Physics of the Rate–Distortion Function

$$\Longrightarrow$$

Rate–Distortion Function via MMSE Estimation

Neri Merhav

Department of Electrical Engineering

Technion—Israel Institute of Technology

Haifa 32000, Israel

ISIT 2010, Austin TX, June 2010

Background

Rate–distortion functions have closed–form expressions in few cases only.

Lower Bounds

- The Shannon lower bound (SLB):
  discrete SLB, continuous SLB, vector SLB,...
- The Wyner–Ziv lower bound (for source with memory):
  sometimes combined with SLB.
- The autoregressive lower bound.

Upper Bounds

Can be obtained from analyzing performance a specific scheme, or a random coding argument, e.g., the Gaussian upper bound.

Some Notation

$X \in \mathcal{X}$ – source symbol; $X \sim p$

$Y \in \mathcal{Y}$ – reproduction symbol;

$d(X, Y)$ – distortion function.

Define

$$R_q(D) = \min\{I(X;Y) : \mathbf{E}\{d(X,Y)\} \leq D,\ Y \sim q\}.$$

Of course,

$$R(D) = \min_q R_q(D).$$

Main Basic Result – MMSE Formula

Parametric representation via a parameter $s \geq 0$:

$$R_s \stackrel{\triangle}{=} R_q(D_s) = \int_0^s \mathrm{d}\hat{s} \cdot \hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X)$$
$$= R_q(D_\infty) - \int_s^\infty \mathrm{d}\hat{s} \cdot \hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X).$$

where $D_\infty \stackrel{\triangle}{=} \mathbf{E}\{\min_y d(X,y)\}$ and

$\mathrm{mmse}_s(\Delta|X) =$ MMSE of estimating $\Delta \stackrel{\triangle}{=} d(X,Y)$ based on X w.r.t. the joint pmf

$$p_s(x,y) = p(x) w_s(y|x) = p(x) \cdot \frac{q(y)e^{-sd(x,y)}}{Z_x(s)}$$

with

$$Z_x(s) = \sum_y q(y) e^{-sd(x,y)}.$$

Main Result (Cont'd)

Similarly,

$$
\begin{aligned}
D_s &= D_0 - \int_0^s \mathrm{d}\hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X) \\
&= D_\infty + \int_s^\infty \mathrm{d}\hat{s} \cdot \mathrm{mmse}_{\hat{s}}(\Delta|X).
\end{aligned}
$$

where

$$D_0 = \sum_{x,y} p(x)q(y)d(x,y).$$

A Few Technical Comments

The result is based on the relation

$$R_q(D) = -\min_{s \geq 0} \left[sD + \sum_{x \in \mathcal{X}} p(x) \ln Z_x(s) \right].$$

which is the large deviations rate function of

$$\text{Pr}\{d(\boldsymbol{x}, \boldsymbol{Y}) \leq nD\}, \quad \boldsymbol{x} \in \mathcal{T}_p, \quad \boldsymbol{Y} \sim q^n.$$

Meanings of s:
(i) Negative local slope of the curve $R_q(D)$: $s = -R_q'(D_s)$;
(ii) Lagrange multiplier of $\min[I(X;Y) + s\mathbf{E}\{d(X,Y)\}]$.

The Y–marginal induced by

$$p_s(x,y) = \frac{p(x)q(y)e^{-sd(x,y)}}{Z_x(s)}$$

may not agree with the reproduction pmf q.

Using the MMSE Relations for Bounds

As both $R_q(D_s)$ and D_s are integrals of mmse$_{\hat{s}}(\Delta|X)$, upper/lower bounds on $R_q(D)$ can be obtained via corresponding bounds on mmse$_{\hat{s}}(\Delta|X)$:

- Bounds derived from estimation–theoretic considerations.
  - Upper bounds: MSE of a certain suboptimum estimator $\hat{\Delta}(X)$.
  - Lower bounds: Bayesian Cramér–Rao bound, or more advanced bounds, if applicable.
- Technical bounds derived directly on the expression of MMSE.

To be demonstrated later on..

What about $R(D)$?

$$\min_q\{\text{lowerbound}_q(D)\} \le R(D) \le \min_q\{\text{upperbound}_q(D)\}.$$

Comparison to I–MMSE Relations

The MMSE formula of $R(D_s)$ rings the bell of the I–MMSE relation [Guo–Shamai–Verdú 2005]:

$$I(\boldsymbol{X}; \sqrt{\textit{snr}}\boldsymbol{X} + \boldsymbol{N}) = \frac{1}{2}\int_0^{\textit{snr}} \mathrm{mmse}(\boldsymbol{X}|\sqrt{\alpha}\boldsymbol{X} + \boldsymbol{N})\mathrm{d}\alpha; \quad \boldsymbol{X} \perp \boldsymbol{N} \sim \mathcal{N}(0, \sigma^2 I).$$

Letting $Y \sim \mathcal{N}(0, \sigma_y^2)$ and $d(x, y) = (x - y)^2$, then $w_s(y|x)$ induces

$$Y = aX + Z; \quad \alpha = \frac{2s\sigma_y^2}{1 + 2s\sigma_y^2}; \quad \mathbf{E}Z^2 = \frac{\alpha}{2s}$$

- Estimation – based on channel input vs. channel output.
- Integrand of $R(D_s)$ includes a factor of s.
- Integration variables are related nonlinearly:

$$\textit{snr} = \frac{4s^2}{\sigma_y^2(1 + 2s\sigma_y^2)}.$$

Analogous MMSE Relation for Channel Capacity

Large deviations rate function of Pr$\{d(\boldsymbol{X}, \boldsymbol{y}) \leq nD)\}$ for $d(x, y) = -\ln w(y|x)$ and $D = H(Y|X)$ ($\boldsymbol{X} \sim p^n$; $\boldsymbol{y} \in \mathcal{T}_q$):

$$C_p = -\min_{s \geq 0} \left[sH(Y|X) + \sum_y q(y) \ln \left(\sum_x p(x) w^s(y|x)] \right) \right]$$

where the minimum is always attained for $s^* = 1$. Accordingly,

$$C_p = \int_0^1 \mathrm{d}s \cdot s \cdot \mathrm{mmse}_s[\ln w(Y|X)|X],$$

where the MMSE is w.r.t. the joint pmf

$$q_s(x, y) = \frac{p(x)q(y)w^s(y|x)}{\sum_{x'} p(x')w^s(y|x')}.$$

Analogies with Statistical Mechanics

$$Z_x(s) = \sum_y q(y) e^{-sd(x,y)}$$

can be thought of as the partition function of subsystem x in equilibrium;
$s = 1/kT$; Hamiltonian (energy function): $\mathcal{E}_x(y) = d(x, y)$.

$$-R_q(D) = \min_s \left[sD + \sum_x p(x) \ln Z_x(s) \right]$$

$=$ normalized entropy in equilibrium of all $|\mathcal{X}|$ subsystems:

Total energy $= nD$; $|$subsystem $x| = np(x)$ particles.

Minimizing $s \longrightarrow$ equilibrium T.

$\text{mmse}_s(\Delta|X) \Longleftrightarrow$ heat capacity.

Analogies with Stat. Mech. (Cont'd)

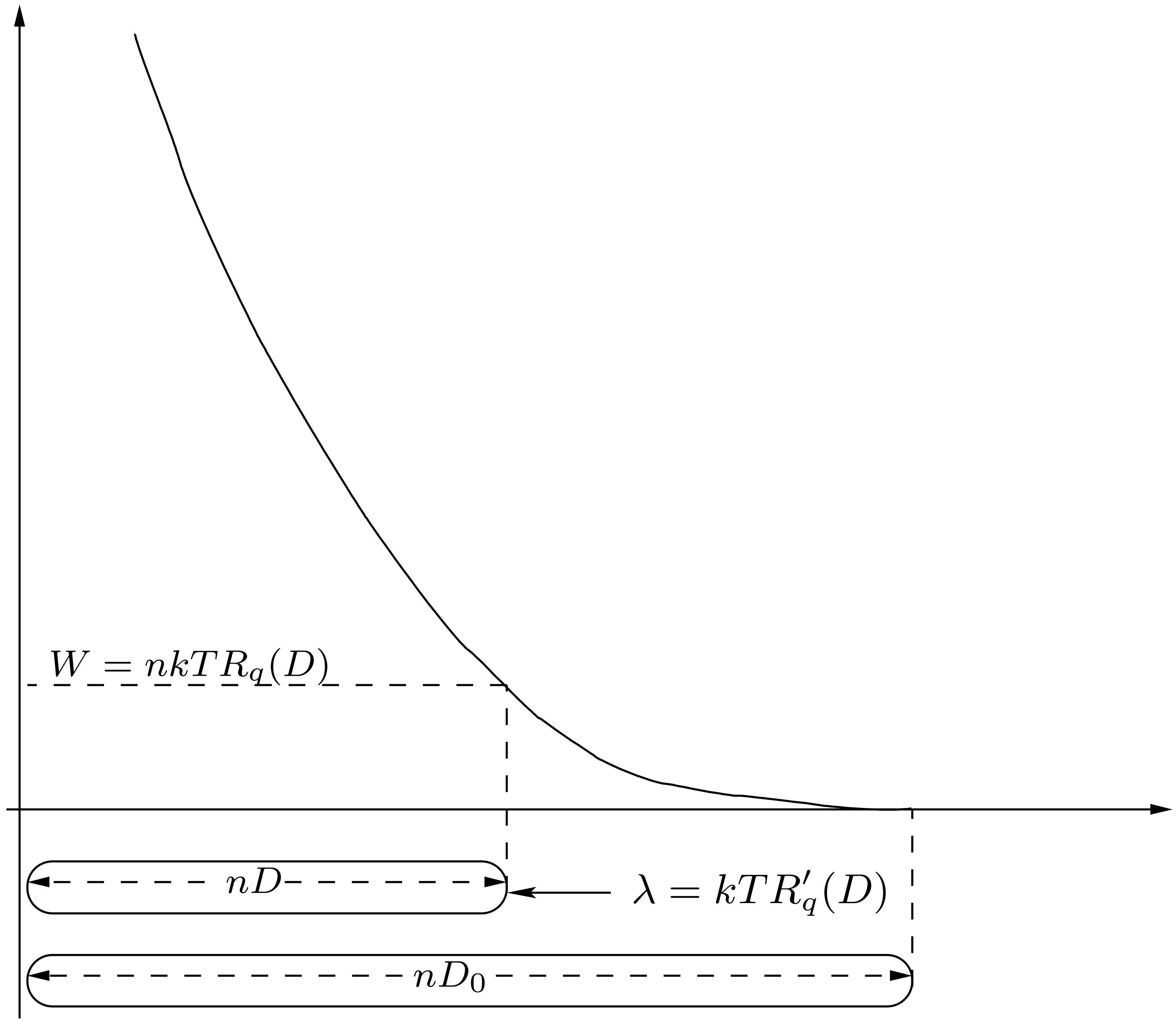
Alternative analogy:

$s \Longleftrightarrow$ force (pressure, magnetic field, ...)

$d(\boldsymbol{x}, \boldsymbol{y}) \Longleftrightarrow$ conjugate physical quantity (volume, magnetization, ...)

$-R_q(D) \Longleftrightarrow$ free energy

MMSE formula $\Longleftrightarrow$ fluctuation–dissipation thm



$W = nkTR_q(D)$
nD
$\lambda = kTR'_q(D)$
nD_0

Example

X has a symmetric pdf around $x = 0$ with $\mathbf{E}(X^2) = \sigma_x^2,\ \mathbf{E}(X^4) = \rho_x^4$.

Quadratic distortion measure: $d(x, y) = (x - y)^2$.

$\mathcal{Y} = \{-a, +a\}; \quad q(+a) = q(-a) = \frac{1}{2}$ (optimum q).

In this case,

$$w_s(y|x) = \frac{e^{-s(x-y)^2}}{e^{-s(x-a)^2} + e^{-s(x+a)^2}} = \frac{e^{2sxy}}{2\cosh(2asx)}.$$

and then

$$\text{mmse}_s(\Delta|X) = 4a^2 \left[\sigma_x^2 - \mathbf{E}\{X^2 \tanh^2(2asX)\}\right].$$

High Distortion (small s): Bounds on the MMSE are obtained from

$$0 \le \tanh^2(2asx) \le (2asx)^2$$

Example (Cont'd)

LB on $R(D)$: LB on MMSE in $\int_0^s$ of $R(D_s)$ eq. and UB on MMSE in $\int_0^s$ eq. of D_s:

$$R(D) \geq \frac{(D_0 - D)^2}{8a^2\sigma_x^2} - \frac{\rho_x^4 (D_0 - D)^4}{64a^4\sigma_x^8}; \quad D_0 = \sigma_x^2 + a^2.$$

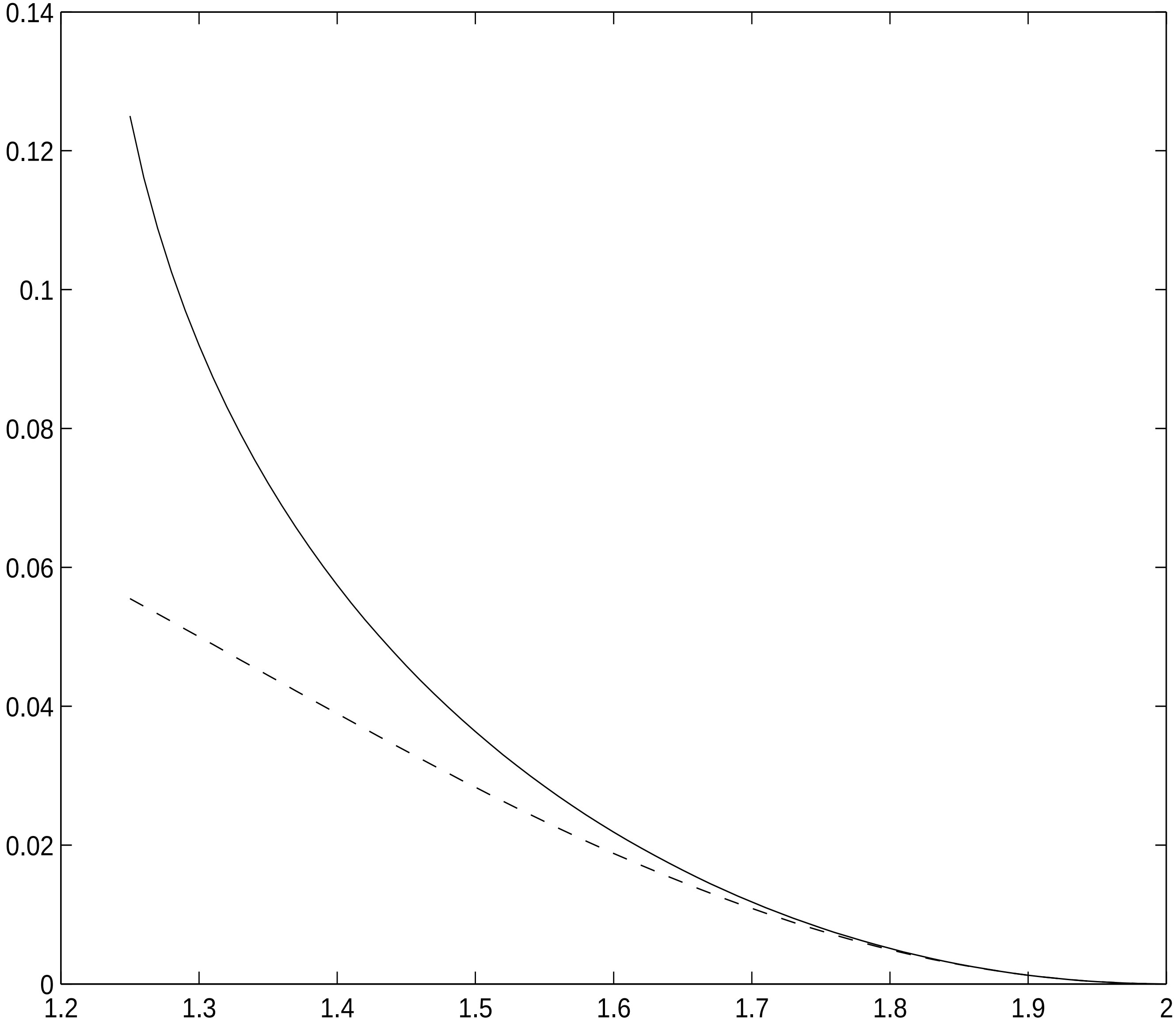
Outperforms SLB, which $= 0$ at $D = (2\pi e)^{-1} e^{2h(X)} \leq \sigma_x^2$.

UB on $R(D)$: the other way around:

$$R(D) \leq \frac{2\sigma_x^4}{\rho_x^4} \sin^2 \left[\frac{1}{3} \sin^{-1} \left(\frac{3\rho_x^2 (D_0 - D)}{4a\sigma_x^3} \right) \right].$$

Bounds are applicable in some distortion intervals; Have the same leading term as $D \uparrow D_0$.

$$R(D) = \frac{(D_0 - D)^2}{8a^2\sigma_x^2} \quad \text{for } D \text{ close to } D_0.$$


0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2

Example (Cont'd)

Low Distortion (large s): For a lower bound on $R(D)$, we use an upper bound on MMSE in $\int_s^\infty$ of the rate and a lower bound on the MMSE in $\int_s^\infty$ of the distortion. Assuming $X \sim$ Laplacian(θ), we get:

$$R(D) \geq 1 - \frac{\sqrt{6C_1(D - D_\infty)}}{2\cos\left[\frac{1}{3}\sin^{-1}\left(2C_2\sqrt{\frac{6(D-D_\infty)}{C_1}}\right) + \frac{\pi}{6}\right]}.$$

For an upper bound, we do the opposite:

$$R(D) \leq 1 - \sqrt{2C_1(D - D_\infty)} + C_2(D - D_\infty)$$

where C_1 and C_2 are constants that depend only on θ/a.

The bounds sandwich the low distortion behavior:

$$R(D) \approx 1 - \sqrt{2C_1(D - D_\infty)}.$$

Conclusion

- An MMSE parametric representation was introduced.
- Reminds I–MMSE relations, but not quite..
- Can be useful for deriving bounds on rate–distortion functions.
- It would be interesting to use estimation–theoretic bounds on MMSE.
- Extensions to more complicated models: successive refinement, side info, ...