Relations Between Redundancy Patterns of the Shannon Code and Wave Diffraction Patterns of Partially Disordered Media

Neri Merhav

Department of Electrical Engineering

Technion—Israel Institute of Technology

Haifa 32000, Israel

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Background and Motivation

The analysis of redundancy of lossless coding has been studied extensively: Capocelli & De Santis ('92), Gallager ('78), Jacquet & Szpankowski ('95), Krichevsky ('68), Louchard & Szpankowski ('97), Savari & Gallager ('97),...

Szpankowski ('00) analyzed the asymptotic (unnormalized) redundancy of the Shannon code, the Huffman code and others for a DMS, and discovered a weird behavior:

For a binary DMS, let $\alpha \stackrel{\triangle}{=} \log_2 \frac{1-p}{p}$. Then, for the Shannon code

$$R_n \stackrel{\triangle}{=} \mathbf{E}\{L(X_1, \ldots, X_n)\} - nH = \begin{cases} \frac{1}{2} + o(n) & \alpha \text{ irrational} \\ \text{oscillatory} & \alpha \text{ rational} \end{cases}$$

The frequency of the oscillations in the 2nd line – dictated by the integer denominator of α.

Background and Motivation (Cont'd)

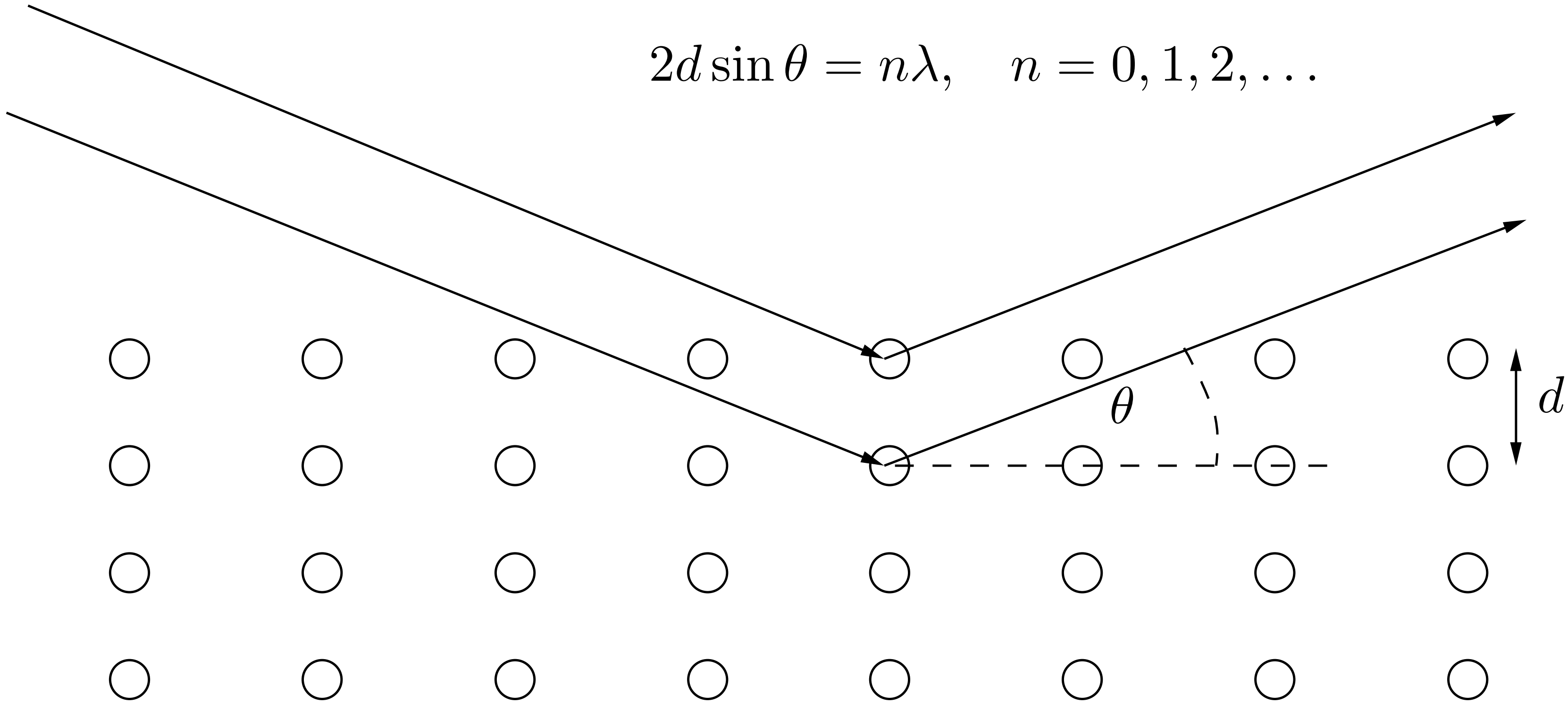
Q: What is the explanation for this erratic behavior?

Our purpose in this work is to try to provide some insight by drawing an analogy with the theory of wave diffraction:

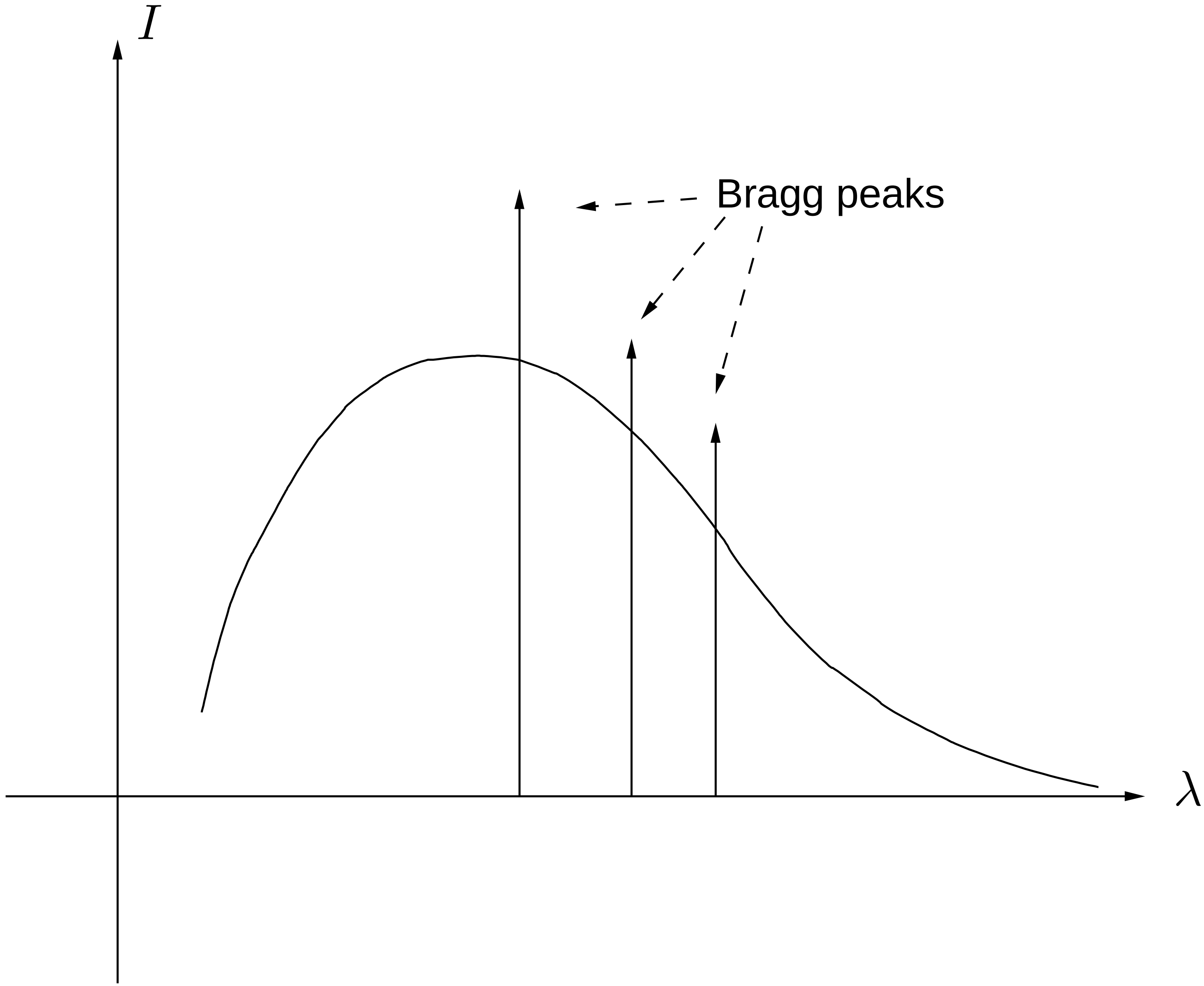
- Perfect crystal (periodic lattice) $\Rightarrow$ Bragg peaks.
- Disordered medium $\Rightarrow$ continuous diffraction pattern (no Bragg peaks).

The first case is parallel to rational α, and the second – to irrational α.

Bragg Diffraction



Bragg Diffraction (Cont'd)



The Hendricks–Teller (HT) Model (1942)

Assume that the distances between consecutive layers are selected independently at random from a finite set.

For example, distance d_0 w.p. p and distance d_1 w.p. $1-p$.

d_0 and d_1 are commensurate (d_1/d_0 – rational) $\Rightarrow \exists$ wavelength λ_1 s.t. both $2d_0 \sin\theta$ and $2d_1 \sin\theta$ are multiples of λ_1: Bragg peaks (constructive interference) appear at all wavelengths $\lambda_n = \lambda_1/n$.

d_0 and d_1 incommensurable $\Rightarrow$ no such wavenlengths exist – no Bragg peaks.

- p and $1-p$ of the source are the same as those of the HT model.
- $\alpha = \log_2 \frac{1-p}{p}$ is analogous to d_1/d_0.
- In the oscillatory case, the fundamental frequency of the oscillations of R_n is related to the fundamental wavenumber of the Bragg peaks.

The Main Common Mathematical Facts

Let $p_0, p_1, \ldots, p_{M-1}$ be probabilities, and define

$$C_m = p_0 + \sum_{j=1}^{M-1} p_j \exp\{2\pi i m \alpha_j\}, \qquad \alpha_j \in \mathbb{R}.$$

- $|C_m| \leq 1$ for all m.
- $C_m = 1$ for some $m \neq 0$ if $\{\alpha_j\}$ are all rational. Otherwise, $|C_m| < 1 \;\; \forall\, m$.
- In the commensurate case, the smallest $m \neq 0$ for which $C_m = 1$ is the common denominator m_0 of all $\{\alpha_j\}$.
- $C_m = 1$ for multiples of m_0 and only for these integers.

Sketch of the Redundancy Analysis (Szpankowski ‘00)

Szpankowski derived the following asymptotic formula for the binary source whose extension to the M–ary source is as follows:

$$R_n = \begin{cases} \frac{1}{2} + \frac{1}{m_0}\left(\frac{1}{2} - \langle \beta m_0 n \rangle\right) + o(1) & \text{all } \{\alpha_j\} \text{ are rational} \\ \frac{1}{2} + o(1) & \text{otherwise} \end{cases}$$

where $\langle x \rangle = x - \lfloor x \rfloor =$ fractional part of x,

$$\alpha_j = \log_2 \frac{p_0}{p_j},$$

$$\beta = -\log_2 p_0.$$

and

$$m_0 = \text{common denominator of all } \{\alpha_j\}$$

Sketch of Analysis (Cont'd)

Consider the Fourier series expansion of the periodic function $\langle x \rangle$:

$$\langle x \rangle = \frac{1}{2} - \sum_{m \neq 0} \frac{\exp\{2\pi i m x\}}{2\pi i m}$$

Applying to $R_n = \mathbf{E}\lceil -\log_2 P(X_1, \ldots, X_n) \rceil - nH$:

$$\begin{aligned} R_n &= \frac{1}{2} + \sum_{m \neq 0} \frac{e^{-2\pi i m n \log p_0}}{2\pi i m} \left[p_0 + \sum_{j=1}^{M-1} p_j \exp\left\{2\pi i m \log(p_0/p_j)\right\} \right]^n \\ &= \frac{1}{2} + \sum_{m \neq 0} \frac{e^{-2\pi i m n \log p_0}}{2\pi i m} \cdot (C_m)^n \end{aligned}$$

where

$$C_m = p_0 + \sum_{j=1}^{M-1} p_j \exp\left\{2\pi i m \log(p_0/p_j)\right\}.$$

Sketch of Analysis (Cont'd)

$$R_n = \frac{1}{2} + \sum_{m \neq 0} \frac{e^{2\pi i m n \beta}}{2\pi i m} \cdot (C_m)^n$$

- Not all $\alpha_j = \log p_0/p_j$ rational:
  $\Rightarrow \; |C_m| < 1 \; \forall \; m \; \Rightarrow \; C_m^n \to 0 \; \Rightarrow \; R_n \to 1/2$.
- All $\alpha_j = \log p_0/p_j$ rational $\Rightarrow \; C_{km_0} = 1$:

$$\begin{aligned} R_n &\approx \frac{1}{2} + \sum_{k \neq 0} \frac{e^{2\pi i k m_0 \beta n}}{2\pi i k m_0} \\ &= \frac{1}{2} + \frac{1}{m_0} \sum_{k \neq 0} \frac{e^{2\pi i k \beta m_0 n}}{2\pi i k} \\ &= \frac{1}{2} + \frac{1}{m_0} \left(\frac{1}{2} - \langle \beta m_0 n \rangle \right), \end{aligned} \tag{1}$$

Fundamental frequency of oscillations: $\omega_0 = 2\pi m_0 \beta$.

The Hendrick–Teller Model (1D)

Let the atoms be in locations $Z_0, Z_1, Z_2, \ldots, Z_{n-1}$ with spacings $\Delta_j = Z_j - Z_{j-1}$, $j = 1, 2, \ldots, n-1$ being i.i.d. RV's taking values in $\{d_0, \ldots, d_{M-1}\}$ with probabilities $\{p_0, \ldots, p_{M-1}\}$.

Each point (atom) at Z_i contributes a scattered wave designated by the phasor e^{-iqZ_j} with $q = 2\pi/\lambda$ being understood as a wavenumber. Thus, the superposition

$$U(q) = \sum_j e^{-iqZ_j}$$

gives rise to the structure function (intensity):

$$I(q) = \mathbf{E}\{|U(q)\|^2\} = \sum_{k,\ell} \mathbf{E}\{e^{iq(Z_k - Z_\ell)}\} = n + I_0(q) + I_0^*(q)$$

with

$$I_0(q) = \sum_{k>\ell} \mathbf{E}\{e^{iq(Z_k - Z_\ell)}\} = \sum_{k>\ell} [\mathbf{E}\{e^{iq\Delta_1}]^{k-\ell}.$$

The Hendrick–Teller Model (Cont'd)

$$I_0(q) = \sum_{k>\ell} [\mathbf{E}\{e^{iq\Delta_1}\}]^{k-\ell} = \sum_{r=1}^{n-1} (n-r)[C(q)]^r.$$

with

$$C(q) = \sum_{j=0}^{M-1} p_j e^{iqd_j}.$$

The intensity is then given by

$$I(q) \approx n \cdot \frac{1 - |C(q)|^2}{|1 - C(q)|^2}.$$

The Hendrick–Teller Model (Cont'd)

There are singularities ($=$ Bragg peaks) if there are values of q with $C(q) = 1$. Let $q_m = 2\pi m/d_0$:

$$C_m = C(q_m) = p_0 + \sum_{j=1}^{M-1} p_j e^{2\pi i m d_j/d_0},$$

and we have the same C_m but now

$$\alpha_j = \frac{d_j}{d_0},$$

which are all rational iff $\{d_j\}$ are commensurate.

Bragg peaks at all multiples of $q_{m_0} = 2\pi m_0/d_0$.

Summarizing the Analogy

- Letter probabilities $\{p_j\}$ $\Leftrightarrow$ Distance probabilities $\{p_j\}$.
- Log-probability ratios $\alpha_j = \log p_0/p_j$ $\Leftrightarrow$ distance ratios $\alpha_j = d_j/d_0$.
- Oscillatory behavior or R_n $\Leftrightarrow$ Bragg peaks.
- Fund. frequency $\omega_0 = 2\pi m_0 \beta$ $\Leftrightarrow$ fund. wavenumber $q_0 = 2\pi m_0/d_0$.

Ongoing work: Extension from i.i.d. to the Markov case (with W. Szpankowski).

Thank You!