Lower Bounds on Parameter Modulation–Estimation Under Bandwidth Constraints

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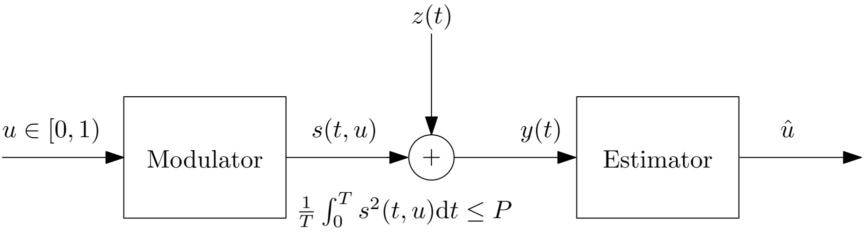
Introduction



Waveform communication:

- Parameter u conveyed via an AWGN channel.
- How well can we estimate u for the best modulator?
- Problem $\in$ {IT} $\cap$ {Estimation theory}
  - IT: Joint source–channel coding - source block length$= 1$.
  - Estimation theory: Most bounds - for a given modulator.

Known Results

For AWGN channel without bandwidth constraints:

- Achievability: PPM/FPM/Uniform quantization and optimal channel code

$$\mathbb{E}\,|\hat{u}-u|^2 \dot{\leq} \exp\left(-\frac{1}{3}\cdot\frac{PT}{N_0}\right).$$

- Converse:
  - Data processing theorem bound

$$\mathbb{E}\,|\hat{u}-u|^2 \dot{\geq} \exp\left(-2\cdot\frac{PT}{N_0}\right).$$

  - Cohn (1970)

$$\mathbb{E}\,|\hat{u}-u|^2 \dot{\geq} \exp\left(-\frac{1}{2.89}\cdot\frac{PT}{N_0}\right).$$

  - Burnashev (1984, 1985)

$$\mathbb{E}\,|\hat{u}-u|^2 \dot{\geq} \exp\left(-\frac{1}{3}\cdot\frac{PT}{N_0}\right).$$

- Upper and lower bounds match.

Bandwidth Constraints

What if the input signals are band-limited to W?

- Data processing bound ($R(D) \leq CT$):

$$\mathbb{E}\,|\hat{u} - u|^2 \stackrel{\cdot}{\geq} \exp\left[-T \cdot 2W \log\left(1 + \frac{P}{N_0 W}\right)\right].$$

- In this work: three better lower bounds + upper bound.

Problem Formulation

- Modulation-estimation system $\mathcal{S}_T$.
- Modulator $u \to s(t,u)$, $u \in [0,1), 0 \le t < T$.
  - Time limitation $t \in [0,T)$.
  - Bandwidth limitation W (almost).
  - Power limitation

$$\frac{1}{T}\int_0^T s^2(t,u)\mathrm{d}t \le P, \qquad \forall u \in [0,1).$$

- AWGN channel, $Z(t)$ with spectral density $N_0/2$,

$$Y(t) = s(t,u) + Z(t).$$

- Estimator $\hat{u} \triangleq g\{Y(t),\, 0 \le t \le T\}$.

Performance Criterion

- The mean power-α error (MPαE)

$$e_\alpha(\mathcal{S}_T) \triangleq \sup_{u \in [0,1)} \mathbb{E}_u\left\{|\hat{u} - u|^\alpha\right\}.$$

- The MPαE exponent

$$E_\alpha\left(\frac{P}{N_o}, W\right) \triangleq \max_{\mathcal{S}_T} \limsup_{T \to \infty}\left[-\frac{1}{T} \cdot \log e_\alpha(\mathcal{S}_T)\right].$$

- The MPαE exponent per unit bandwidth

$$E_\alpha\left(\frac{P}{N_o}, W\right) = W \cdot F_\alpha(\Gamma),$$

where $\Gamma \triangleq \frac{P}{N_0 W}$ is the SNR.

MPαE Exponent - Unlimited Bandwidth

- The unlimited-bandwidth MPαE

$$\min_{\mathcal{S}_T} e_\alpha(\mathcal{S}_T) \doteq e^{-\gamma_\alpha \cdot PT/N_0}.$$

- Burnashev (1985)

$$\gamma_\alpha \leq \begin{cases} \frac{1}{(1+\alpha)} \min\{\alpha, \psi(\alpha)\}, & 0 < \alpha \leq \alpha_0 \\ \frac{\alpha}{2(1+\alpha)} \left[1 + \frac{\alpha+5-4\sqrt{\alpha+1}}{3\alpha+1}\right], & \alpha_0 \leq \alpha \leq 2, \\ \frac{\alpha}{2(1+\alpha)}, & \alpha \geq 2 \end{cases}$$

where $\alpha_0 \approx 1.5875$ and

$$\psi(\alpha) \triangleq 1 + \alpha - \max_{q \geq 1/2} \left[2\alpha q + 4q\sqrt{(1-q)q(1+\alpha)} - q^2(3\alpha+1)\right].$$

- Achieved for $\alpha \geq 2$ (PPM/FMM/Channel code).

MPαE Exponent - Limited Bandwidth

- Lower bounds:
  - The channel coding converse bound.
  - The spherical cap bound.
  - The spectrum replication bound.
- Upper bound:
  - The channel coding achievability bound.

Methodology

- Spherical cap/spectrum replication bounds:

  1. Consider an arbitrary band-limited system.
  2. Construct from (1) a new wideband system.
  3. MPαE exponent of new system $\gamma_\alpha \cdot \frac{P}{N_0}$.
  4. Relate the MPαE's of the two systems.
  5. Obtain a bound for (1).

- Channel coding bounds: Adapted from Merhav (2014), originally derived for DMCs.

The Channel Coding Converse Bound

- Gallager's random coding function

$$E_0(\rho,\Gamma) \triangleq \frac{1}{2}\left[(1-\beta_0)(1+\rho)+\Gamma+\log\left(\beta_0-\frac{\Gamma}{1+\rho}\right)+\rho\log\beta_0\right],$$

where

$$\beta_0 \triangleq \frac{1}{2}\left(1+\frac{\Gamma}{1+\rho}\right)\left[1+\sqrt{1-\frac{4\Gamma\rho}{(1+\rho+\Gamma)^2}}\right].$$

Proposition (channel coding converse bound)

The MPαE exponent per unit bandwidth is upper bounded as

$$F_\alpha(\Gamma) \leq \min\{2E_0(\alpha,\Gamma),\gamma_\alpha\Gamma\}.$$

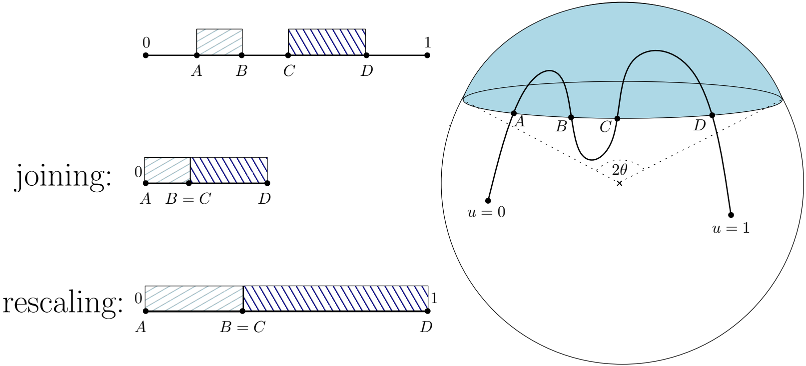
The Spherical Cap Bound

Theorem (spherical cap bound)

The MPαE exponent per unit bandwidth is upper bounded as

$$F_\alpha(\Gamma) \leq \begin{cases} \gamma_\alpha \Gamma, & \Gamma < \frac{\alpha}{\gamma_\alpha} \\ \alpha \left[\log\left(\frac{\gamma_\alpha \Gamma}{\alpha} \right) + 1 \right], & \Gamma \geq \frac{\alpha}{\gamma_\alpha} \end{cases}.$$

The Spherical Cap Bound - Proof Outline



- Signals $\in 2WT$-dimensional sphere, radius $\sqrt{PT}$.
- $\exists$ cap of angle θ corresponding to "significant" portion of $[0,1)$.
- Construct a new system for this portion: join and rescale.

$$e_\alpha(\mathcal{S}_T) \geq \max_\theta \left[\frac{\text{Area}(\text{cap}_\theta)}{\text{Area}(\text{sphere})} \right]^\alpha \cdot e^{-\gamma_\alpha PT \sin^2(\theta)/N_0}$$

The Spectrum Replication Bound

For $\rho \in [0,1]$ define

$$\Phi(\rho,\Gamma) \triangleq \rho[\eta - 1 - \log \eta] + \eta + \Gamma + \log\left[\frac{\sqrt{4\eta\Gamma+1}+1}{2\eta}\right] - \sqrt{4\eta\Gamma+1},$$

with

$$\eta = \frac{\Gamma + \sqrt{\Gamma^2 - 4(\rho^2+1)(\rho+1)^2}}{2(\rho+1)^2},$$

and

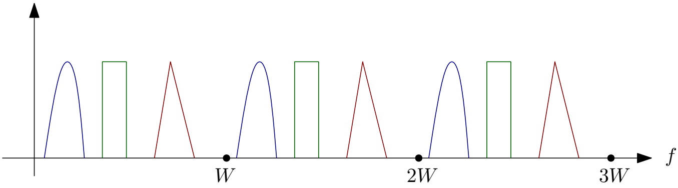
$$\Lambda_\alpha(\Gamma) \triangleq \sup_{0<\rho\leq 1}\left\{\frac{\Phi(\rho,\Gamma) - \gamma_\alpha\Gamma}{\rho}\right\}.$$

Theorem (spectrum replication bound)

The MPαE exponent per unit bandwidth is upper bounded as

$$F_\alpha(\Gamma) \leq \gamma_\alpha\Gamma - \left[\alpha\Lambda_\alpha(\Gamma)\right]_+ .$$

The Spectrum Replication Bound - Proof Outline



- $\mathcal{S}_T$ - limited to $[0, W]$.
- Construct a wideband system $\tilde{\mathcal{S}}_T$: Duplicate signals by frequency translations $\{mW\}_{m=1}^{M-1}$.
- Modulator:
  - Coarse part of $u \leftrightarrow$ frequency band m.
  - Fine part of $u \leftrightarrow$ signal within band.
- Estimator:
  - Decode band index $\hat{m}$- estimate coarse part.
  - Estimate fine part using the estimator of $\mathcal{S}_T$.

$$\mathbb{P}\left[\hat{m} \neq m\right] + \frac{2}{M^\alpha} e_\alpha(\mathcal{S}_T) \geq e_\alpha(\tilde{\mathcal{S}}_T) \geq e^{-\gamma_\alpha PT/N_0}$$

- Optimize over M.

The Channel Coding Achievability Bound

- Gallager's expurgated function

$$E_x(\rho,\Gamma) \triangleq (1-\beta_x)\rho + \frac{\Gamma}{2} + \frac{\rho}{2}\log\left[\beta_x\left(\beta_x - \frac{\Gamma}{2\rho}\right)\right],$$

where

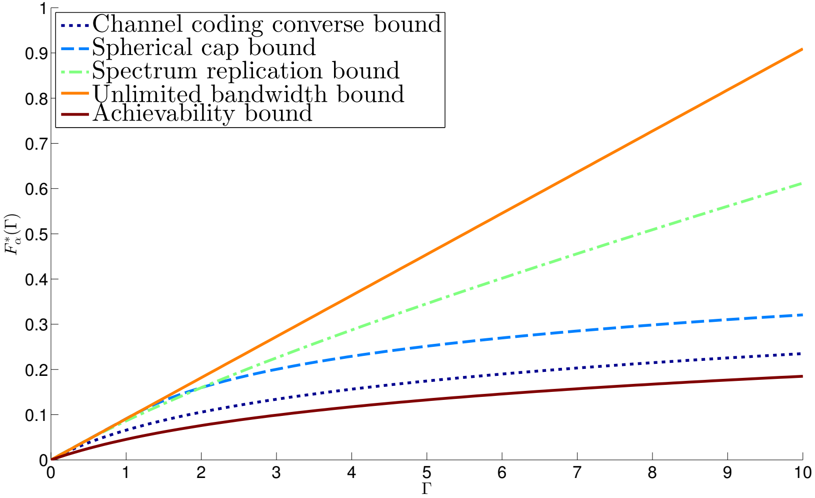
$$\beta_x \triangleq \frac{1}{2} + \frac{\Gamma}{4\rho} + \frac{1}{2}\sqrt{1+\frac{\Gamma^2}{4\rho^2}}.$$

Proposition (achievability bound)

The MPαE exponent per unit bandwidth is lower bounded as

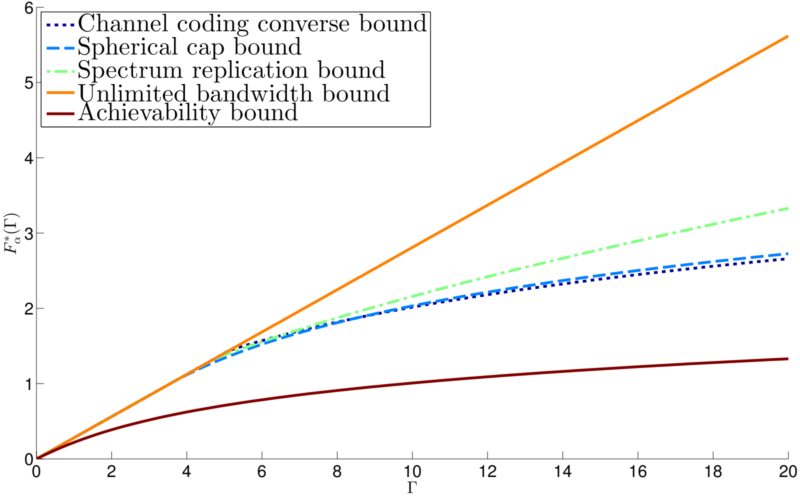
$$F_\alpha(\Gamma) \geq 2\cdot\max\left\{\sup_{0\leq\rho\leq 1}\frac{\alpha E_0(\rho,\Gamma)}{\rho+\alpha}, \sup_{\rho\geq 1}\frac{\alpha E_x(\rho,\Gamma)}{\rho+\alpha}\right\}.$$

Results - $\alpha = 0.1$



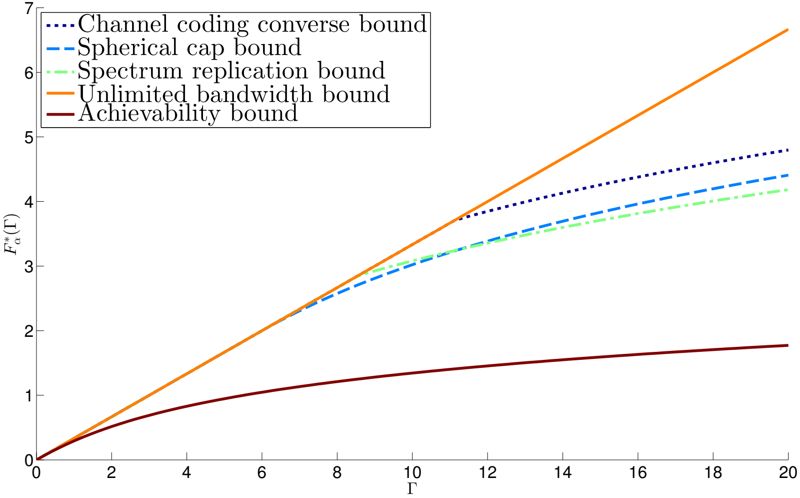
Channel coding converse bound dominates (tight for $\alpha \downarrow 0$).

Results - $\alpha = 1$



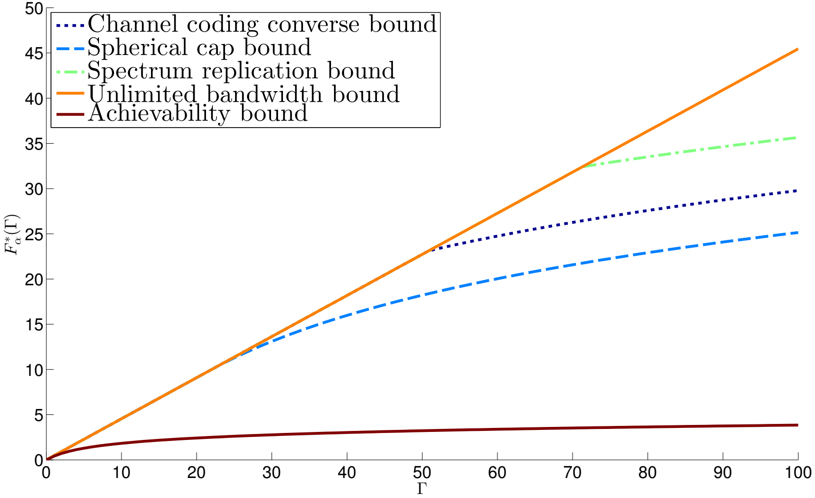
Channel coding converse bound - best for most SNRs.

Results - $\alpha = 2$



Spectrum replication bound - best for most SNRs.

Results - $\alpha = 10$



The spherical cap bound dominates.

High SNR Scaling

Proposition

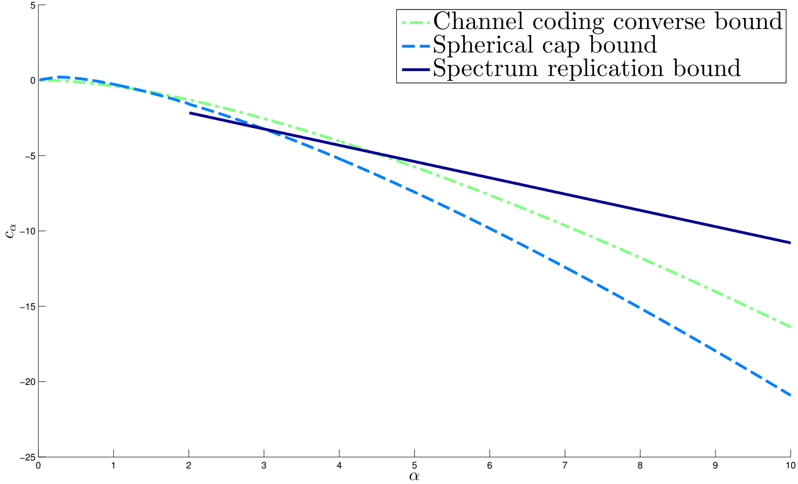
The converse bounds at $\Gamma \to \infty$ are given by

$$F_\alpha(\Gamma) \leq \alpha \log \Gamma + c_\alpha + o(\Gamma)$$

with

$$c_\alpha = \begin{cases} \alpha - (1+\alpha)\log(1+\alpha), & \textit{channel coding converse bound} \\ \alpha \log\left(\frac{\gamma_\alpha}{\alpha}\right) + \alpha, & \textit{spherical cap bound} \\ \alpha \log\left(\frac{e}{8}\right), & \textit{spectrum replication bound, } \alpha \geq 2 \end{cases}.$$

High SNR Scaling - Value of c_α



Summary

- Two new bounds under bandwidth constraints.
- Challenge: close the gap between the bounds!

N. Weinberger and N. Merhav,
“Lower Bounds on Parameter Modulation–Estimation Under Bandwidth Constraints,”
arXiv:1606.06576, June 2016.

Thank You !