Relations Between Work Extraction and Entropy Production of Information Driven FSMs

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Information Thermodynamics – Some Background

Common theme: the presence of information “violates” the second law.

Systems with measurement and feedback control:

Systems controlled by parameters that depend on past (noisy) measurements.

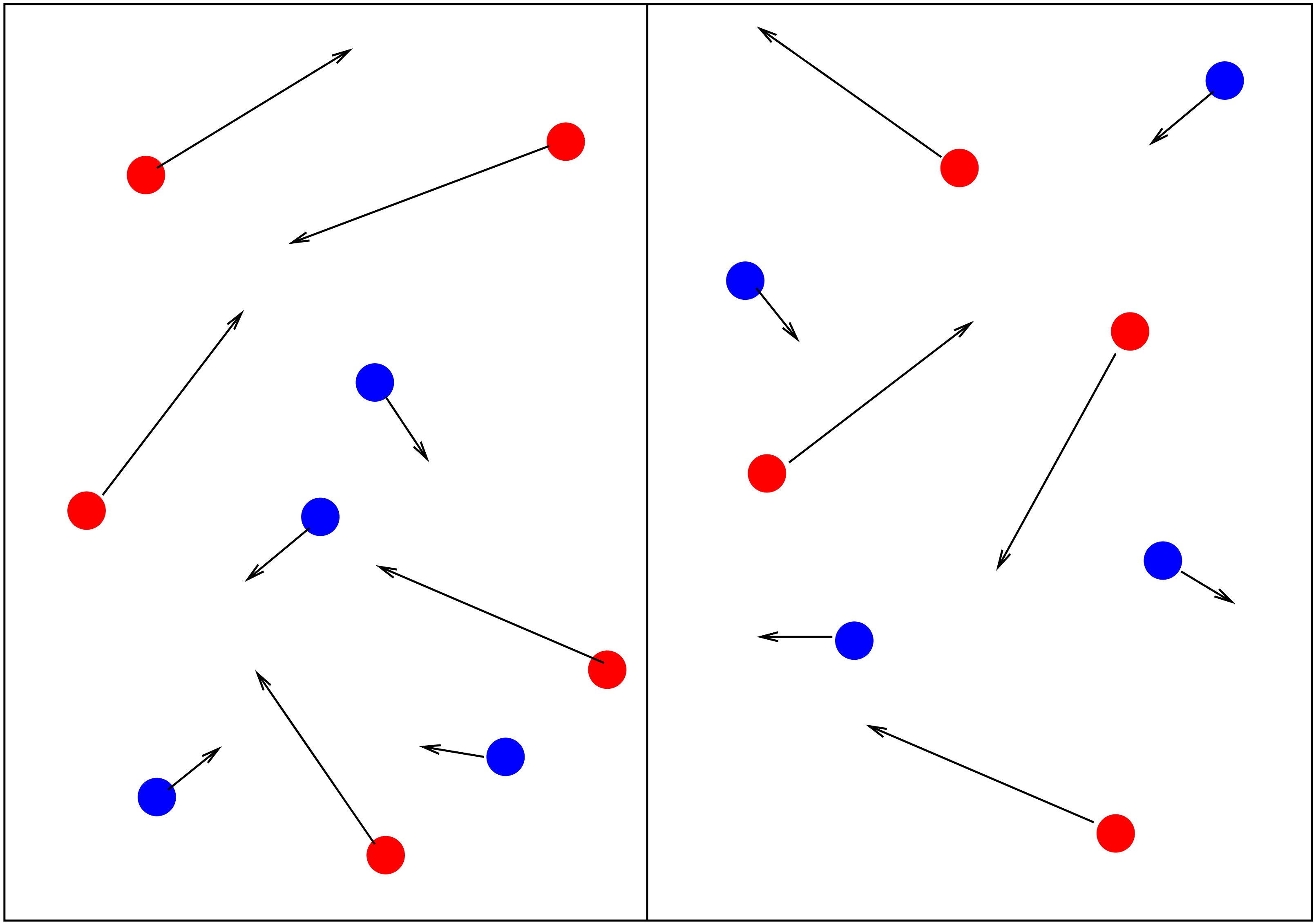
Barato & Seifert (2013, 2014); Deffner (2013); Deffner & Jarzynski (2013); Hoppenau & Angel (2014); Sagawa & Ueda (2011, 2012, 2013); Parrondo, Horowitz & Sagawa (2015); ...

Systems with information reservoirs:

Systems that interact with an informational device (memory, digital tape, etc.).

Barato & Seifert (2014); Mandal & Jarzynski (2012, 2014); Boyd, Mandal & Crutchfield (2015); Mandal, Quan & Jarzynski (2013); Horowitz & Esposito (2014); ...

The Maxwell Demon (1867)



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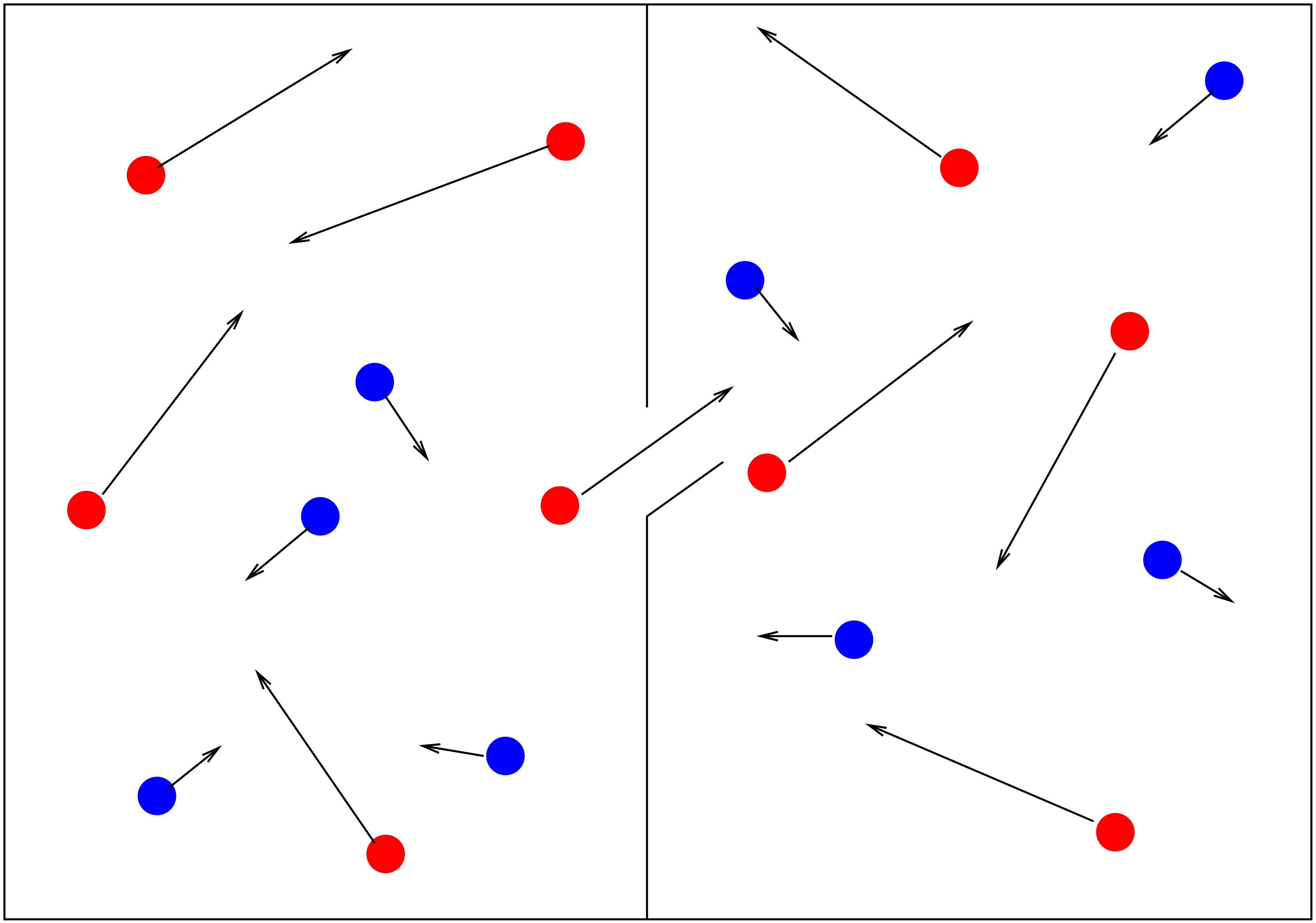

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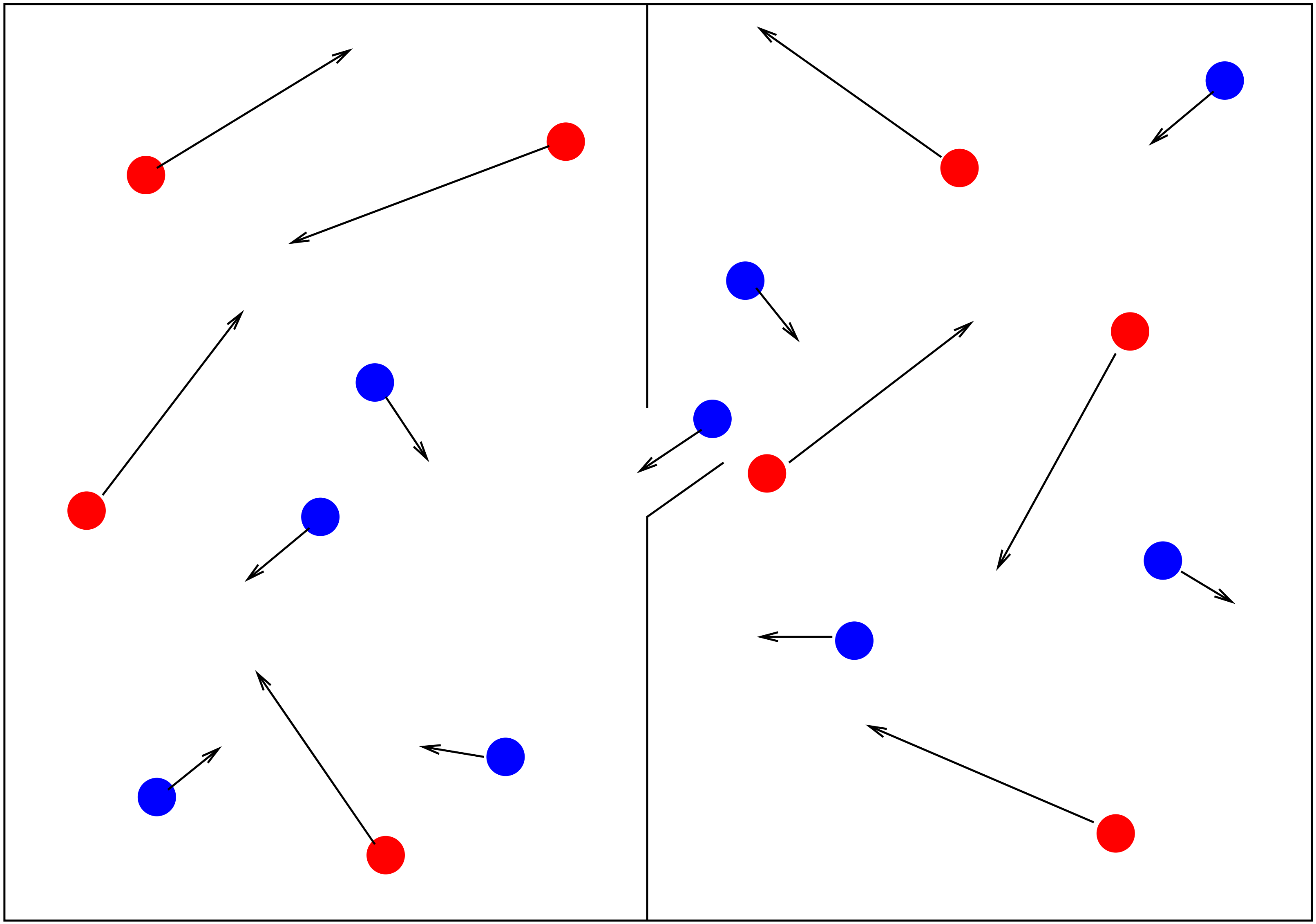

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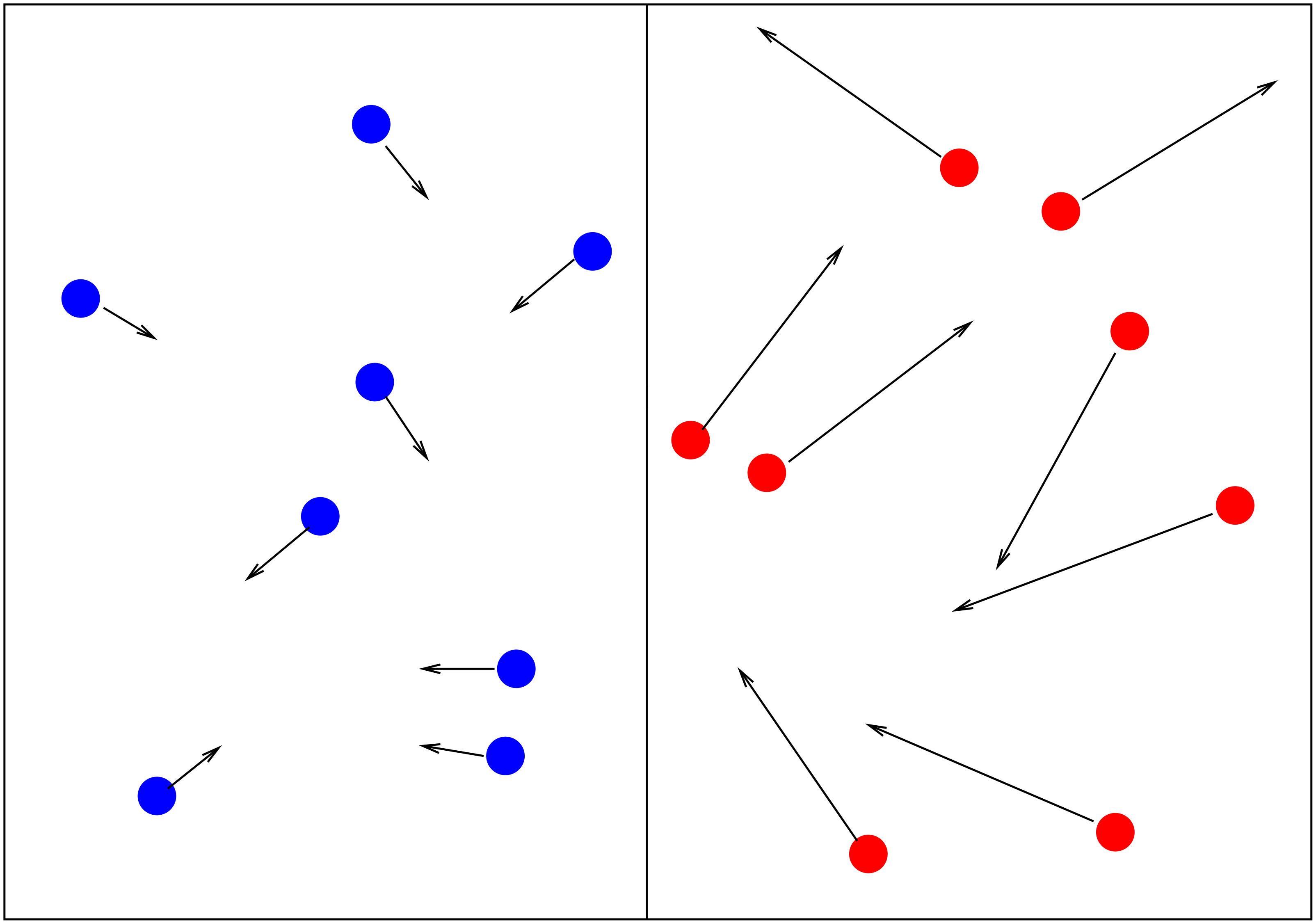
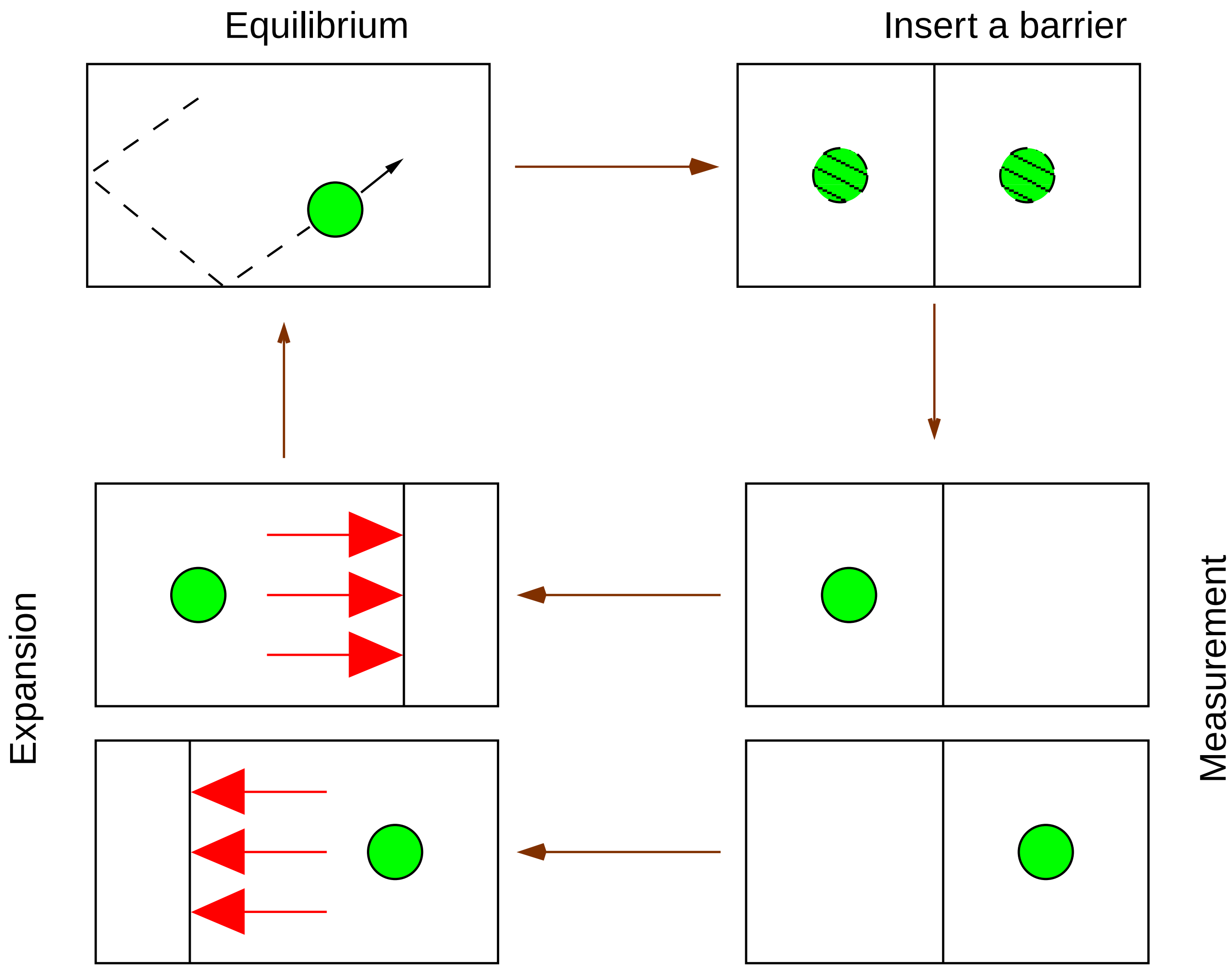
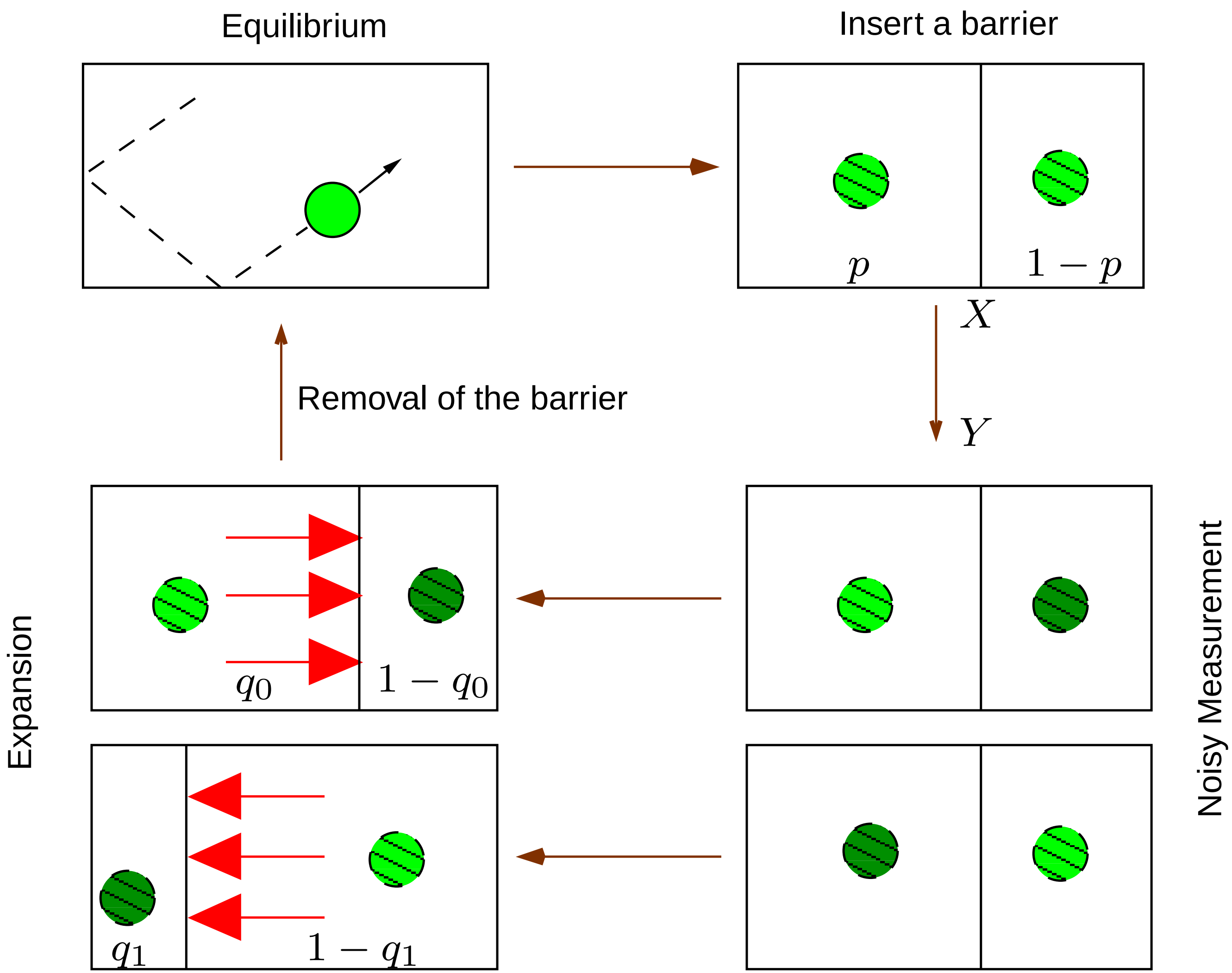


The Szilard Engine (1929)



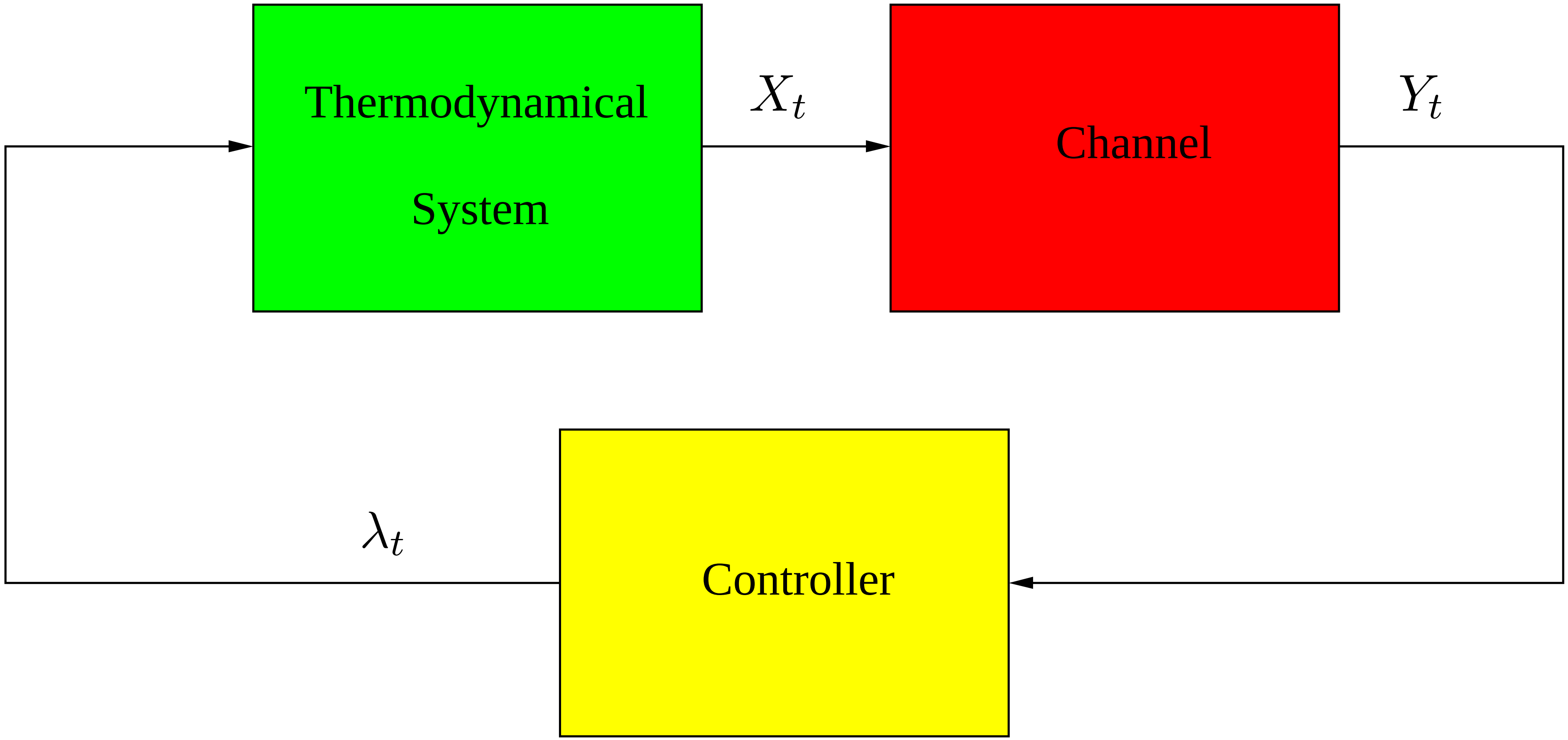
Generalized Szilard Engine (Sagawa & Ueda, 2011)



$\max_{q_0,q_1} W = kT \cdot I(X;Y)$

Vinkler, Permuter and Merhav (2014): relation to gambling.

System with Measurement and Feedback Control



Sagawa and Ueda (2008–2011) : Extracted work $\leq -\Delta F + kT \cdot nI(X;Y)$

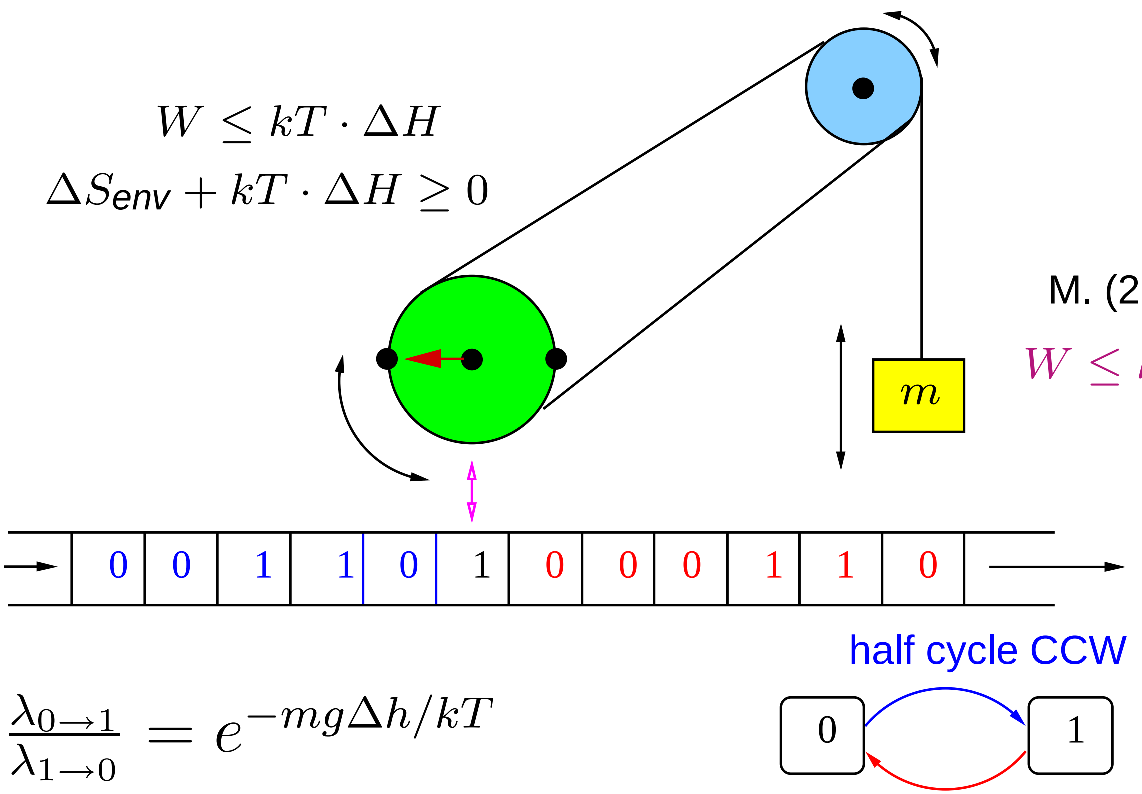
More generally, $nI(X;Y) \to I(X^n \to Y^n)$ = capacity of channel w. feedback.

Physical Systems with an Informational Device:

Mandal & Jarzynski (2012): system converting thermal fluctuations to work while writing info.

$W \leq kT \cdot \Delta H$

$\Delta S_{\text{env}} + kT \cdot \Delta H \geq 0$



M. (2015):

$W \leq kT \cdot F[\text{LZ complexity}]$

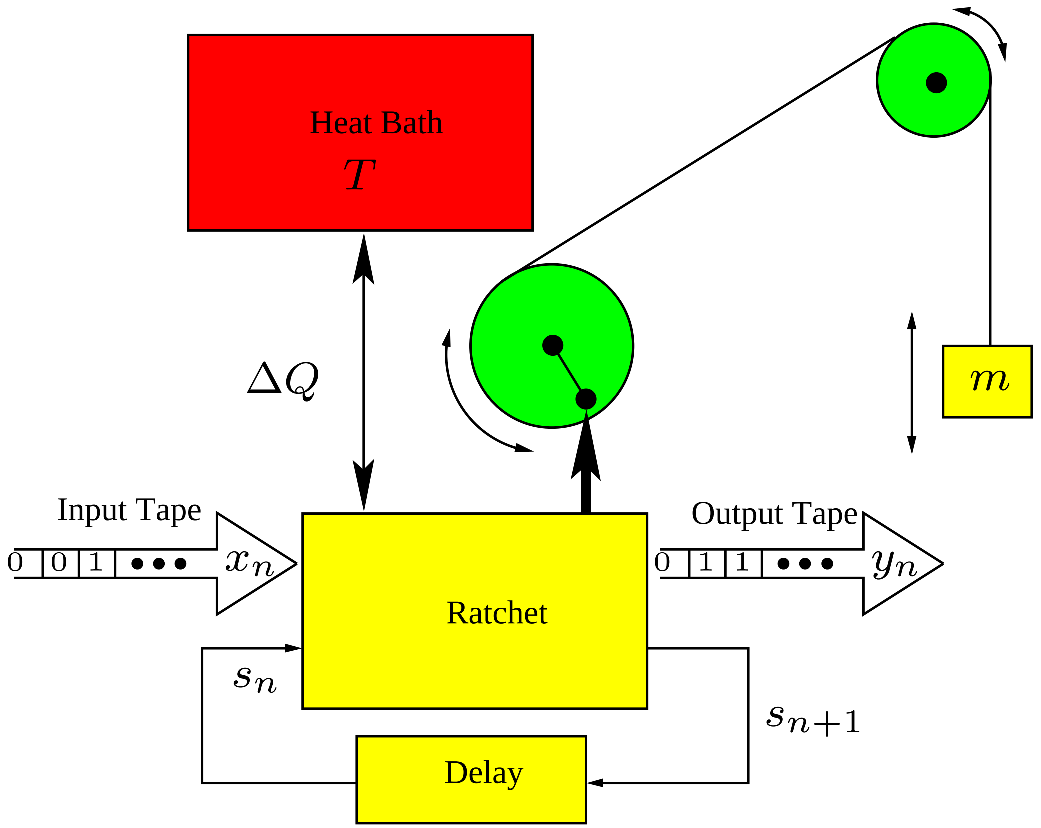
$$\frac{\lambda_{0\to1}}{\lambda_{1\to0}} = e^{-mg\Delta h/kT}$$

More generally, Deffner and Jarzynski (2013):

- A system (device) + a heat bath (heat reservoir),
- An information reservoir, e.g., a memory device with N bits (2^N states).

An extended 2nd law: $\Delta S_{\text{dev}} + \Delta S_{\text{heat-res}} + \Delta S_{\text{info-res}} \geq 0$.

More General System Model



- $\{x_n\}$ – finite-alphabet input process. Each symbol interacts for τ seconds.
- In the n-th cycle: $\{(\xi_t, \sigma_t),\ n\tau \leq t < (n+1)\tau\}$ = Markov jump process.
- $(\xi_{n\tau}, \sigma_{n\tau}) = (x_n, s_n)$; $(y_n, s_{n+1}) = (\xi_{(n+1)\tau-}, \sigma_{(n+1)\tau-})$.
- Energy $E(x,s) = mg\Delta h(x,s)$; stat. dist. $\propto e^{-E(x,s)/kT}$ (detailed balance).

Recent Related Work + This Work

- Boyd, Mandal and Crutchfield (2015, 2016):

$$\lim_{N\to\infty} \overline{W}_N \leq kT[\overline{H}(Y) - \overline{H}(X)].$$

- Several open issues (if not gaps..) in the proof:
  - 2nd law using Shannon entropy (true only under some conditions).
  - Confusing Shannon entropy with thermodynamic entropy.
  - Assuming independence between different parts of the system.
- In this work:
  - Simple (and rigorous) approach; mild asssumptions.
  - Exact results for every number of cycles, N.
  - Bounds – simple to calculate and potentially tight.
  - The state plays a role.

The Basic Idea

Using the inequality [Cover & Thomas, 2006]:

$$D(P_\tau \| P_{\text{eq}}) \le D(P_0 \| P_{\text{eq}}),$$

with

$$P_{\text{eq}}(x, s) = \frac{e^{-E(x,s)/kT}}{Z},$$

we readily obtain

$$\mathbf{E}\{\Delta W_n\} \equiv \mathbf{E}\{E(Y_n, S_{n+1}) - E(X_n, S_n)]\} \le kT[H(Y_n, S_{n+1}) - H(X_n, S_n)].$$

The Basic Idea (Cont'd)

Therefore,

$$
\begin{aligned}
\overline{W}_N &\leq kT \cdot \sum_{n=1}^{N} [H(Y_n, S_{n+1}) - H(X_n, S_n)] \\
&= kT \cdot \sum_{n=1}^{N} [H(Y_n | S_{n+1}) - H(X_n | S_n)] + kT \cdot \sum_{n=1}^{N} [H(S_{n+1}) - H(S_n)] \\
&= kT \cdot \sum_{n=1}^{N} [H(Y_n | S_{n+1}) - H(X_n | S_n)] + kT[H(S_{N+1}) - H(S_1)]
\end{aligned}
$$

First term $=$ input/output entropy production given the past (via the state);

Second term $=$ entropy production of the state.

Tightness

If P_0 is very close to P_{eq} in the sense that

$$P_0(\xi,\sigma) = P_{\text{eq}}(\xi,\sigma)[1+\epsilon(\xi,\sigma)], \qquad \epsilon = \max_{\xi,\sigma} |\epsilon(\xi,\sigma)| \ll 1$$

and if P_τ is similarly within $1 \pm \epsilon^2$ away from P_{eq}, then the bound is tight in the sense that

$$\frac{\mathbf{E}\{\Delta W_n\}}{kT[H(Y_n,S_{n+1}) - H(X_n,S_n)]} = 1 - O(\epsilon).$$

If P_0 and P_{eq} are far apart, one can make many intermediate steps,

$$P_0 \rightarrow P_1 \rightarrow \ldots \rightarrow P_L \rightarrow P_{\text{eq}}$$

in between and still saturate the bound up to a term of $O(1/L)$. This requires a cascade of L systems.

Memoryless/Markov Inputs

For a memoryless input, $H(X_n|S_n) = H(X_n)$, and

$$\frac{\overline{W}_N}{kT} \leq \sum_{n=1}^{N}[H(Y_n) - H(X_n)] + H(S_{N+1}) - H(S_1) - \sum_{n=1}^{N} I(S_{n+1};Y_n)$$

Conclusion: memoryless inputs are best processed by memorlyess machines.

For a Markov input:

- $\{(X_n, S_n)\}$ is Markov as well.
- If it has a stationary distribution $P_\infty(x, s)$, then

$$P(x, s, y, s') = P_\infty(x, s)P(y, s'|x, s)$$

  completely dictates the steady–state work and the entropy production.
- Easier to calculate than $kT[\overline{H}(Y) - \overline{H}(X)]$, as $\{Y_n\}$ is an HMM.

Conditional Entropy Bounds

For a given n, let $V_n = f_n(X^{n-1}, Y^{n-1}, S^n)$, then

$$\overline{W}_N \leq kT \sum_{n=1}^{N} [H(Y_n, S_{n+1}|V_n) - H(X_n, S_n|V_n)],$$

with the freedom of choosing f_n, but the best choice is $V_n = \emptyset$......
However, for $V - n = (X^{n-1}, Y^{n-1})$, we can obtain a lower bound on the output entropy:

$$H(Y^N) \geq H(X^N) + \frac{\overline{W}_N}{kT} - N \ln |\mathcal{S}|.$$

If $\{X_n\}$ is memoryless/Markov, then $\{Y_n\}$ is an HMM, but the lower bound is easy to calculate.

Summary

- Simple relations between work and Δ-entropy (generalized 2nd law).
- The bounds are (potentially) tight.
- Memory of the past – via the state; a state entropy–production term.
- Relatively easy to calculate (examples in the paper).
- Lower bound on the entropy of HMM.
- Generalization – monotonicity of the f–divergence, $D_f(P_\tau \| P_{\text{eq}})$.