Threshold Effects in Parameter Estimation as Phase Transitions in Statistical Physics

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The Threshold Effect

Consider the model

$$r(t) = s_m(t) + n(t), \qquad -T/2 \leq t < T/2,$$

where:

$s_m(t) =$ a waveform parameterized by m;

$n(t) =$ AWGN with spectral density $N_0/2$.

Conveying information via a parameter m by modulating it in $s_m(t)$:

Shannon–Kotel'nikov mappings (Floor '08, Floor & Ramstad '09, Hekland '07, Ramstad '02 + references).

Nonlinear modulation $\Longrightarrow$ threshold effet:

Below some critical SNR, anomalous errors dominate the MSE.

The Threshold Effect (Cont'd)

- Not an artifact of a particular modulation/estimation scheme: it cannot be avoided.
- In the wideband regime, the threshold effect becomes abrupt: Pr$\{$anomaly$\}$ jumps from ~ 0 to ~ 1.

In this talk, we relate the abrupt threshold effect to phase transitions in statistical physics.

Motivation: the physical perspective gains our insight on the problem.

Some Background (W & J, ‘65)

In the simple case of a linear model

$$r(t) = m \cdot s(t) + n(t), \qquad -T/2 \leq t < T/2$$

the ML estimator always achieves

$$\text{MSE} = \text{CRLB} = \frac{N_0}{2E},$$

where E is the energy of $\{s(t)\}$: $\Leftrightarrow$ No threshold effect.

The only way to improve: non–linear modulation of $s_m(t)$.

Background (Cont'd) – Nonlinear Modulation

Now MSE depends not only on E: Let

$$s_m(t) \approx s_{m_0}(t) + (m - m_0) \cdot \dot{s}_{m_0}(t).$$

like the linear case with $\dot{s}_{m_0}(t)$ in the role of $s(t)$. Thus, at high SNR,

$$\text{MSE} \approx \text{CRLB} \approx \frac{N_0}{2\dot{E}},$$

where $\dot{E} =$ energy of $\dot{s}_{m_0}(t)$, which depends on more details.

For example, if $s_m(t) = s(t - m)$, $\dot{E} = W^2 E$, where

$$W = \sqrt{\frac{1}{E} \int_{-\infty}^{\infty} \mathrm{d}f \cdot (2\pi f)^2 S(f)} \qquad \text{Gabor bandwidth}$$

Why not increase W without a limit?

Background (Cont'd) – Geometry of Anomalous Errors

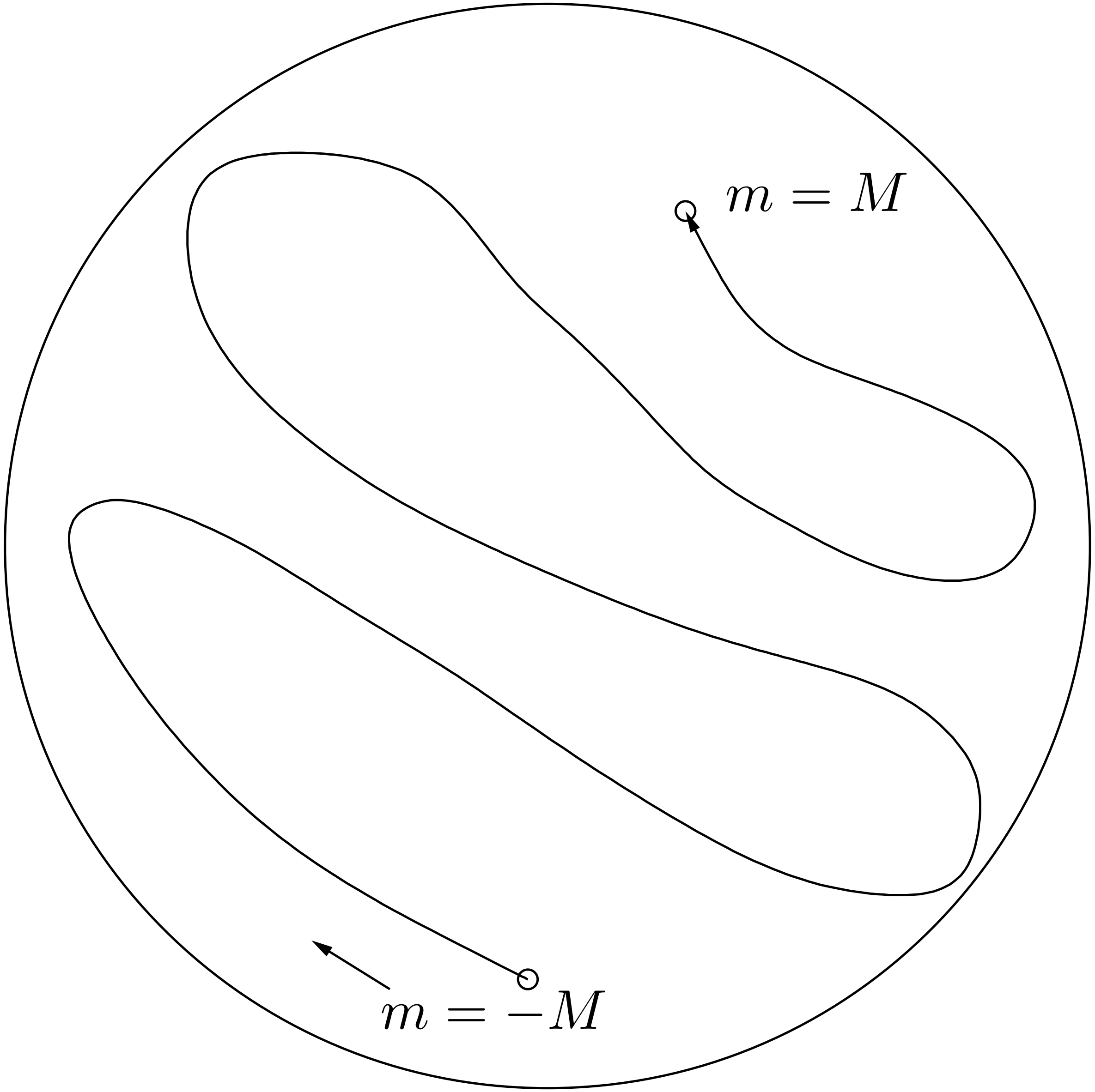
Let $\bar{s}(m) = (s_1(m), \ldots, s_K(m))$ = representation of $s_m(t)$ by K orthonormal basis functions and consider the locus of $\{\bar{s}(m), \ -M \leq m \leq M\}$ in $\mathbb{R}^K$.

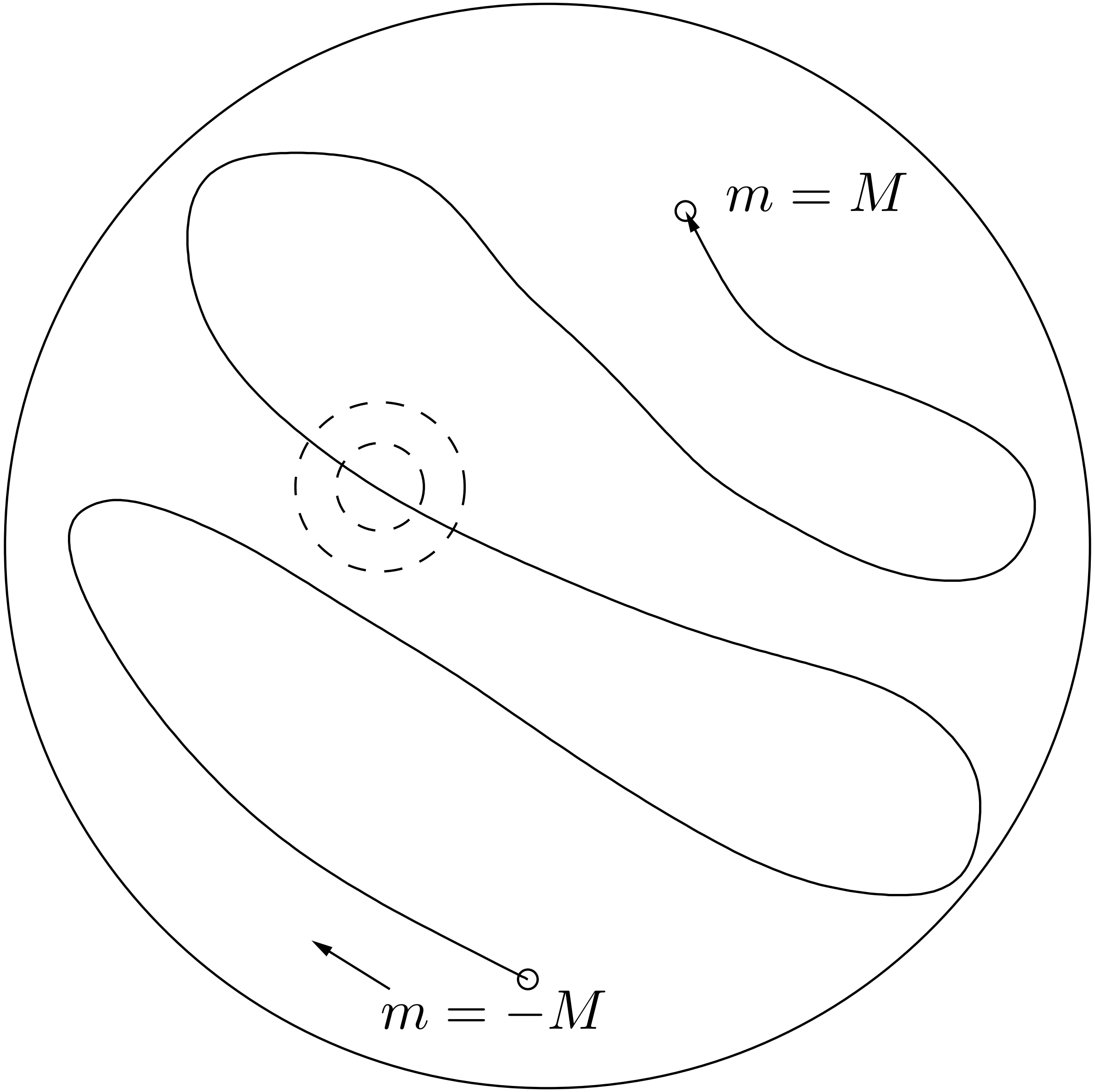
Assuming that E is independent of m, the locus lies on the hypersurface of the K–dimensional sphere of radius $\sqrt{E}$.

The length of the curve

$$L = \int_{-M}^{M} \mathrm{d}m \sqrt{\sum_i \dot{s}_i^2(m)} = 2M\sqrt{\dot{E}}.$$

High–SNR MSE $\downarrow$ with $\dot{E}$, we want $\dot{E} \uparrow$, thus $L \uparrow$.


$m = M$
$m = -M$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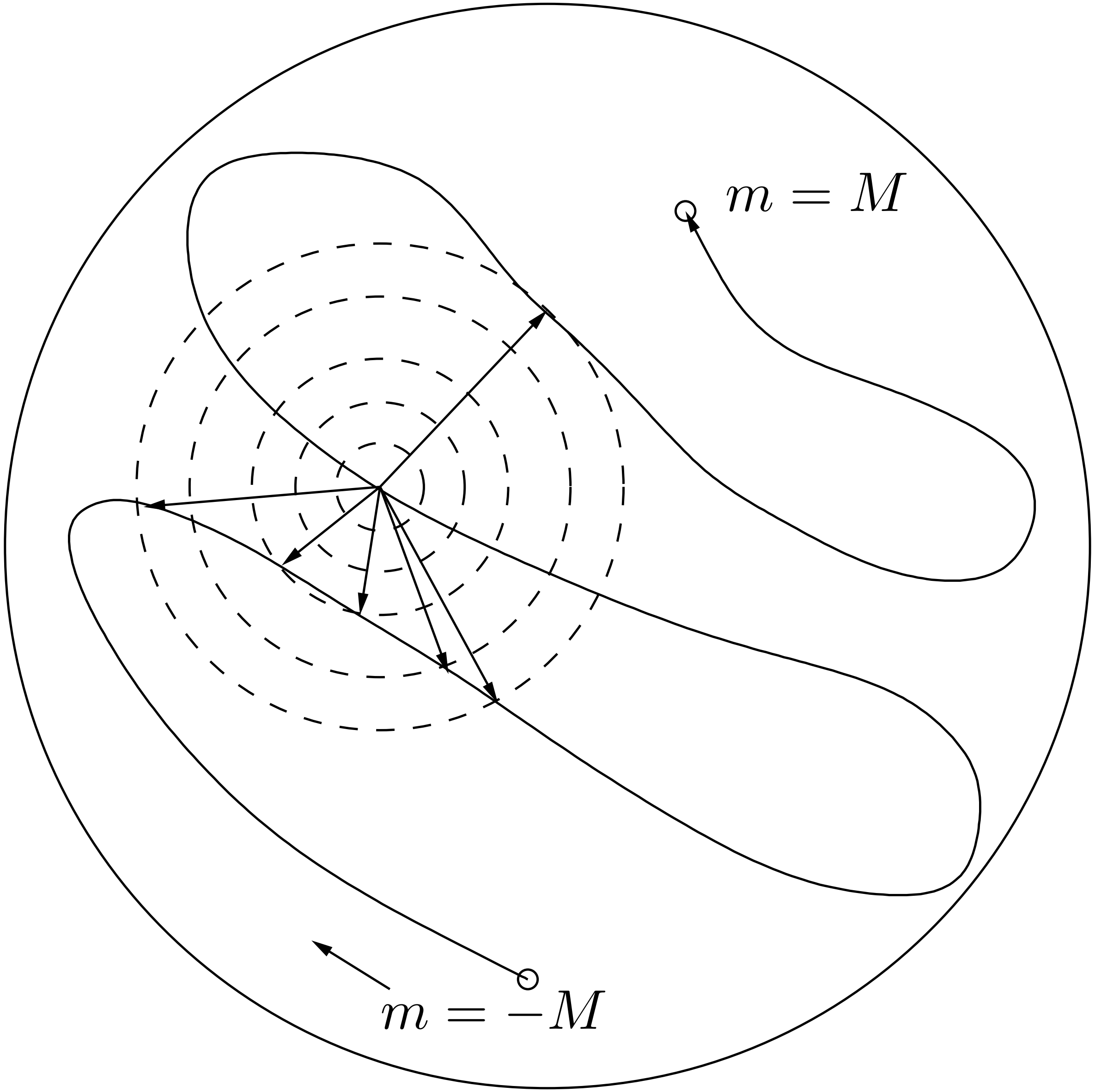


$m = M$
$m = -M$

Anomalous Errors (Cont'd)

L – limited by the need of safe distances between folds – hot dog packing.
Maximum achieable $L \sim e^{CT}$, $C = P/N_0$ (PPM).
For PPM, $K \sim 2WT$,

$$\text{MSE} \approx \underbrace{\frac{N_0}{2W^2E}}_{\text{small error}} + \underbrace{4M^2 \cdot 2WT \cdot e^{-E/(2N_0)}}_{\text{anomalous error}}$$

For fixed W, anomalous error term $\uparrow$ gracefully as $E/N_0 \downarrow$.
For a better balance between terms – let $W \sim e^{RT}$.

$$\text{MSE} \approx \frac{N_0}{2E} e^{-2RT} + 4M^2 \cdot e^{-TE(R)} \qquad R < C$$

where $E(R) =$ reliability function of AWGN channel.
For $W \sim e^{RT}$, anomalous error $\uparrow$ abruptly as $E/N_0 \downarrow$, like a phase transition.

Purpose of this work: study the abrupt threshold effect from the viewpoint of the physics of phase transitions.

A Physical Perspective on the Threshold Effect

Consider the PPM model

$$r(t) = s(t - mT) + n(t), \quad |t| \leq T/2, \quad |m| \leq M, \quad M < \frac{1}{2}.$$

Imagine that $m \sim U[-M/2, +M/2]$, then

$$\begin{aligned} P(m|\mathcal{R}) &= \frac{P(m)P(\mathcal{R}|m)}{\int \mathrm{d}m' P(m')P(\mathcal{R}|m')} \\ &= \frac{\exp\left\{\frac{2}{N_0}\int_{-T/2}^{T/2} r(t)s(t-mT)\mathrm{d}t\right\}}{\int_{-M}^{M} \mathrm{d}m' \exp\left\{\frac{2}{N_0}\int_{-T/2}^{T/2} r(t)s(t-m'T)\mathrm{d}t\right\}} \end{aligned}$$

where $\mathcal{R} = \{r(t), \ |t| \leq T/2\}$. This can be viewed as the Boltzmann distribution with inverse temperature $\beta = 2/N_0$ and Hamiltonian (energy function)

$$\mathcal{E}(m) = -\int_{-T/2}^{T/2} r(t)s(t - mT)\mathrm{d}t.$$

A Physical Perspective ... (Cont'd)

Borrowing from the concept of finite–temperatue decoding [Ruján '93], define

$$P_\beta(m|\mathcal{R}) = \frac{\exp\left\{\beta \int_{-T/2}^{T/2} r(t)s(t-mT)\mathsf{d}t\right\}}{\int_{-M}^{M} \mathsf{d}m' \exp\left\{\beta \int_{-T/2}^{T/2} r(t)s(t-m'T)\mathsf{d}t\right\}}.$$

Motivation: a degree of freedom in case of uncertainty; simulated annealing, analysis of ML estimation.

Meaningful choices of β:
$\beta = 0$ – prior; $\beta = 2/N_0$ – natural; $\beta \to \infty$ – ML estimator dominates.

Define a partition function:

$$\zeta(\beta) = \int_{-M}^{M} \mathsf{d}m \exp\left\{\beta \int_{-T/2}^{T/2} r(t)s(t-mT)\mathsf{d}t\right\}.$$

A Physical Perspective ... (Cont'd)

Assume $s(t - mT)$ has duration Δ and divide $[-M, M]$ to $K = 2MT/\Delta$ subintervals $\mathcal{M}_i$.

ML estimation: find $\epsilon_i = \max_{m \in \mathcal{M}_i} \int_{-T/2}^{T/2} r(t)s(t - mT)\mathrm{d}t$, then $\max_i \epsilon_i$.

Define another paritition function

$$Z(\beta) = \sum_i e^{\beta \epsilon_i}.$$

The RV's $\{\epsilon_i\}$ are alternately independent, with a density known for some waveforms, e.g., rectangular pulses (Slepian '62, Shepp '66, Zakai & Ziv '69).

For T large, the tail is $\approx$ Gaussian.

The contribution $e^{\beta \epsilon_0}$ of the subinterval that includes the signal should be handled separately.

The Random Energy Model (REM)

The analysis of $Z(\beta)$ – very similar to that of the random energy model (REM) in statistical physics (Derrida ‘80,‘81):

$$Z(\beta) = \int \mathrm{d}\epsilon \cdot N(\epsilon) \cdot e^{\beta\epsilon},$$

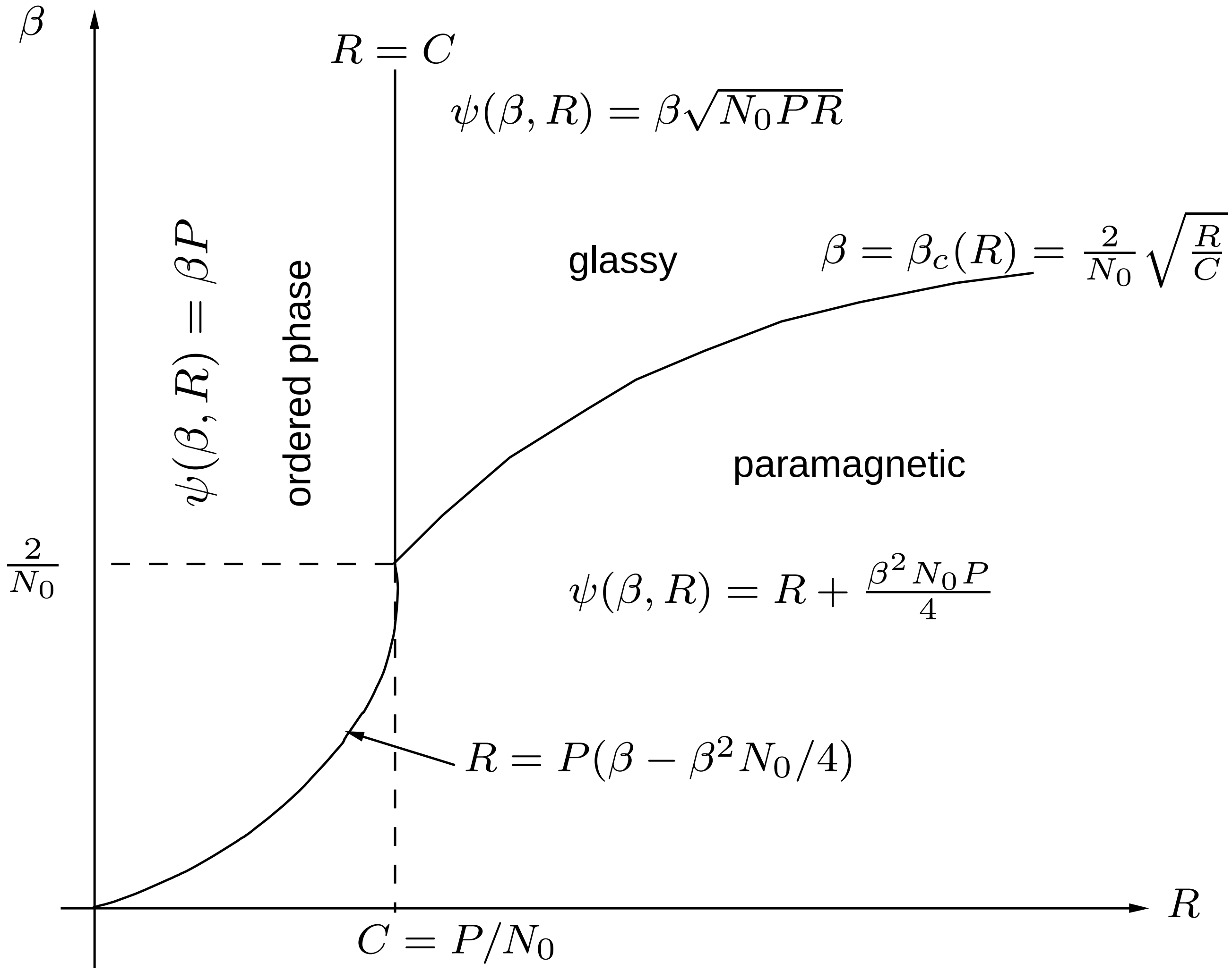
where typically

$$N(\epsilon)\mathrm{d}\epsilon \sim \begin{cases} 0 & f(\epsilon)\mathrm{d}\epsilon \ll 1/K \\ K \cdot f(\epsilon)\mathrm{d}\epsilon & \text{elsewhere} \end{cases}$$

For $W \sim e^{RT}$, we select $\Delta \sim e^{-RT} \implies K \sim e^{RT}$.

Accordingly, from now on, we denote the partition function by $Z(\beta, R)$ and define

$$\psi(\beta, R) = \lim_{T\to\infty} \frac{\ln Z(\beta, R)}{T}.$$


β
$R = C$
$\psi(\beta, R) = \beta\sqrt{N_0 P R}$
$\psi(\beta, R) = \beta P$
ordered phase
glassy
$\beta = \beta_c(R) = \frac{2}{N_0}\sqrt{\frac{R}{C}}$
paramagnetic
$\frac{2}{N_0}$
$\psi(\beta, R) = R + \frac{\beta^2 N_0 P}{4}$
$R = P(\beta - \beta^2 N_0/4)$
R
$C = P/N_0$

Some Extensions

- Mismatched estimation: suppose that $r(t)$ is correlated with $\tilde{s}(t-mT)$ instead of $s(t-mT)$: Same phase diagram, except that that C is degraded by a factor of ρ^2 and β by a factor of ρ, where ρ is the correlation between $s(\cdot)$ and $\tilde{s}(\cdot)$.
- Other pulse shapes: essentially the same results.

Joint ML Estimation of Amplitude and Delay

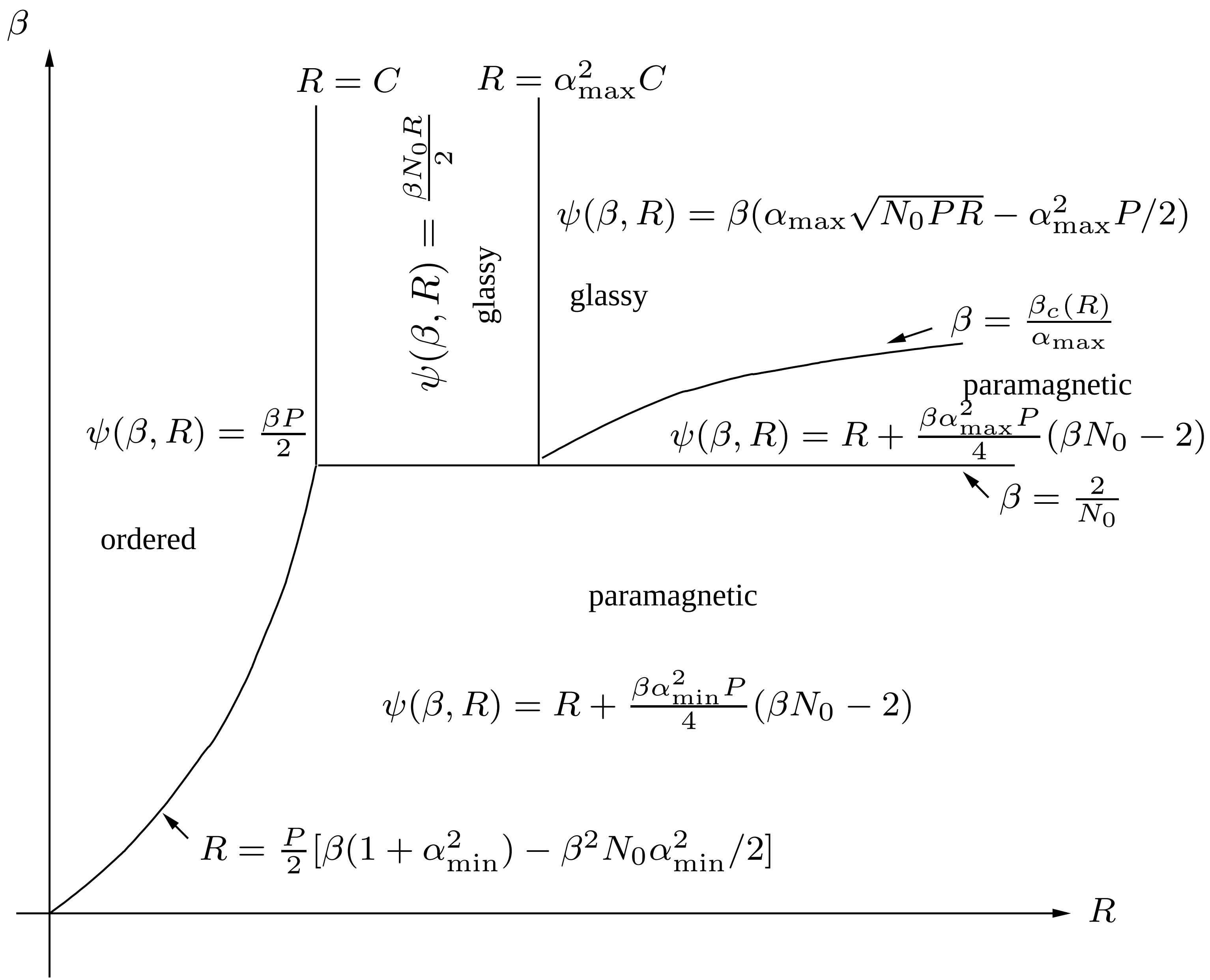
Consider now the model

$$r(t) = \alpha \cdot s(t - mT) + n(t),$$

where now both m and α have to be estimated. It is assumed that $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

- The parameter α alone does not contribute any phase transitions.
- The parameter m alone generates three phases.

Q: How many phases would there be in the joint estimation of m and α?



β
$R = C$
$R = \alpha_{\max}^2 C$
$\psi(\beta, R) = \frac{\beta N_0 R}{2}$
glassy
$\psi(\beta, R) = \beta(\alpha_{\max}\sqrt{N_0 P R} - \alpha_{\max}^2 P/2)$
glassy
$\beta = \frac{\beta_c(R)}{\alpha_{\max}}$
paramagnetic
$\psi(\beta, R) = \frac{\beta P}{2}$
$\psi(\beta, R) = R + \frac{\beta \alpha_{\max}^2 P}{4}(\beta N_0 - 2)$
$\beta = \frac{2}{N_0}$
ordered
paramagnetic
$\psi(\beta, R) = R + \frac{\beta \alpha_{\min}^2 P}{4}(\beta N_0 - 2)$
$R = \frac{P}{2}[\beta(1 + \alpha_{\min}^2) - \beta^2 N_0 \alpha_{\min}^2/2]$
R

Discussion and Conclusion

- The behavior is much more complicated than when only m should be estimated.
- One ordered phase (non–anomalous errors) and four anomalous phases.
- Although α alone does not generate phase transitions, its interaction with m generates more phases than those of m alone.
- Anomalies in α have a different behavior.

The physical point of view helps to gain insight on the behavior.