\boldsymbol{y}|\boldsymbol{x})]}}{\sum_{\boldsymbol{x}' \in \mathcal{C}} e^{-\ln[1/P(\boldsymbol{y}|\boldsymbol{x}')]}}.$$

Suggests a Boltzmann family:

$$P_\beta(\boldsymbol{x}|\boldsymbol{y}) = \frac{e^{-\beta \ln[1/P(\boldsymbol{y}|\boldsymbol{x})]}}{\sum_{\boldsymbol{x}' \in \mathcal{C}} e^{-\beta \ln[1/P(\boldsymbol{y}|\boldsymbol{x}')]}} = \frac{P^\beta(\boldsymbol{y}|\boldsymbol{x})}{\sum_{\boldsymbol{x}' \in \mathcal{C}} P^\beta(\boldsymbol{y}|\boldsymbol{x}')}.$$

REM & Random Coding (Cont'd)

Motivations:

- $\beta =$ degree of freedom for channel uncertainty.
- Annealing: find ground–state by 'cooling'.
- Finite–temperature decoding (Rujframework 1993):

$$\hat{x}_t = \text{argmax}_a P_\beta(x_t = a|\boldsymbol{y})$$

$\beta = 1 \Rightarrow$ minimum bit–error probability
$\beta = \infty \Rightarrow$ minimum block–error probability.

- $Z(\beta|\boldsymbol{y}) = \sum_{\boldsymbol{x}\in\mathcal{C}} P^\beta(\boldsymbol{y}|\boldsymbol{x})$ $\exists$ in bounds on P_e. Random $\mathcal{C} \Longleftrightarrow$ REM: Phase transitions 'inherited' from REM.

Statistical Physics of Code Ensembles

$\boldsymbol{x}_0 =$ correct codeword; $B = \ln \frac{1-p}{p}$:

$$\begin{aligned} Z(\beta|\boldsymbol{y}) &= (1-p)^{n\beta} \sum_{\boldsymbol{x}\in\mathcal{C}} e^{-\beta B d(\boldsymbol{x},\boldsymbol{y})} \\ &= (1-p)^{n\beta} e^{-\beta B d(\boldsymbol{x}_0,\boldsymbol{y})} + (1-p)^{n\beta} \sum_{\boldsymbol{x}\in\mathcal{C}\setminus\{\boldsymbol{x}_0\}} e^{-\beta B d(\boldsymbol{x},\boldsymbol{y})} \\ &\stackrel{\Delta}{=} Z_c(\beta|\boldsymbol{y}) + Z_e(\beta|\boldsymbol{y}). \end{aligned}$$

$d(\boldsymbol{x}_0, \boldsymbol{y}) \approx np \Rightarrow Z_c(\beta|\boldsymbol{y}) \approx (1-p)^{n\beta} e^{-\beta B np}$.

$$Z_e(\beta|\boldsymbol{y}) = (1-p)^{n\beta} \sum_{\delta=0}^{n} N_{\boldsymbol{y}}(n\delta) e^{-\beta B n\delta}$$

with $\overline{N_{\boldsymbol{y}}(n\delta)} \doteq e^{nR} \cdot e^{n[h(\delta)-\ln 2]}$.
$R + h(\delta) - \ln 2 < 0 \Rightarrow N_{\boldsymbol{y}}(n\delta) \sim 0$. Happens for $\delta < \delta_{GV}(R)$ and $\delta > 1 - \delta_{GV}(R)$, where $\delta_{GV}(R) =$ solution $\delta \leq 1/2$ of $R + h(\delta) - \ln 2 = 0$.

Stat. Phys. of Code Ensembles (Cont'd)

Similar to the REM:

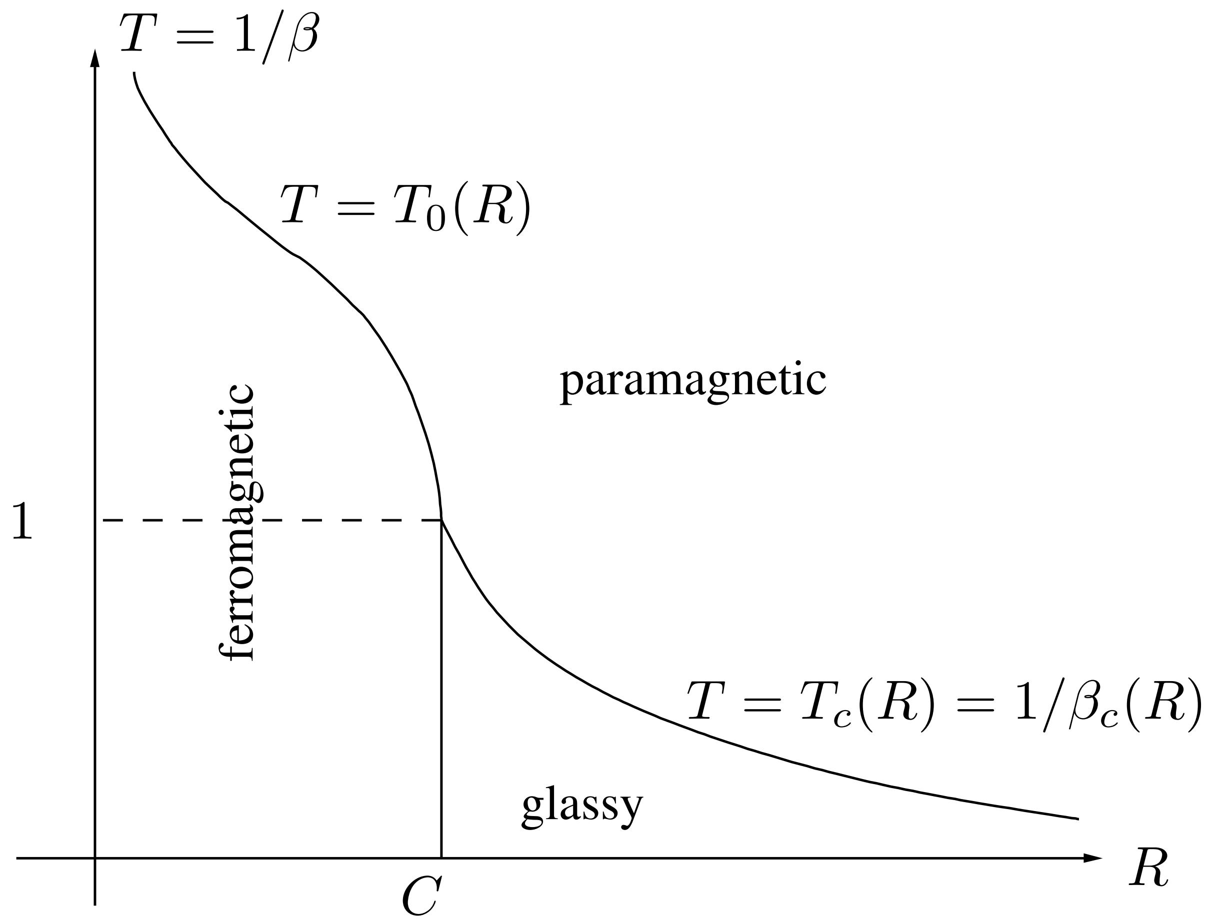
$$Z_e(\beta|\boldsymbol{y}) \doteq \exp\{n \max_{\delta\in[\delta_{GV}(R),1-\delta_{GV}(R)]}[R + h(\delta) - \ln 2 - \beta B\delta]\} \stackrel{\Delta}{=} e^{n\phi(\beta,R)}$$

$$\phi(\beta, R) = \begin{cases} R + \ln[p^\beta + (1-p)^\beta] - \ln 2 & \beta < \beta_c(R) \quad \text{paramagnetic} \\ \beta[\delta_{GV}(R)\ln p + (1-\delta_{GV}(R))\ln(1-p)] & \beta \geq \beta_c(R) \quad \text{spin–glass} \end{cases}$$

$$\beta_c(R) = \frac{\ln[(1-\delta_{GV}(R)]/\delta_{GV}(R)]}{\ln[(1-p)/p]}.$$

$Z_c(\beta|\boldsymbol{y}) \Rightarrow$ ordered phase = ferromagnetic phase.
Ferro–glassy boundary: $R = C$.
Ferro–para boundary: $T = T_0(R) = 1/\beta_0(R)$, solution to:

$$\beta h(p) = \ln 2 - R - \ln[p^\beta + (1-p)^\beta].$$



Phase diagram of finite–temperature decoding (Mezard & Montanari, 2008).

The Correct Decoding Exponent (M. 2007)

$$
\begin{aligned}
\overline{P_c} &= \boldsymbol{E}\left\{\frac{1}{M}\sum_{\boldsymbol{y}}\max_m P(\boldsymbol{y}|\boldsymbol{X}_m)\right\} \\
&= \boldsymbol{E}\left\{\frac{1}{M}\sum_{\boldsymbol{y}}\lim_{\beta\to\infty}\left[\sum_{m=0}^{M-1} P^{\beta}(\boldsymbol{y}|\boldsymbol{X}_m)\right]^{1/\beta}\right\} \\
&= \frac{1}{M}\sum_{\boldsymbol{y}}\lim_{\beta\to\infty}\boldsymbol{E}\left\{Z_e(\beta|\boldsymbol{y})^{1/\beta}\right\}
\end{aligned}
$$

$R > C$ and $\beta \to \infty \Rightarrow$ calculating $\boldsymbol{E}\{Z_e(\beta|\boldsymbol{y})^{1/\beta}\}$ in the spin-glass phase.

The Correct Decoding Exponent (Cont'd)

$$\begin{aligned}
\boldsymbol{E}\{Z_e(\beta|\boldsymbol{y})^{1/\beta}\} &= \boldsymbol{E}\left\{\left[(1-p)^{n\beta}\sum_{\delta} N_{\boldsymbol{y}}(n\delta)e^{-\beta Bn\delta}\right]^{1/\beta}\right\} \\
&\doteq (1-p)^n \boldsymbol{E}\left\{\sum_{\delta} N_{\boldsymbol{y}}^{1/\beta}(n\delta)e^{-Bn\delta}\right\} \\
&= (1-p)^n \sum_{\delta} \boldsymbol{E}\left\{N_{\boldsymbol{y}}^{1/\beta}(n\delta)\right\}\cdot e^{-Bn\delta}
\end{aligned}$$

$$\boldsymbol{E}\left\{N_{\boldsymbol{y}}^{1/\beta}(n\delta)\right\} = \begin{cases} \exp\{n[R+h(\delta)-\ln 2]\} & \delta \le \delta_{GV}(R) \text{ or } \delta \ge 1-\delta_{GV}(R) \\ \exp\{n[R+h(\delta)-\ln 2]/\beta\} & \delta_{GV}(R) < \delta < 1-\delta_{GV}(R) \end{cases}$$

The Correct Decoding Exponent (Cont'd)

Intuition: Below $\delta_{GV}(R)$

$$\begin{aligned} \boldsymbol{E}\left\{N_{\boldsymbol{y}}^{1/\beta}(n\delta)\right\} &\approx 0^{1/\beta}\cdot \Pr\{N_{\boldsymbol{y}}(n\delta)=0\}+1^{1/\beta}\cdot \Pr\{N_{\boldsymbol{y}}(n\delta)=1\} \\ &\doteq \exp\{n[R+h(\delta)-\ln 2]\} \end{aligned}$$

Above $\delta_{GV}(R) \Rightarrow$ double–exponentially fast concentration:

$$\boldsymbol{E}\left\{N_{\boldsymbol{y}}^{1/\beta}(n\delta)\right\}\approx\left[\boldsymbol{E}\{N_{\boldsymbol{y}}(n\delta)\}\right]^{1/\beta}\approx\left(e^{n[R+h(\delta)-\ln 2]}\right)^{1/\beta}$$

Putting into $\boldsymbol{E}\{Z_e^{1/\beta}(\beta|\boldsymbol{y})\}$ & taking the dominant δ:

$$\overline{P_c}\doteq\exp\{-n[R-\ln 2-F_g]\}$$

$F_g =$ free energy of glassy phase:

$$F_g=\delta_{GV}(R)\ln\frac{1}{p}+(1-\delta_{GV}(R))\ln\frac{1}{1-p}.$$

Correct Decoding Exponent (Cont'd)

Alternative expression: use $\ln 2 - R \equiv h(\delta_{GV}(R))$:

$$\begin{aligned} \overline{P_c} &\doteq \exp\{-nD(\delta_{GV}(R)\|p)\} \\ &= \text{Pr}\{\boldsymbol{x}_0 \text{ at distance } < \delta_{GV}(R)\} \end{aligned}$$

$\delta_{GV}(R) =$ typical distance of wrong codewords dominating the spin–glass phase.

Main ideas of the analysis technique:

- Summations over exponentially many codewords $\Rightarrow$ summations over polynomially few terms of distance enumerators, $\{N_{\boldsymbol{y}}(\cdot)\}$.
- Power of $\sum \doteq \sum$ of powers.
- Moments of $\{N_{\boldsymbol{y}}(n\delta)\}$: treated differently depending on whether or not $\delta \in [\delta_{GV}(R), 1 - \delta_{GV}(R)]$.

The Random Coding Error Exponent

Gallager's bound:

$$P_{e|m=0} \leq \sum_{\boldsymbol{y}} P(\boldsymbol{y}|\boldsymbol{x}_0)^{1/(1+\rho)} \left[\sum_{m\geq 1} P(\boldsymbol{y}|\boldsymbol{x}_m)^{1/(1+\rho)}\right]^{\rho}$$

$$= \sum_{\boldsymbol{y}} P(\boldsymbol{y}|\boldsymbol{x}_0)^{1/(1+\rho)} \cdot Z_e^{\rho}\left(\frac{1}{1+\rho}|\boldsymbol{y}\right)$$

Jensen $\Rightarrow$ $\boldsymbol{E}\{Z_e^{\rho}(1/(1+\rho)|\boldsymbol{y})\} \leq [\boldsymbol{E}Z_e(1/(1+\rho)|\boldsymbol{y})]^{\rho}$. Calculation in paramagnetic regime $\Rightarrow$ $E_r(R)$ is related to paramagnetic F:

$$\bar{P}_e \leq \exp\left\{-n\left[\frac{\rho}{1+\rho}F_p\left(\frac{1}{1+\rho}\right) - \ln(p^{1/(1+\rho)} + (1-p)^{1/(1+\rho)})\right]\right\}.$$

Another Application: Decoding with Erasures

Decoder with an option not to decide (erasure): Decision rule $=$ partition into $(M+1)$ regions:

$$\boldsymbol{y} \in \mathcal{R}_0 \quad \text{erase}$$

$$\boldsymbol{y} \in \mathcal{R}_m \quad (m \geq 1) \quad \text{decide } \boldsymbol{x}_m.$$

Performance – tradeoff between

$$\text{Pr}\{\mathcal{E}_1\} = \frac{1}{M} \sum_m \sum_{\boldsymbol{y} \in \mathcal{R}_m^c} P(\boldsymbol{y}|\boldsymbol{x}_m) \quad \text{erasure} + \text{undetected error}$$

$$\text{Pr}\{\mathcal{E}_2\} = \frac{1}{M} \sum_m \sum_{\boldsymbol{y} \in \mathcal{R}_m} \sum_{m' \neq m} P(\boldsymbol{y}|\boldsymbol{x}_{m'}) \quad \text{undetected error}$$

Decoding with Erasures (Cont'd)

Optimum decoder: decide message m iff

$$\frac{P(\boldsymbol{y}|\boldsymbol{x}_m)}{\sum_{m' \neq m} P(\boldsymbol{y}|\boldsymbol{x}_{m'})} \geq e^{nT} \quad (T \geq 0 \text{ for the erasure case}).$$

Erasure: if this holds for no m.

Forney's lower bounds on err. exponents of $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$E_1(R,T) = \max_{0 \leq s \leq \rho \leq 1} [E_0(s,\rho) - \rho R - sT] \quad \text{where}$$

$$E_0(s,\rho) = -\ln \left[\sum_y \left(\sum_x P(x) P^{1-s}(y|x) \right) \cdot \left(\sum_{x'} P(x') P^{s/\rho}(y|x') \right)^{\rho} \right],$$

$$E_2(R,T) = E_1(R,T) + T.$$

and $P(x) =$ random coding distribution.

Decoding with Erasures (Cont'd)

Main step in [Forney68]:

$$\boldsymbol{E}\left\{\left(\sum_{m'\neq m} P(\boldsymbol{y}|\boldsymbol{X}_{m'})\right)^{s}\right\} \quad \text{upper bounded by}$$

$$\boldsymbol{E}\left\{\left(\sum_{m'\neq m} P(\boldsymbol{y}|\boldsymbol{X}_{m'})^{s/\rho}\right)^{\rho}\right\}, \quad \rho \geq s,$$

and then Jensen.

Our technique: 1st expression exponentially tight, no need for ρ.

- A simpler bound (under some symmetry condition), at least as tight.
- Sometimes (e.g., BSC): optimum s in closed form.

Also: exact exponent (complicated) – joint work with A. Somekh–Baruch.

Two Extensions

- The REM in a uniform magnetic field and joint source–channel coding.
- The generalized REM (GREM) and hierarchical coding structures.

Back to Physics: REM in a Magnetic Field (Derrida)

Earlier we modeled only interaction energies, $\{\mathcal{E}_I(\boldsymbol{x})\}$ as $\mathcal{N}(0, nJ^2/2)$.

When an external magnetic field H is applied

$$\mathcal{E}(\boldsymbol{x}) = \mathcal{E}_I(\boldsymbol{x}) - H \cdot \sum_{i=1}^{n} x_i = \mathcal{E}_I(\boldsymbol{x}) - n \cdot m(\boldsymbol{x})H$$

where $m(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ = magnetization of $\boldsymbol{x}$.

$$\begin{aligned} Z(\beta, H) &= \sum_{\boldsymbol{x}} e^{-\beta[\mathcal{E}_I(\boldsymbol{x}) - nm(\boldsymbol{x})H]} \\ &= \sum_{m} \left[\sum_{\boldsymbol{x}:\ m(\boldsymbol{x})=m} e^{-\beta\mathcal{E}_I(\boldsymbol{x})} \right] \cdot e^{n\beta mH} \\ &\stackrel{\Delta}{=} \sum_{m} \zeta(\beta, m) e^{n\beta mH} \end{aligned}$$

The REM in a Magnetic Field (Cont'd)

$\zeta(\beta, m) = \sum_{\boldsymbol{x}:\ m(\boldsymbol{x})=m} e^{-\beta\mathcal{E}_I(\boldsymbol{x})}$: similar to REM with $H = 0$ with only $\exp[nh((1+m)/2)]$ terms.

Using the same technique, we compute $\zeta(\beta, m) \doteq e^{n\psi(\beta,m)}$ and

$$\phi(\beta, H) = \max_m [\psi(\beta, m) + \beta m H],$$

where $m^* = m(\beta, H) =$ mean (typical) magnetization.

The REM in a Magnetic Field (Cont'd)

Results: Let $\beta_c(H)$ solve the equation

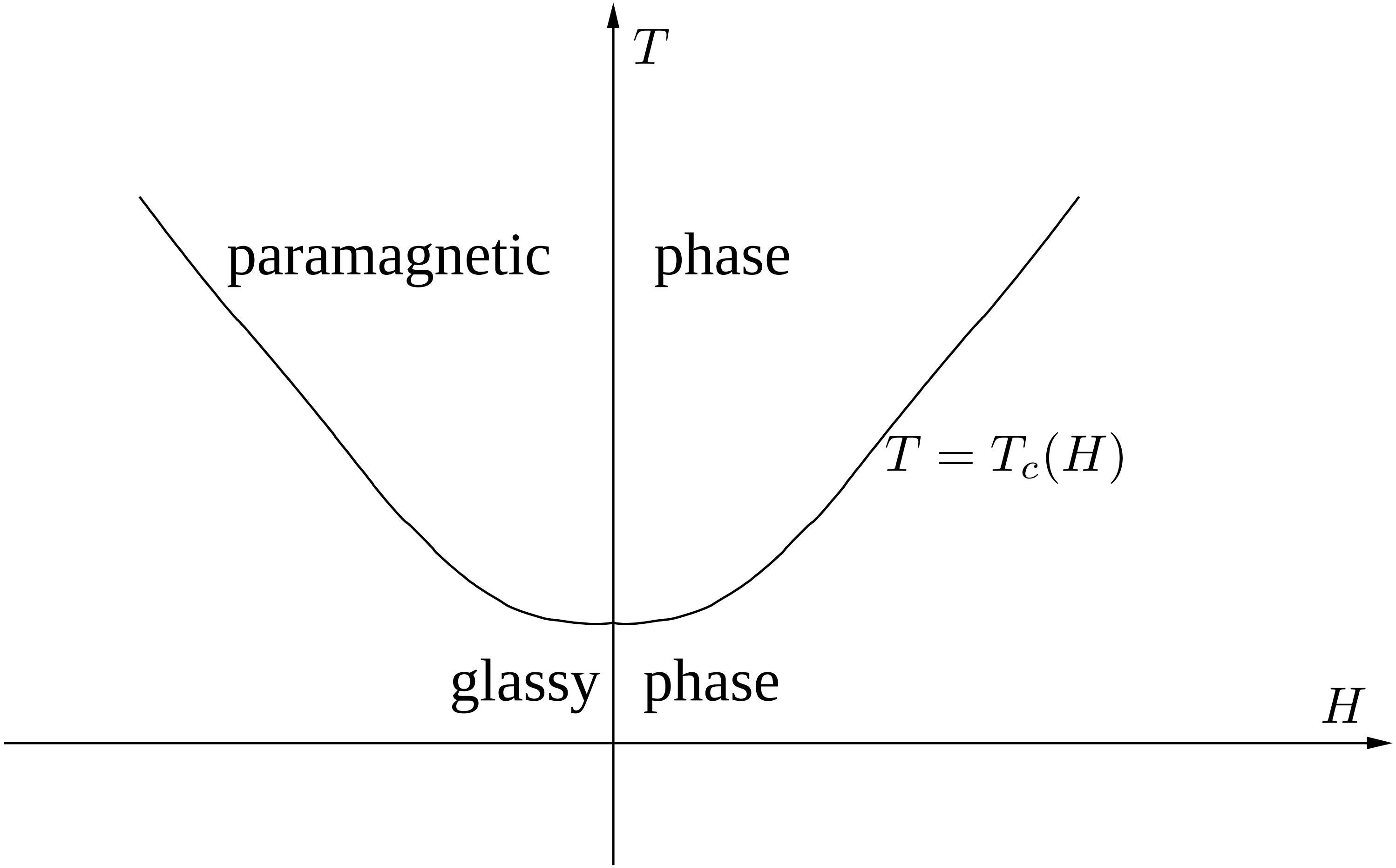
$$\beta^2 J^2 = 4h\left(\frac{1+\tanh(\beta H)}{2}\right).$$

Phase transition at $\beta = \beta_c(H)$:

$$m(\beta, H) = \begin{cases} \tanh(\beta H) & \beta < \beta_c(H) \quad \text{paramagentic phase} \\ \tanh(\beta_c(H) \cdot H) & \beta \geq \beta_c(H) \quad \text{spin glass phase} \end{cases}$$

Free energy: $F = -\phi/\beta$, where:

$$\phi(\beta, H) = \begin{cases} \frac{\beta^2 J^2}{4} + h\left(\frac{1+\tanh(\beta H)}{2}\right) + \beta H \tanh(\beta H) & \beta < \beta_c(H) \\ \beta\left[J\sqrt{h\left(\frac{1+\tanh(\beta_c(H)H)}{2}\right)} + H \cdot \tanh(\beta_c(H)H)\right] & \beta \geq \beta_c(H) \end{cases}$$



T
paramagnetic phase
$T = T_c(H)$
glassy phase
H

REM in a Magnetic Field & JSC Coding

- Binary source: $U_1, U_2, \ldots, U_i \in \{-1, +1\}$, $q = \Pr\{U_i = 1\}$.
- source-rate/channel-rate$= \theta$.
- JSC code: $\boldsymbol{u} = (u_1, \ldots, u_{n\theta}) \Rightarrow \boldsymbol{x}(\boldsymbol{u})$ of length n.
- Random coding: Draw $2^{n\theta}$ binary n-vectors $\{\boldsymbol{x}(\boldsymbol{u})\}$ by fair coin tossing.

Finite–temperature decoder:

$$\hat{u}_i = \text{argmax}_{u \in \{-1,+1\}} \sum_{\boldsymbol{u}:\ u_i = u} [P(\boldsymbol{u})P(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{u}))]^\beta, \quad i = 1, 2, \ldots, n\theta.$$

$$\begin{aligned} Z &= \sum_{\boldsymbol{u}} [P(\boldsymbol{u})P(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{u}))]^\beta \\ &= [P(\boldsymbol{u}_0)P(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{u}_0))]^\beta + \sum_{\boldsymbol{u} \neq \boldsymbol{u}_0} [P(\boldsymbol{u})P(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{u}))]^\beta \\ &\stackrel{\Delta}{=} Z_c + Z_e \end{aligned} \tag{1}$$

REM in a Magnetic Field & JSC Coding (Cont'd)

$P(\boldsymbol{u}) = [q(1-q)]^{n\theta/2} e^{n\theta m(\boldsymbol{u})H}$ where $H = \frac{1}{2}\ln\frac{q}{1-q}$. Thus,

$$
\begin{aligned}
Z_e &= [q(1-q)]^{n\beta\theta/2} \sum_m \left[\sum_{\boldsymbol{x}(\boldsymbol{u}):\ m(\boldsymbol{u})=m} e^{-\beta \ln[1/P(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{u}))]} \right] e^{n\beta m H} \\
&= [q(1-q)]^{n\beta\theta/2} (1-p)^{n\beta} \sum_m \left[\sum_{\boldsymbol{x}(\boldsymbol{u}):\ m(\boldsymbol{u})=m} e^{-\beta B d(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{y})} \right] e^{n\beta\theta m H} \\
&\stackrel{\Delta}{=} [q(1-q)]^{n\beta\theta/2} (1-p)^{n\beta} \sum_m \zeta(\beta, m) e^{n\beta\theta m H}
\end{aligned}
$$

Statistical physics of $Z_e \sim$ REM in a magnetic field. Similar analysis $\Rightarrow$:

Let $\beta_{pg}(H)$ solve:

$$
\ln 2 - h(p_\beta) = \theta h\left(\frac{1+\tanh(\beta H)}{2}\right), \quad p_\beta \stackrel{\Delta}{=} \frac{p^\beta}{p^\beta + (1-p)^\beta}.
$$

REM in a Magnetic Field & JSC Coding (Cont'd)

Magnetization of Z_e (incorrectly decoded patterns):

$$m(\beta, H) = \begin{cases} \tanh(\beta H) & \beta < \beta_{pg}(H) \\ \tanh(\beta_{pg}(H) \cdot H) & \beta \geq \beta_{pg}(H) \end{cases}$$

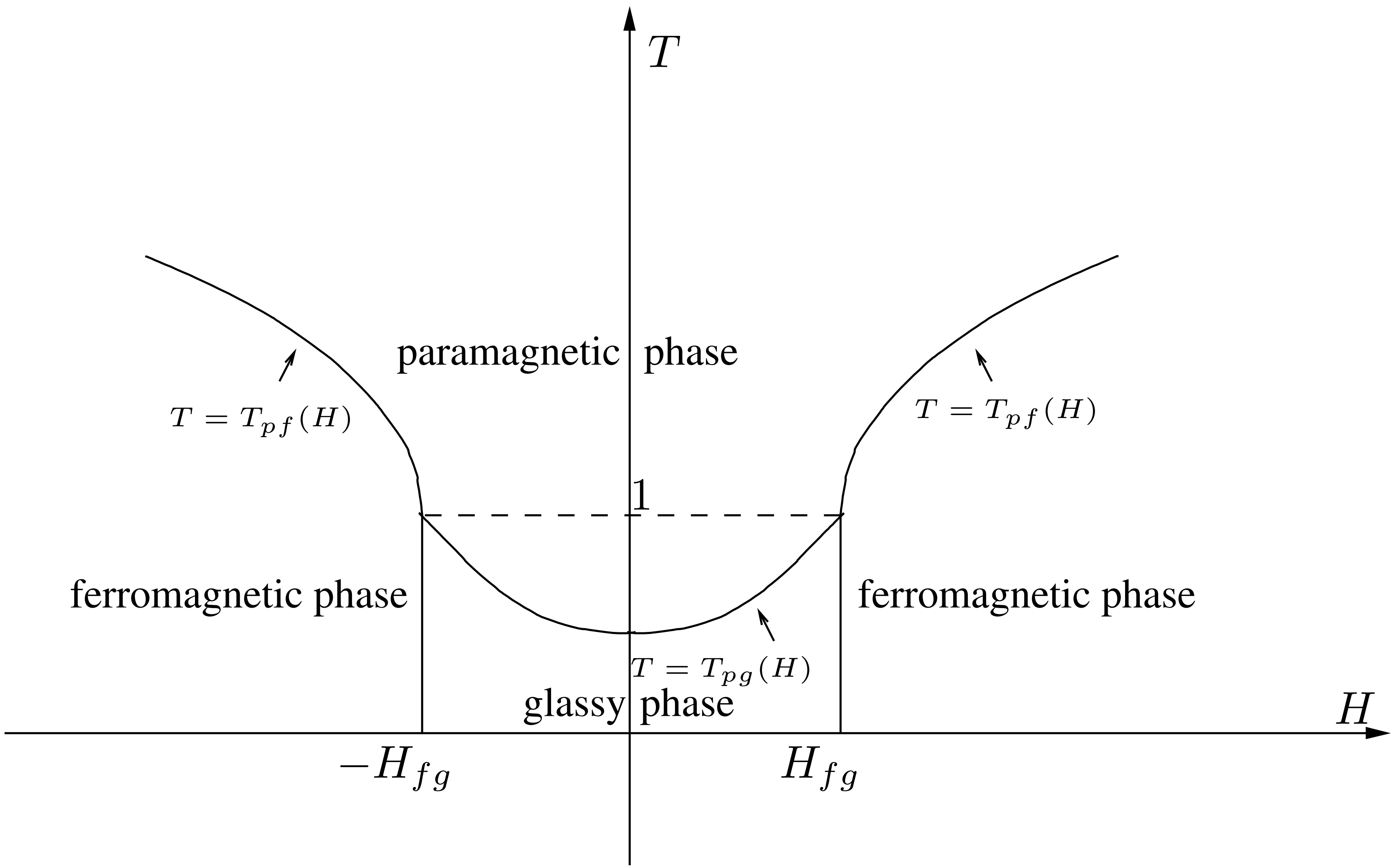
$Z_c \Rightarrow$ 3rd phase.
The ferro–glassy boundary is $H = H_{fg}$ where

$$H_{fg} = \frac{1}{2} \ln \frac{q^*}{1 - q^*} \quad \theta h(q^*) = \ln 2 - h(p).$$

REM in a Magnetic Field & JSC Coding (Cont'd)

Discussion

- Correct decoding for large $|H|$.
- Low temp.: (sub–exponentially few) typical patterns of erroneneously decoded $\{\boldsymbol{u}\}$ have m dictated by the frozen phase, i.e., $m_g(H) = \tanh(\beta_{pg}(H) \cdot H)$ independently of temp.
- For $|H| < H_{fg}$, $\beta_{pg}(H) > 1$, means that m of a typical erroneously decoded $\boldsymbol{u}$ is $> m$ of a typical (correct) $\boldsymbol{u}$, $m_f = \tanh(H)$.
  - If $T < T_{pg}(0)$, remains true no matter small $|H|$ is.
  - If $T_{pg}(0) < T < 1$, then when $|H| \downarrow$ the m of (exponentially many) erroneously decoded $\{\boldsymbol{u}\}$ is $m_p(\beta, H) = \tanh(\beta H)$: still $> m$ of typical $\boldsymbol{u}$, but now temperature–dependent.
  - Analysis of P_e and P_c – similar as before.



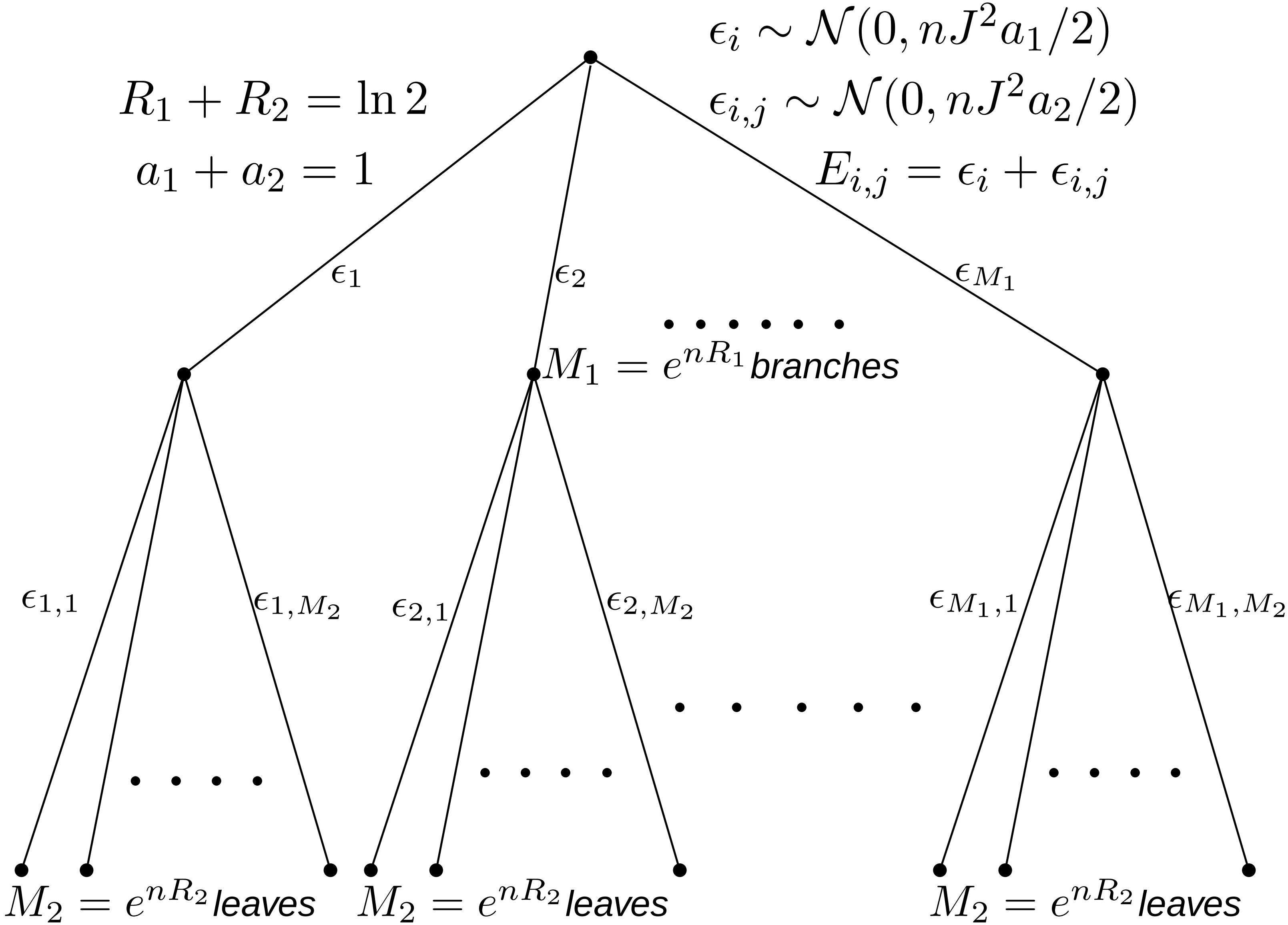
T
paramagnetic phase
$T = T_{pf}(H)$
$T = T_{pf}(H)$
1
ferromagnetic phase
ferromagnetic phase
$T = T_{pg}(H)$
glassy phase
H
$-H_{fg}$
H_{fg}

GREM (Derrida, ‘85) and Hierarchical Code Ensembles

Allowing correlations between $\{\mathcal{E}_I(\boldsymbol{x})\}$ in an hierarchical (tree) structure.

Features:

- More realistic model of dependencies.
- Still (relatively) easy analysis.
- May have > 1 phase transition.
- Analogies with code ensembles with a tree structure.


$R_1 + R_2 = \ln 2$
$a_1 + a_2 = 1$
$\epsilon_i \sim \mathcal{N}(0, nJ^2 a_1/2)$
$\epsilon_{i,j} \sim \mathcal{N}(0, nJ^2 a_2/2)$
$E_{i,j} = \epsilon_i + \epsilon_{i,j}$
ϵ_1
ϵ_2
ϵ_{M_1}
$M_1 = e^{nR_1}$ branches
$\epsilon_{1,1}$
ϵ_{1,M_2}
$\epsilon_{2,1}$
ϵ_{2,M_2}
$\epsilon_{M_1,1}$
ϵ_{M_1,M_2}
$M_2 = e^{nR_2}$ leaves
$M_2 = e^{nR_2}$ leaves
$M_2 = e^{nR_2}$ leaves

Sketch of Analysis for GREM

As before,

$$Z(\beta) = \sum_{\boldsymbol{x}} e^{-\beta\mathcal{E}_I(x)} \approx \int \mathrm{d}E \cdot N(E) e^{-\beta E}$$

Estimating $N(E) \doteq e^{nS(E)}$ for a typical realization: $\forall \boldsymbol{x}$ with energy E: 1st branch – ϵ, 2nd branch: $E - \epsilon$.

$$N_1(\epsilon) \doteq e^{nR_1} \cdot \exp\left\{-\frac{\epsilon^2}{nJ^2 a_1}\right\} = \exp\left\{n\left[R_1 - \frac{1}{a_1}\left(\frac{\epsilon}{nJ}\right)^2\right]\right\},$$

“alive” for $|\epsilon| \leq \epsilon_0 \stackrel{\Delta}{=} nJ\sqrt{a_1 R_1}$. Thus,

$$N(E) \doteq \int_{-\epsilon_0}^{+\epsilon_0} \mathrm{d}\epsilon \cdot N_1(\epsilon) \cdot \exp\left\{n\left[R_2 - \frac{1}{a_2}\left(\frac{E-\epsilon}{nJ}\right)^2\right]\right\}.$$

Sketch of Analysis for the GREM (Cont'd)

$$S(E) = \lim_{n\to\infty} \frac{\ln N(E)}{n} = \max_{|\epsilon|\le\epsilon_0}\left[R_1 - \frac{1}{a_1}\left(\frac{\epsilon}{nJ}\right)^2 + R_2 - \frac{1}{a_2}\left(\frac{E-\epsilon}{nJ}\right)^2\right]$$

$$\phi(\beta) = \lim_{n\to\infty}\frac{1}{n}\ln\left[\int \mathrm{d}E \cdot e^{nS(E)} \cdot e^{-\beta E}\right] = \max_E [S(E) - \beta E].$$

Two cases:

If $R_1/a_1 > R_2/a_2 \Rightarrow$ behavior exactly like in the REM.

Otherwise: two phase transitions at $\beta_i = \frac{2}{J}\sqrt{\frac{R_i}{a_i}}$, $i = 1, 2$:

$$\phi(\beta) = \begin{cases} \ln 2 + \frac{\beta^2 J^2}{4} & \beta < \beta_1 \ \text{pure paramagnetic} \\ \beta J\sqrt{a_1 R_1} + R_2 + \frac{a_2\beta^2 J^2}{4} & \beta_1 < \beta \le \beta_2 \ \text{glassy-paramagnetic} \\ \beta J(\sqrt{a_1 R_1} + \sqrt{a_2 R_2}) & \beta > \beta_2 \ \text{pure glassy} \end{cases}$$

GREM and Hierarchical Lossy Source Coding

- BSS $X_1, X_2, \ldots$, $X_i \in \{0,1\}$ and Hamming distortion measure.
- Performance measure $\boldsymbol{E}\{\exp\{-s\text{distortion}\}\}$ – related to Z.
- Tree structured code:
  - $n = n_1 + n_2$ and $nR = n_1 R_1 + n_2 R_2$.
  - 1st–stage code: $M_1 = e^{n_1 R_1}$ n_1–vectors $\{\hat{\boldsymbol{x}}_i\}$.
  - 2nd–stage code: For each i, $M_2 = e^{n_2 R_2}$ n_2–vectors $\{\tilde{\boldsymbol{x}}_{i,j}\}$.
  - Encode $\boldsymbol{x} = (\boldsymbol{x}', \boldsymbol{x}'')$ by $\min\{d(\boldsymbol{x}', \hat{\boldsymbol{x}}_i) + d(\boldsymbol{x}'', \tilde{\boldsymbol{x}}_{i,j})\}$.
  - Decode 1st n_1 symbols using 1st $n_1 R_1$ compressed bits.
  - Overall distortion $\Longleftrightarrow$ overall energy in GREM.
- Hierarchical ensemble:
  - Draw M_1 n_1–vectors $\{\hat{\boldsymbol{x}}_i\}$ by fair coin tossing.
  - For each i, draw M_2 n_2–vectors $\{\tilde{\boldsymbol{x}}_{i,j}\}$ by fair coin tossing.

Results

Evaluate $\boldsymbol{E}\{\exp\{-s\text{distortion}\}\}$, using

$$Z(\beta|\boldsymbol{x}) = \sum_{\boldsymbol{y}\in\mathcal{C}} e^{-\beta d(\boldsymbol{x},\boldsymbol{y})}$$

and then $\lim_{\theta\to\infty} \boldsymbol{E}\{Z^{1/\theta}(s\theta|\boldsymbol{x})\}$.
$\Rightarrow$ calculation in the glassy regime.
For $R_1 \geq R_2$,

- $\phi(\beta) = \lim_n \frac{\ln Z}{n}$ is like in the REM:

$$\phi(\beta) = \begin{cases} R - \ln 2 - \beta + \ln(1+e^{\beta}) & \beta < \beta(R) \\ -\beta\delta_{GV}(R) & \beta \geq \beta(R) \end{cases}$$

  where $\beta(R) = \ln[(1-\delta(R))/\delta(R)]$.

- $\boldsymbol{E}\{\exp\{-s\text{distortion}\}\}$ like in an optimum code for $s \in (0, s_0)$ with $s_0 = \infty$ when $R_1 = R_2$.

Results (Cont'd)

For $R_1 < R_2$,

- Two phase transitions: Defining $\lambda = \lim_n n_1/n$ and $v(\beta, R) = \ln 2 - R + \beta - \ln(1 + e^{\beta})$:

$$\phi(\beta) = \begin{cases} -v(\beta, R) & \beta < \beta(R_1) \\ -\beta\lambda\delta_{GV}(R_1) - (1-\lambda)v(\beta, R_2) & \beta(R_1) \leq \beta < \beta(R_2) \\ -\beta[\lambda\delta_{GV}(R_1) + (1-\lambda)\delta_{GV}(R_2)] & \beta \geq \beta(R_2) \end{cases}$$

- $\boldsymbol{E}\{\exp\{-s\text{distortion}\}\}$ behaves like in two decoupled codes in the two segments.

Conclusion

- Analogies between certain mathematical models in stat. mech. and IT.
- Inspiring alternative analysis techniques of code performance (error exponents).
- Applied to error– and correct decoding exponets in channel coding, joint source channel coding, and decoding with erasures.
- Potentially applicable to other situations, e.g., the IFC (joint work with Ordentlich and Etkin).