On the Data Processing Theorem in the Semi–Deterministic Setting

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Background

During the 70s and 80s, Ziv (partly, with Lempel) has developed the individual–sequence approach:

- 1978: Fixed–rate, almost–lossless compression using FSM's.
- 1978: The LZ algorithm.
- 1980: Lossy compression for noiseless/noisy transmission.
- 1984: Almost lossless comp. with side info @ decoder (Slepian–Wolf).
- 1986: Lossless compression of 2D data using FSM's.

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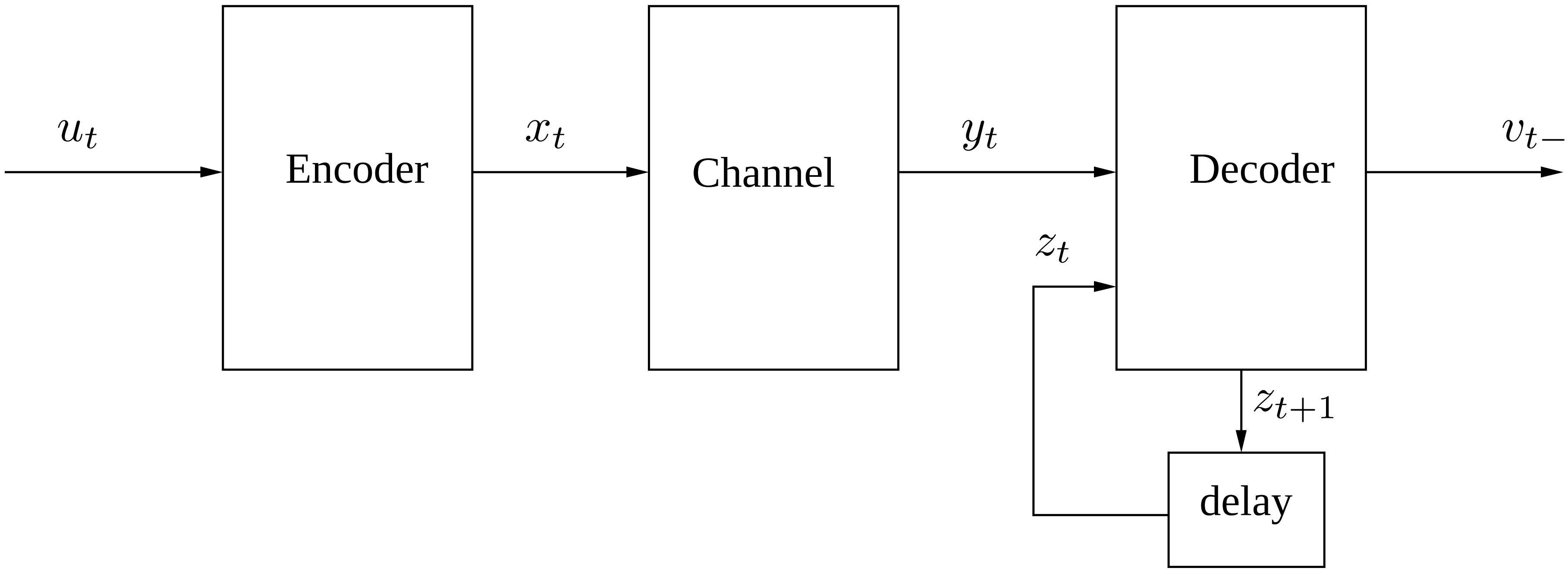
Objectives

- Data processing thm of [Ziv80] revisited – the semi–deterministic setting:
  1. Correcting a few imprecise points.
  2. Strengthening the lower bound:
     - No limitations on the encoder (not necessarily an FSM).
     - Allowing a modulo–ℓ counter (periodically time–varying decoder).
- Tighter bound under a common reconstruction constraint.
- Bounds that depend on the LZ complexity of the input/output sequence.
- Analogous results for linear encoders/decoders in the continuous case.

The Communication System



$$v_{t-d} = f(z_t, y_t), \qquad t = d+1, d+2, \ldots$$
$$z_{t+1} = g(z_t, y_t), \qquad t = 1, 2, \ldots$$

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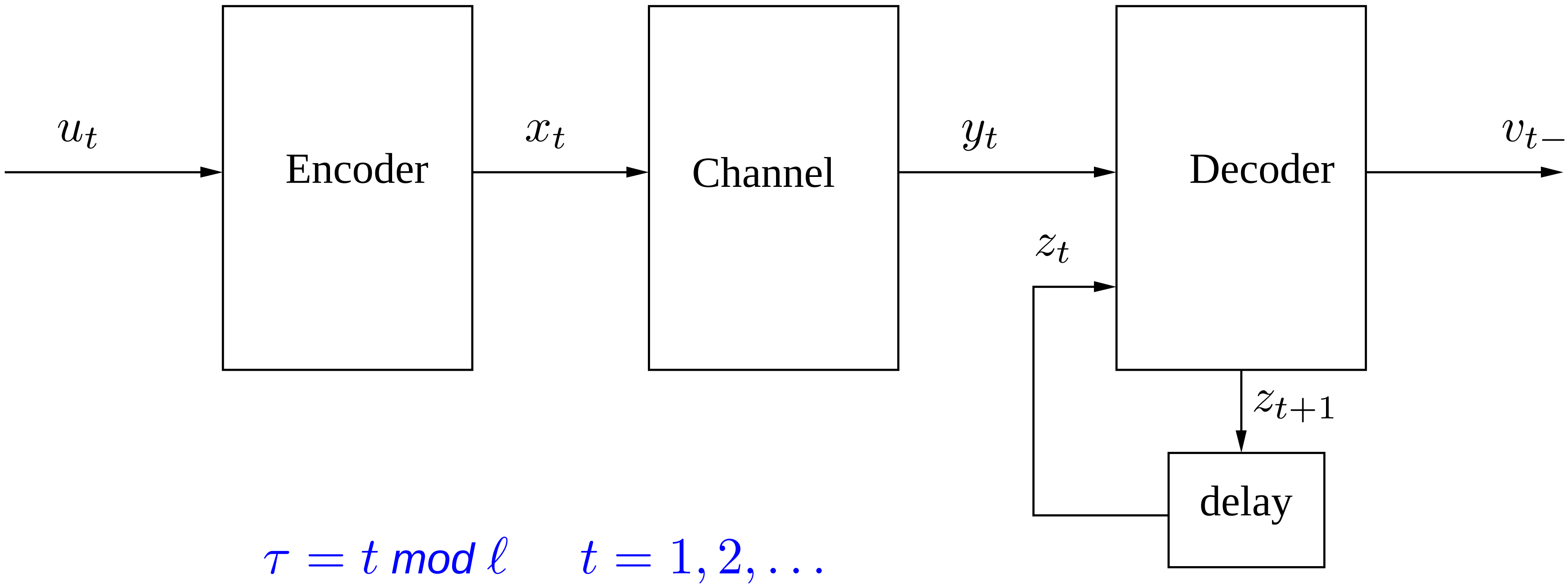

$\tau = t \text{ mod } \ell \qquad t = 1, 2, \ldots$

$$v_{t-d} = f_\tau(z_t, y_t), \qquad t = d+1, d+2, \ldots$$
$$z_{t+1} = g_\tau(z_t, y_t), \qquad t = 1, 2, \ldots$$

Empirical Statistics Defined

Let ℓ divide n and consider the partition of $\boldsymbol{u} = (u_1, \ldots, u_n)$ into n/ℓ ℓ–blocks

$$\boldsymbol{u}_i = (u_{i\ell+1}, \ldots, u_{i\ell+\ell}), \qquad i = 0, 1, \ldots, n/\ell - 1$$

and similarly for other vectors (with $v_{n-d+1}, \ldots, v_n$ taking arbitrary values in $\mathcal{V}$).

We define $\hat{P}_{U^\ell X^\ell Y^\ell V^\ell Z}$ as the empirical joint distribution generated by counting the relative frequency of

$$\{\boldsymbol{u}_i = u^\ell,\ \boldsymbol{x}_i = x^\ell,\ \boldsymbol{y}_i = y^\ell,\ \boldsymbol{v}_i = v^\ell,\ z_{i\ell+1} = z\}, \qquad i = 0, 1, \ldots, n/\ell - 1$$

for all $u^\ell \in \mathcal{U}^\ell$, $x^\ell \in \mathcal{X}^\ell$, $y^\ell \in \mathcal{Y}^\ell$, $v^\ell \in \mathcal{V}^\ell$, and $z \in \mathcal{Z}$.

Some Concerns About the DPT in [Ziv80]

- Joint distribution defined s.t.
  - $U^\ell \perp (Z, Z')$ ($Z' =$ encoder state), but Z' responds to U^ℓ.
  - Y^ℓ is independent of Z' given X^ℓ, but Z responds to Y^ℓ.
- DPT: $U^\ell \perp V^\ell$ given (Z, X^ℓ), but $\exists$ empirical dependencies.
- v_{t-d} is renamed as v_t, but:
  - DPT applies to $\{u_t, v_{t-d}\}$
  - distortion is defined between $\{u_t, v_t\}$.
  - Consequence: Lower bound is independent of d.

Distortion Bound Without Common Reconstruction

Since $V^{\ell-d}$ is a deterministic function of Y^ℓ and Z,

$$\hat{I}(U^\ell; V^{\ell-d}|Z) \leq \hat{I}(U^\ell, X^\ell; Y^\ell|Z) \leq \hat{H}(Y^\ell) - \hat{H}(Y^\ell|U^\ell, X^\ell) + \log s$$

but on the other hand,

$$\hat{I}(U^\ell; V^{\ell-d}|Z) \geq \hat{I}(U^\ell; V^\ell) - \log s - d \log M.$$

Taking expectations and using $\mathbf{E}\hat{H}(Y^\ell|U^\ell, X^\ell) \sim H(Y^\ell|X^\ell)$,

$$\text{upper bound} \leq \ell C + \log s,$$

whereas

$$\text{lower bound} \geq \ell \cdot \hat{R}_{U^\ell}(\mathbf{E}\rho(U^\ell, V^\ell)) - \log s - d \log M.$$

Bound Without Common Reconstruction (Cont'd)

Theorem 1:

$$\frac{1}{n}\sum_{t=1}^{n}\mathbf{E}\{\rho(u_t,V_t)\} \geq \hat{D}_{U^\ell}\left(C + \frac{2\log s + d\log M}{\ell} + \delta_1(\ell,n)\right)$$

where

$$\delta_1(\ell,n) = \frac{(JK)^\ell \log L}{\sqrt{n}} + \frac{(JKL)^\ell \log e}{2n} + o\left(\frac{1}{\sqrt{n}}\right)$$

and

$$\hat{D}_{U^\ell}(R) = \min\left\{\frac{\mathbf{E}\rho(U^\ell,\tilde{V}^\ell)}{\ell} \,:\, \hat{I}(U^\ell;\tilde{V}^\ell) \leq \ell R\right\}.$$

- Blue term $=$ "effective extra capacity" due to memory and allowed delay.
- In the absence of a mod–ℓ counter, maximize the r.h.s. over ℓ.
- Dependence on ℓ – complicated.

Alternative Bound for Difference Distortion Measures

Assume $\rho(u, v) = \varrho(v - u)$, and define

$$\Phi(D) = \max\{H(W) : \mathbf{E}\rho(W) \leq D\}$$

$$\Psi(t) = \begin{cases} \Phi^{-1}(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Theorem 2:

$$\frac{1}{n}\sum_{t=1}^{n} \mathbf{E}\{\varrho(V_t - u_t)\} \geq \Psi\left(\frac{c(u^n)\log c(u^n)}{n} - C - \frac{2\log s + d\log M}{\ell} - \delta_2(\ell, n)\right),$$

where $c(u^n)$ is the number of LZ phrases of u^n and

$$\delta_2(\ell, n) = \delta_1(\ell, n) + \frac{2\ell(1 + \log J)^2}{(1 - \epsilon_n)\log n} + \frac{2\ell J^{2\ell}\log J}{n} + \frac{1}{\ell},$$

Common Reconstruction

For a given $\epsilon_n \to 0$, suppose that $\exists\ q$ s.t.

$$\mathbf{E}\hat{\text{Pr}}\{V^\ell \neq \hat{V}^\ell\} \equiv \frac{\ell}{n} \sum_{i=0}^{n/\ell-1} \text{Pr}\{V_{i\ell+1}^{i\ell+\ell} \neq \hat{v}_{i\ell+1}^{i\ell+\ell}\} \leq \epsilon_n,$$

where $\hat{V}^\ell = q(U^\ell)$.

Theorem 3:

$$\frac{1}{n}\sum_{t=1}^{n} \mathbf{E}\{\rho(u_t, V_t)\} \geq \tilde{D}_{U^\ell}\left(C + \frac{2\log s + d\log M}{\ell} + \delta_2(\ell, n) + 2\Delta(\epsilon_n)\right) - \rho_{\max}\epsilon_n,$$

where $\Delta(\epsilon_n) = h_2(\epsilon_n) + \epsilon_n \ell \log J$, $h_2(\cdot)$ being the binary entropy function, and

$$\tilde{D}_{U^\ell}(R) = \min_q \left\{\frac{1}{\ell}\hat{\mathbf{E}}\rho(U^\ell, q(U^\ell)) :\ \hat{H}(q(U^\ell)) \leq \ell R\right\}.$$

$\hat{H}(q(U^\ell))$ can be further lower bounded in terms of $c(\hat{v}^n)\log c(\hat{v}^n)$.

Linear Encoders/Decoders

Analogous results can be obtained in the linear/Gaussian/quadratic case.

Let $\rho(u, v) = (u - v)^2$ and consider a linear encoder

$$x_t = \sum_{i=1}^{\infty} a_i x_{t-i} + \sum_{i=0}^{\infty} b_i u_{t-i},$$

a Gaussian memoryless channel

$$Y_t = x_t + N_t, \qquad N_t \sim \mathcal{N}(0, \sigma^2),$$

and a linear decoder defined by

$$\begin{aligned} v_{t-d} &= \alpha z_t + \beta y_t \\ z_{t+1} &= \gamma z_t + \delta y_t. \end{aligned}$$

Linear Encoders/Decoders (Cont'd)

Main idea: Assume (for a moment) a Gaussian input process:

- All processes in the system are jointly Gaussian.
- All information measures depend solely on auto/cross–correlations.
- DPT gives inequality relations between these auto/cross–correlations.
- Relations between auto/cross–correlations do not rely on Gaussianity.
- The input does not have to be Gaussian.
- Use the empirical cov. matrix of U^ℓ to derive semi–deterministic bounds.
- Due to the linear–Gaussian structure, the test channel is $V^\ell = GU^\ell + W^\ell$.

Linear/Gaussian Lower Bound

Theorem 4:

$$\frac{1}{n}\sum_{t=1}^{n}\mathbf{E}(V_t - u_t)^2 \geq \hat{D}\left(C + \frac{1}{\ell}\log\frac{\sigma_Z^2}{\epsilon_Z^2} + \frac{d}{2\ell}\log\frac{\sigma_V^2}{\epsilon_V^2} + \epsilon_n\right)$$

where

$$\hat{D}(R) = \min_{G,\Sigma_W}\Big\{\mathrm{tr}\{(G-I)\hat{\Sigma}_U(G^T - I) + \frac{1}{\ell}\Sigma_W\} : \frac{1}{2\ell}\log|I + \Sigma_W^{-1}G\hat{\Sigma}_U G^T| \leq R\Big\}$$

and

$$\hat{\Sigma}_U = \frac{\ell}{n}\sum_{i=0}^{n/\ell-1}\boldsymbol{u}_i\boldsymbol{u}_i^T.$$