On Optimum Parameter Modulation–Estimation From a Large Deviations Perspective

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Communications and Information Theory Seminar, May 3, 2012.

Special thanks to Yariv Ephraim for many useful discussions.

Thanks also to Tsachy Weissman and Yonina Eldar for interesting conversations.

Background

Consider the model

$$y(t) = x(t, u) + n(t), \quad 0 \leq t < T,$$

where:

$x(t, u) =$ a waveform parametrized by u;

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Nonlinear modulation $\Longrightarrow$ threshold effect:

Below some critical SNR, anomalous errors dominate the MSE.

Background (Cont'd) - The Threshold Effect

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the ML estimator always achieves

$$\text{MSE} = \text{CRLB} = \frac{N_0}{2\mathcal{E}},$$

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Only way to improve (at high SNR): non–linear modulation $x(t, u)$.

Background (Cont'd) – Nonlinear Modulation

Let

$$x(t, u) \approx x(t, u_0) + (u - u_0) \cdot \dot{x}(t, u_0).$$

like the linear case with $\dot{x}(t, u_0)$ in the role of $s(t)$.

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For example, if $x(t,u) = s(t-u)$, $\dot{\mathcal{E}} = W^2\mathcal{E}$, where

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Why not increase W without a limit?

Background (Cont'd) – Geometry of Anomalous Errors

Let $\bar{x}(u) = (x_1(u), \ldots, x_K(u)) =$ representation of $x(t, u)$ by K orthonormal basis functions. Consider the locus of $\{\bar{x}(u),\ a \le u \le b\}$ in $\mathbb{R}^K$.

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The length of the curve

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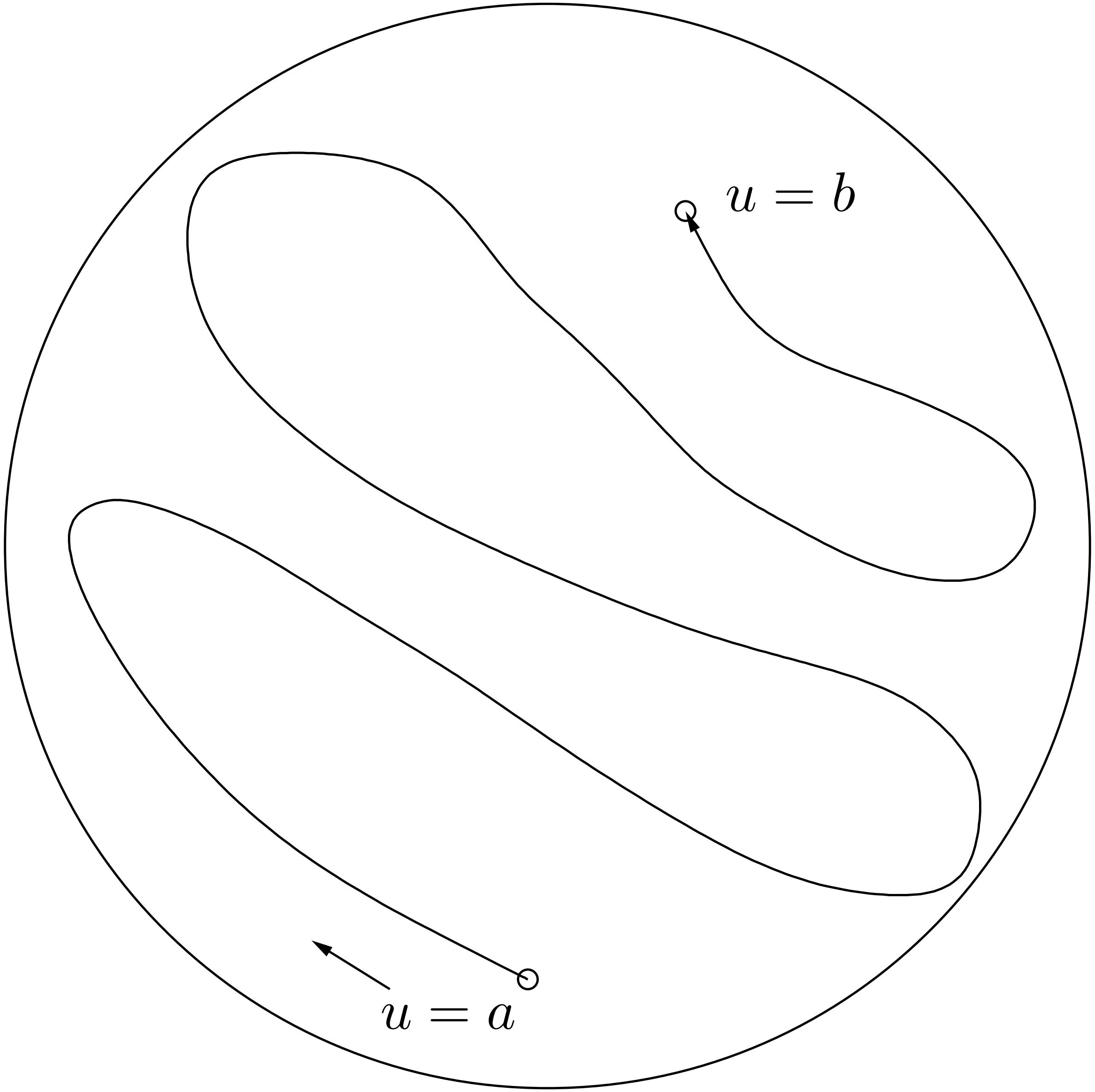
High–SNR MSE $\downarrow$ with $\dot{\mathcal{E}}$, we want $\dot{\mathcal{E}}\ \uparrow$, thus $L\ \uparrow$.



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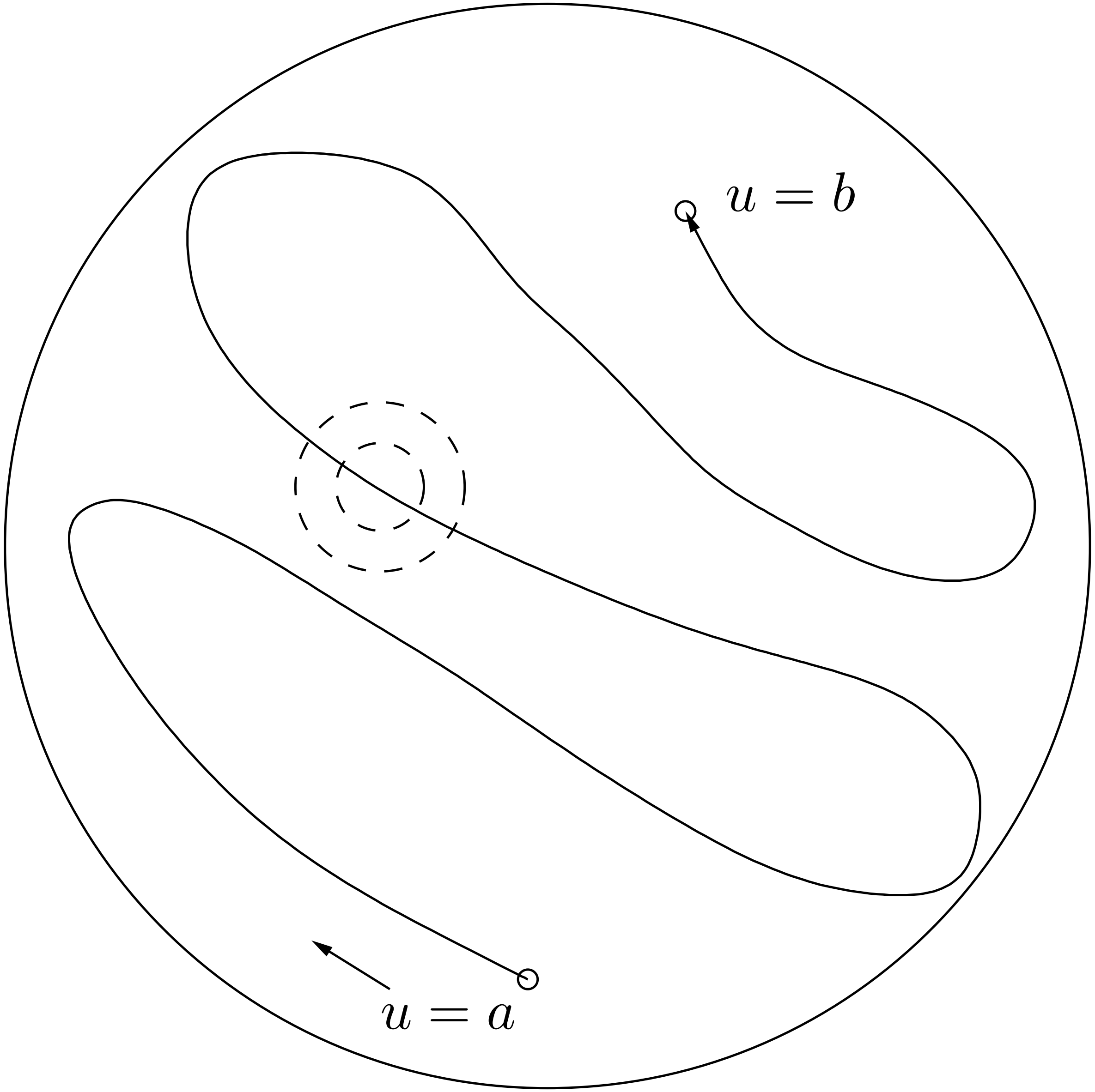

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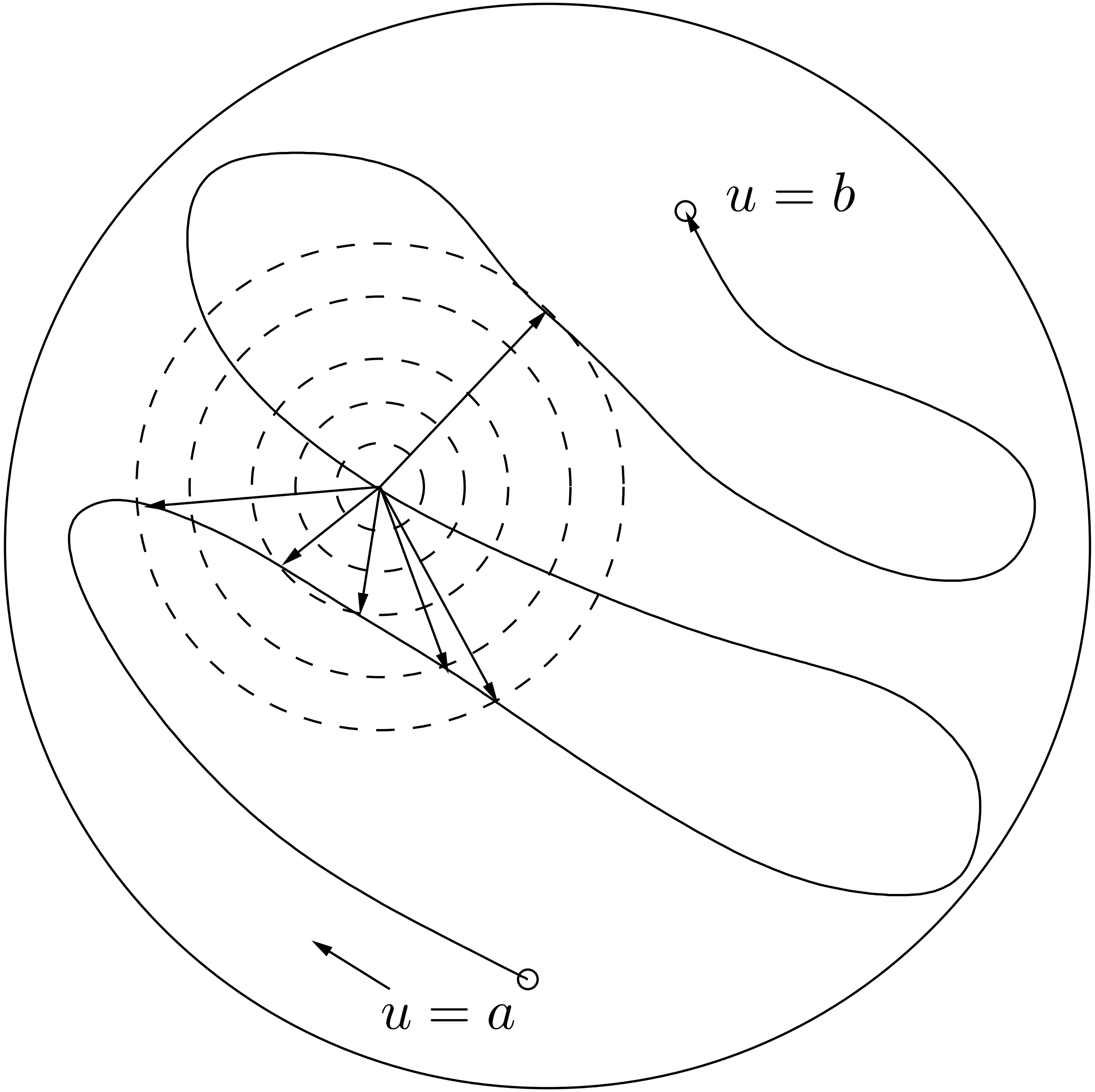
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Optimum compromise: $R = C/6 \quad \Longrightarrow \quad \text{MSE} \sim e^{-CT/3}$.

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- Is there a compatible lower bound?

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Conjecture: “Blame” the lower bound.

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We are interested in

$$E^*(R) = \limsup_{T\to\infty}\left[-\frac{1}{T}\log \inf \mathsf{Pr}\left\{|\hat{U} - U| > e^{-RT}\right\}\right].$$

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MSE does not distinguish between weak–noise errors and anomalous errors.

Basic Result

Theorem: For all $R>0$, the $\limsup$ of $E^*(R)$ is actually $\lim$ and

$$E^*(R) = E(R) = \begin{cases} \frac{C}{2} - R & 0 \le R \le \frac{C}{4} \\ (\sqrt{C} - \sqrt{R})^2 & \frac{C}{4} \le R \le C \\ 0 & R \ge C \end{cases}$$

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Modulator: Form a grid of $M = e^{RT}/2$ points in $[-1/2, +1/2)$:

$$\{-1/2 + 1 \cdot e^{-RT}, -1/2 + 3 \cdot e^{-RT}, -1/2 + 5 \cdot e^{-RT}, \ldots, 1/2 - e^{-RT}\}.$$

Map grid points to orthogonal signals $s_i(t)$ with power S: $x(t,u) = s_i(t)$, where $i =$ index of grid point NN to u.

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Obviously,

$$\Pr\{|\hat{U} - U| > e^{-RT}\} \le \Pr\{\hat{i} \ne i\} \sim e^{-TE(R)}.$$

Converse Part

For a given u, consider the grid

$$\{u, u+2e^{-RT}, u+4e^{-RT}, \ldots, u+2(M-1)e^{-RT}\}, \qquad M = \frac{e^{(R-\epsilon)T}}{2}+1$$

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$$\{u, u + 2e^{-RT}, u + 4e^{-RT}, \ldots, u + 2(M-1)e^{-RT}\}, \qquad M = \frac{e^{(R-\epsilon)T}}{2} + 1$$

and define the hypothesis testing problem:

$$\mathcal{H}_i : \; y(t) = x\left(t, u + 2ie^{-RT}\right) + n(t) \qquad i = 0, 1, \ldots, M-1.$$

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The result is obtained by integrating both sides over u.

The Case $R = 0$

The operational reliability – discontinuous at $R = 0$. For fixed M, P_e is dictated by $d_{\min} = \frac{2M\mathcal{E}}{M-1}$. In particular,

$$P_e \propto Q\left(\sqrt{\frac{\mathcal{E}}{N_0} \cdot \frac{M}{M-1}}\right) \sim \exp\left(-\frac{CT}{2} \cdot \frac{M}{M-1}\right).$$

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In estimation, Pr$\{|\hat{U} - U| > \Delta\}$ for fixed Δ, is according to P_e but with $M \propto 1/\Delta$.

Small gap between upper bound and the lower bound for every fixed Δ, but this gap $\to 0$ as $\Delta \to 0$. In particular,

$$\lim_{\Delta \to 0} \lim_{T \to \infty} \left[-\frac{\ln \text{Pr}\{|\hat{U} - U| > \Delta\}}{T}\right] = \frac{C}{2} = E(0).$$

The Case $R = 0$ (Cont'd)

Relation to the MSE:

$$\mathbf{E}(\hat{U} - U)^2 = 2 \int_0^1 \mathsf{d}\Delta \cdot \Delta \cdot \mathsf{Pr}\{|\hat{U} - U| \geq \Delta\}.$$

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Open question: devise a system independent of Δ, yet minimizes Pr$\{|\hat{U} - U| \geq \Delta\}$ for every Δ.

Discussion

Strong Converse $\Leftrightarrow$ Sharp Threshold Effect

- Both achievability and converse rely on signal detection considerations.
- Strong converse: $\lim_{T \to \infty} P_e$ jumps from 0 to 1 as R crosses C.
- Equivalently, $E^*(R) = 0$ for $R > C$ in the strong sense.
- “Inheriting” strong converse — jump in Pr$\{|\hat{U} - U| > e^{-RT}\}$.
- For an optimum system, $|\hat{U} - U|$ “concentrates” around e^{-CT}.

Alternative Achievability Schemes

Achievability – quantization of U + orthogonal signaling.

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Alternative modulators (+ ML estimation):

FPM: $x(t,u) = \sqrt{2S}\cos[(\omega_0 + u \cdot \Delta\omega)t + \phi] \quad \omega_0, \Delta\omega \propto e^{RT}$.

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In AM:

$$\Pr\{|\hat{U} - U| > e^{-RT}\} = 2Q(e^{-RT}\sqrt{2CT}) \to 1 \quad \forall\, R > 0$$

Relation to Moments of the Estimation Error

By Chebyshev's inequality

$$e^{-T[E(R)+o(T)]} \leq \text{Pr}\{|\hat{U} - U| > e^{-RT}\} \leq \frac{\mathbf{E}(\hat{U} - U)^2}{e^{-2RT}}$$

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For a general moment $\mathbf{E}|\hat{U} - U|^\alpha$ ($\alpha > 0$, arbitrary):

$$\mathbf{E}|\hat{U} - U|^\alpha \geq \begin{cases} e^{-CT/2} & \alpha \geq 1 \\ e^{-\alpha CT/(1+\alpha)} & 0 < \alpha < 1 \end{cases}$$

Relation to Joint Source–Channel Coding

Csiszár (1982): JSC problem under

$$\min \mathsf{Pr} \left\{ \sum_{i=1}^{N} d(U_i, \hat{U}_i) > ND \right\}.$$

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The exponential rate cannot exceed

$$e(D) = \min_R [F(D,R) + E(R)]$$

where

$$F(D,R) = \min_{Q':\ R(D,Q') \geq R} D(Q' \| Q)$$

is the source coding exponent of the source Q (Marton, 1974).

Relation to Joint Source–Channel Coding (Cont'd)

For separate source– and channel coding:

$$e_{sep}(D) = \sup_R \min\{F(D,R), E(R)\} \leq e(D)$$

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Q: How does this settle?

Relation to Joint Source–Channel Coding (Cont'd)

Answer: Let Q^* maximize $R(D,Q)$ (often, uniform).

$$F(D,R) = \min_{Q:\ R(D,Q)\geq R} D(Q\|Q^*) = \begin{cases} 0 & R \leq R(D,Q^*) \\ \infty & R > R(D,Q^*) \end{cases}$$

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Intuition:

- "Cover" source space by $e^{NR(D,Q^*)}$ D–spheres.
- Source encoder does not cause $\sum_i d(U_i,\hat{U}_i) > ND$.
- Excess distortion – only due to channel – w. p. $e^{-NE[R(D,Q^*)]}$.
- This is our case too.

Extensions

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Consider a d–dimensional vector $\boldsymbol{U} = (U_1, \ldots, U_d) \sim$ Unif$[-1/2, +1/2)^d$.

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Think of a grid with $e^{R_i T}$ points in the i–th coordinate $\Rightarrow$ total $= e^{(R_1+\ldots+R_d)T}$.

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Consider the case $R_1 = R_2 = \ldots = R_d \equiv R$:

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Different from the common "curse of dimensionality", which is usually graceful in d.

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- Applicable to bandlimited Gaussian channel with $N = 2WT$ channel uses.
- Unknown channels: universal decoding metrics – applicable for universal estimation.

Rayleigh Fading

Let

$$y(t) = a \cdot x(t, u) + n(t), \qquad 0 \leq t < T$$

where $a =$ realization of A, with density

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For $R = 0$ – decays like $1/T$.

Summary and Conclusion

- Large deviations performance metric – natural for wideband communication.
- Precise characterization of the best achievable exponent.
- Intimately related to signal detection – reliability function.
- Simple considerations; simple to extend in many directions.
- Relation to JSCC: separate source– and channel coding is optimal.
- Open problem: close the gap between upper and lower bounds on the MMSE.

Thank You!