

Analogy Between Gambling and Measurement-Based Work Extraction

Dror A. Vinkler, Haim H. Permuter and Neri Merhav

Abstract

In information theory, one area of interest is gambling, where mutual information characterizes the maximal gain in wealth growth rate due to knowledge of side information; the betting strategy that achieves this maximum is named the Kelly strategy. In the field of physics, it was recently shown that mutual information can characterize the maximal amount of work that can be extracted from a single heat bath using measurement-based control protocols, i.e., using “information engines”. However, to the best of our knowledge, no relation between gambling and information engines has been presented before. In this paper, we briefly review the two concepts and then demonstrate an analogy between gambling, where bits are converted into wealth, and information engines, where bits representing measurements are converted into energy. From this analogy follows an extension of gambling to the continuous-valued case, which is shown to be useful for investments in currency exchange rates or in the stock market using options. Moreover, the analogy enables us to use well-known methods and results from one field to solve problems in the other. We present three such cases: maximum work extraction when the probability distributions governing the system and measurements are unknown, work extraction when some energy is lost in each cycle, e.g., due to friction, and an analysis of systems with memory. In all three cases, the analogy enables us to use known results in order to obtain new ones.

Index Terms

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D. A. Vinkler and H. H. Permuter are with the Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, 84105, Beer-Sheva, Israel (email: vinklerd@post.bgu.ac.il; haimp@bgu.ac.il). N. Merhav is with the Department of Electrical Engineering, Technion - Israel Institute of Technology, Technion City, Haifa 32000, Israel (email: merhav@ee.technion.ac.il).

Directed information, Gambling, Kelly betting, Maxwell's demon, Szilard Engine, Universal investment, Work extraction.

I. INTRODUCTION

While both work extraction from feedback controlled systems and information-theoretic analysis of gambling are old concepts, to the best of our knowledge the relation between them has not been highlighted before. This relation includes a straightforward mapping of concepts from one field to the other, e.g., measurements are analogous to side information and control protocols to betting strategies. Fundamental formulas in one field apply to the other after simple replacement of variables according to the derived mapping. This allows us to gain insights with regard to one field from known results from the other one.

The relationship between work extraction and information was first suggested by Maxwell [1] in a thought experiment consisting of an intelligent agent, later named Maxwell's demon. The agent measures the velocity of gas molecules in a box that is divided into two parts by a barrier. Although both parts have the same temperature to begin with, the molecules inside the box have different velocities. The demon opens a small hole in the barrier only when a faster-than-average molecule arrives from the left part of the box, allowing it to pass to the right part, and when a slower-than-average molecule arrive from the right part of the box, allowing it to pass to the left part. By doing this, the demon causes molecules of higher energy to concentrate in the right part of the box and those of lower energy to concentrate in the left part. This causes the right part to heat up and the left part to cool down, thus enabling work extraction when the system returns to equilibrium in apparent contradiction to the second law of thermodynamics. This experiment shows how information on the speed and location of individual molecules can be transformed into extracted energy, setting the basis for what is now known as "information engines".

Extensive research and debate has centered around Maxwell's demon since its inception, expanding the concept to more general cases of feedback control based on measurements, where work is extracted at the price of writing bits [2]–[6]. However, it was not until recently that Sagawa *et al.* reached an upper bound on the amount of work that can be extracted [7], [8], owing to the development of fluctuation theorems. That upper bound was found to be closely related to Shannon's mutual information, hinting at a possible relation to problems in information theory, a relation that has not yet been explored in full.

Gambling is another field where bits of information were given concrete value, through

the analysis of optimal gambling strategies using tools from information theory, an analysis that was first done by Kelly [9]. The setting consisted of consecutive bets on some random variable, where all the money won in the previous bet is invested in the current one. Kelly showed that maximizing over the expectation of the gambler's capital would lead to the loss of all capital with high probability after sufficiently many rounds. However, this problem is resolved when maximization is done over the expectation of the logarithm of the capital. Moreover, the logarithm of the capital is additive in consecutive bets, which means that the law of large numbers applies. Under these assumptions, the optimal betting strategy is to place bets proportional to the probability of each result, a strategy referred to as the "Kelly strategy". Kelly also showed that, given some side information on the event, the profit that can be made compared to the one with no side information is given by Shannon's mutual information. This serves as another hint at a possible relation between information engines and gambling, as the amount of work that can be extracted using measurements, compared to that which can be extracted without measurements, is also given by mutual information.

In this paper, we present an analogy between the analysis of feedback controlled physical systems and the analysis of gambling in information theory. We show that the optimal control protocol in various systems is analogous to the Kelly strategy, which is the optimal betting strategy. Furthermore, the amount of work extracted after n cycles of an information engine is shown to be analogous to the capital gained after n rounds of gambling. The analogy is then shown on two models: the Szilard Engine, where the particle's location is a discrete random variable, and a particle in some potential field, where the location can be a continuous random variable. The latter prompts us to consider an extension of Kelly gambling to cases with continuous-valued random variables, which is shown to be useful for investment in currency exchange rates or in the stock market using binary options.

This analogy enables us to develop a simple criterion to determine the best control protocol in cases where an optimal protocol is inapplicable and an optimal protocol when the probabilities governing the system are not known. Moreover, well known results for gambling with memory and causal knowledge of side information are applied in the field of physical systems with memory, yielding the bounds on extracted work in such systems.

The remainder of the paper is organized as follows: in Section II, we review the problem of horse race gambling, including the optimal betting strategy and maximal gain from side information. In Section III, we review the operation of the Szilard Engine, its optimal control protocol and maximal work extraction. Then, in Section IV, we present the analogy between

these two problems and discuss briefly the implications of such an analogy. In Section V, we review the mechanism for work extraction from a particle in an external potential field, and present the extension of Kelly gambling to cases with continuous-valued random variables which arises from that physical system. Finally, in Section VI, we discuss in more detail some of the implications and uses of the analogy.

II. THE HORSE RACE GAMBLING

The problem of gambling, as presented in [9] and [10], consists of n experiments whose results are denoted by the random vector $X^n = (X_1, \dots, X_n)$, e.g., the winning horse in n horse races. We are concerned with the case where the gambler has some side information about the races, denoted $Y^n = (Y_1, \dots, Y_n)$. The following notation is used:

- P_X - the probability vector of X , the winning horse.
- $P_{X,Y}$ - the joint probability of X and Y .
- $P_{X|Y}$ - the conditional probability of X given Y .
- $P_{X|y}$ - the probability vector of X given an observation $Y = y$ of the side information.
- $b_{X|Y}$ - the betting strategy on X given Y , describing the fraction of the gambler's capital invested in each horse.
- $b_{X|y}$ - a vector describing the betting strategy for $Y = y$.
- o_X - a vector describing the amount of money earned for each dollar invested in the winning horse, for each horse.
- S_n - the gambler's capital after n experiments.

$P_X(x)$, $P_{X,Y}(x, y)$ and $P_{X|Y}(x|y)$ denote the probability mass function (PMF) of X , the joint PMF of X and Y and the conditional PMF of X given Y , respectively, for the observations x and y . Similarly, $b_{X|Y}(x|y)$ and $o_X(x)$ denote the fraction of capital invested and odds, respectively, for $X = x$ and $Y = y$. Unless stated otherwise, we assume $\{(X_i, Y_i)\}_{i=1}^n$ are i.i.d, i.e., $P_{X^n, Y^n}(x^n, y^n) = \prod_{i=1}^n P_{X,Y}(x_i, y_i)$, and that the gambler invests all of his capital in each round.

Without loss of generality, we will set $S_0 = 1$, namely, the gambling starts with 1 dollar. S_n is then given by:

$$S_n = \prod_{i=1}^n b_{X|Y}(X_i|Y_i) o_X(X_i), \quad (1)$$

and maximization will be done on $\log S_n$. We define the profit at round i as

$$\log S_i - \log S_{i-1} = \log [b_{X|Y}(X_i|Y_i) o_X(X_i)]. \quad (2)$$

The wealth growth rate is defined as

$$\mathcal{W} = \frac{1}{n} E[\log S_n], \quad (3)$$

where the expectation is with respect to P_{X^n, Y^n} . The maximal wealth growth rate will be denoted as \mathcal{W}^* .

Since the races are assumed i.i.d., the same betting strategy will be used in every round, i.e., $b_{X_i|Y_i} = b_{X|Y}$ for all i . As shown in [10, Chapter 6], the optimal betting strategy is given by:

$$b_{X|Y}^* = \arg \max_{b_{X|Y}} E[\log S_n] = P_{X|Y}. \quad (4)$$

Substituting $b_{X|Y}^*$ into (1) yields the following formula for \mathcal{W}^* :

$$\mathcal{W}^* = \sum_{x,y} P_{X,Y}(x,y) \log [P_{X|Y}(x|y) o_X(x)]. \quad (5)$$

As defined in [9], the bet is “fair” if $o_X(x) = 1/P_X(x)$. It can be seen from (5) that without side information, no money can be earned in that case. For a fair bet where side information is available, (5) can be written as

$$\mathcal{W}^* = I(X; Y), \quad (6)$$

where $I(X; Y)$ is Shannon’s mutual information. In this paper, we are mostly concerned with fair bets. Another point of interest is a constrained bet, which is a fair bet where for each y $b_{X|y}$ is limited to some set \mathcal{B} of possible vectors. For such a bet, the maximal wealth growth rate is given by:

$$\begin{aligned} \mathcal{W}^* &= \max_{b_{X|Y} \in \mathcal{B}} \frac{1}{n} E[\log S_n] \\ &= I(X; Y) - \sum_{y \in \mathcal{Y}} P_Y(y) \min_{b_{X|y} \in \mathcal{B}} D(P_{X|y} || b_{X|y}), \end{aligned} \quad (7)$$

where $D(\cdot || \cdot)$ is the Kullback-Leibler divergence and for each y the optimal $b_{X|y} \in \mathcal{B}$ is the one that minimizes $D(P_{X|y} || b_{X|y})$.

III. THE SZILARD ENGINE

We now describe the Szilard Engine [11], which involves a single particle of an ideal gas enclosed in a box of volume V and attached to a heat bath of temperature T . The engine’s cycle consists of the following stages (see Fig. 1):

- 1) The particle moves freely in equilibrium with the heat bath.

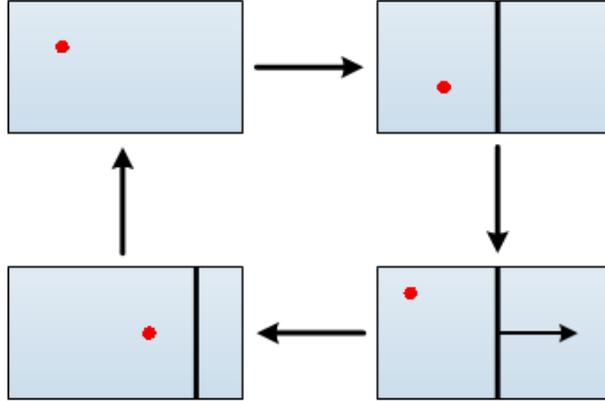


Fig. 1: The cycle of the Szilard Engine, starting at the upper left corner.

- 2) A divider is inserted, dividing the box into two parts of volumes V_0^L and V_0^R ($V_0^L + V_0^R = V$). The part of the box that contains the particle is denoted by X , with the alphabet $\mathcal{X} = \{L, R\}$.
- 3) A noisy measurement of the particle's location is made; the result is denoted Y with $\mathcal{Y} = \{L, R\}$.
- 4) Given $Y = y$, the divider is moved quasi-statically¹ until the volumes of the parts are set to the prescribed volumes $V_f^L(y)$ and $V_f^R(y)$.
- 5) The divider is removed from the box.

Denote the initial normalized volume as $v_0(x)$, which equals V_0^L/V for $x = L$ and V_0^R/V otherwise. Similarly, the final normalized volume $v_f(x|y)$ is equal to $V_f^L(y)/V$ for $x = L$ and $V_f^R(y)/V$ otherwise. Since the particle starts each cycle in equilibrium with its environment, different cycles of the engine are independent of each other. Moreover, since the particle has no potential energy, the particle's location has a uniform distribution across the volume of the box. Thus, assuming $v_0(x)$ to be the same for each cycle, X^n are i.i.d. with $P_X(x) = v_0(x)$. Following the analysis in [12], the work extracted for $Y = y$ is given by:

$$W = k_B T \ln \frac{v_f(X|Y)}{P_X(X)}, \quad (8)$$

where k_B is the Boltzmann constant. It was also shown in [12] that, for every $y \in \mathcal{Y}$, the optimal v_f is

$$v_f^*(\cdot|y) = \arg \max_{v_f} E[W|Y = y] = P_{X|Y}(\cdot|y), \quad (9)$$

¹Infinitesimally slowly, keeping the system close to equilibrium at all times.

TABLE I: Analogy of gambling and the Szilard Engine

Gambling	The Szilard Engine
X_i - result of horse race in round i .	X_i - location of the particle in cycle i . Namely, left or right.
Side information.	Measurement results, possibly with noise.
Y_i - some side information on round i .	Y_i - noisy measurement of the particle's location in cycle i .
P_X - probability vector of the result.	P_X - probability vector of the particle's location.
$P_{X y}$ - probability vector of the result given side information y .	$P_{X y}$ - probability vector of the particle's location given measurement y .
$o_X(x)$ - amount of money earned for every dollar gambled.	$1/v_0(x)$ - the reciprocal of the initial volume of the box's parts.
Placing bets on different horses.	Moving the dividers to their final positions.
Choosing the optimal race to bet on.	Choosing the optimal initial location for the divider.
$b_{X y}(x)$ - amount of money gambled on each result, given y .	$v_f(x y)$ - the normalized final volume of the box's parts, given y .
Logarithm of the capital.	Extracted work.
$\log S_n$ - log of the acquired money after n rounds of gambling.	$W_n/(k_B T)$ - total work extracted after n cycles of the engine.
Transforming bits to wealth.	Transforming bits to energy.
(2), (4), (6) - Profit in round i , optimal betting strategy and maximum profit.	(8), (9), (10) - Work extracted in round i , optimal control protocol and maximum work extraction.

and the maximal amount of work extracted after n cycles is

$$\begin{aligned} \max_{v_f} E[W_n] &= nk_B T E \left[\ln \frac{P_{X|Y}(X|Y)}{P_X(X)} \right] \\ &= nk_B T I(X; Y). \end{aligned} \quad (10)$$

Note that the initial location of the barrier $v_0(x)$ can also be optimized, leading to the following formula:

$$\max_{v_f, v_0} E[W_n] = nk_B T \max_{P_X} I(X; Y). \quad (11)$$

IV. ANALOGY

An analogy between the Szilard Engine and gambling arises from this analysis, as presented in Table I. The equations defining both problems, (2) and (8), are the same if one renames $b_{X|Y}$ as v_f and o_X as $1/P_X$. The analogy also holds for the optimal strategy in both problems, presented in (4) and (9), and the maximum gain, presented in (6) and (10), where $\log S_n$ is renamed $W_n/k_B T$.

Analogous to the way bits of side information are converted into wealth in gambling, bits of measurements are converted into work in the Szilard Engine. Moreover, the actions of the controller in the Szilard Engine are analogous to a gamble on the location of the particle. More specifically, in both problems the goal is to allocate a limited resource (box's volume, gambler's capital) in a way that maximizes the gain (extracted work, increase in capital).

Specifically, the Szilard Engine is analogous to a fair bet, since $v_0(x) = P_X(x)$ and this is analogous to $o_X(x) = 1/P_X(x)$. As stated previously, in a fair bet no money can be earned without side information. In an analogous manner, no work can be extracted from the Szilard Engine without measurements; this conforms with the second law of thermodynamics. However, the option to maximize over P_X in the Szilard Engine has no analogy in gambling as formulated in [9] and [10]. This prompts us to consider an extension to horse race gambling, where the gambler can choose between several different race tracks. This means that (6) can be maximized over all distributions $\{P_X\}$ across some set of distributions \mathcal{P} , yielding

$$\mathcal{W}^* = \max_{P_X \in \mathcal{P}} I(X; Y). \quad (12)$$

Using the analogy, the Szilard Engine can also be extended to a configuration with multiple dividers. The dividers are inserted simultaneously, dividing the box into m parts with the normalized volume of the i th part denoted by $v_0(i)$. The alphabet of X and Y is then given by $\mathcal{X} = \mathcal{Y} = \{1, \dots, m\}$. This setting is analogous to a horse race with m horses, where the optimal betting strategy is given by (4). Thus, the optimal control protocol for a measurement y will consist of moving the dividers quasi-statically until for every i the normalized volume of the i th part is $v_f^*(i|y) = P_{X|Y}(i|y)$. The work extracted by this scheme, when $v_0(i)$ is optimized, is given by:

$$\begin{aligned} \max_{v_0(i)} E[W_n] &= nk_B T \max_{P_X} \sum_{i,j=1}^m P_Y(i) P_{X|Y}(j|i) \ln \frac{v_f^*(j|i)}{v_0(j)} \\ &= nk_B T \max_{P_X} \sum_{i,j=1}^m P_Y(i) P_{X|Y}(j|i) \ln \frac{P_{X|Y}(j|i)}{P_X(j)} \\ &= nk_B T \max_{P_X} I(X; Y), \end{aligned} \quad (13)$$

i.e., the amount of work extracted is equal to the maximum given in [7], [8].

V. A PARTICLE IN AN EXTERNAL POTENTIAL FIELD AND CONTINUOUS-VALUED GAMBLING

In this section, we review the optimal control protocol for work extraction from a single particle in an external potential field. This case is similar to the Szilard Engine in that for both cases the optimal protocol depends solely on $P_{X|Y}$. Moreover, in both cases the protocol consists of making changes to the system that cause the probability distribution of X after the change to be equal to $P_{X|y}$, where y is the measurement result.

We show that the analogy derived in the previous section holds for this case as well. Since the location of the particle in this case can be a continuous random variable, it prompts us to consider an extension of Kelly gambling to scenarios with continuous random variables. This extension is then shown to describe investment in the stock market using options.

A. A Particle in an External Potential Field

We now consider a system of one particle that has the Hamiltonian (energy function):

$$H(x, p) = \frac{p^2}{2M} + \mathcal{E}_0(x), \quad (14)$$

where p is the particle's momentum, M is its mass, x is its location and $\mathcal{E}_0(x)$ is the potential energy. Again, the particle is kept at constant temperature T . The probability distribution of X is then the Boltzmann distribution that arises from $\mathcal{E}_0(x)$ and is denoted P_X . Namely,

$$P_X(x) = \frac{1}{Z_0} e^{-\mathcal{E}_0(x)/k_B T}, \quad (15)$$

where Z_0 is a normalization factor (a.k.a. the partition function) given by:

$$Z_0 = \sum_{x \in \mathcal{X}} e^{-\mathcal{E}_0(x)/k_B T}. \quad (16)$$

For now, we will limit ourselves to cases where $P_{X|y}$ is a Boltzmann distribution for every $y \in \mathcal{Y}$, a constraint that will be relaxed later on. This happens, for example, in the Gaussian case, where $X \sim \mathcal{N}(0, k_B T \sigma_X^2)$ and $Y = X + N$, where $N \sim \mathcal{N}(0, k_B T \sigma_N^2)$ and is independent of X , similarly to [13]. The optimal control protocol for such a system was found in [14] and [15] to be as follows:

- Based on the measurement y , $\mathcal{E}_0(x)$ is instantaneously modified by the controller to a different potential field. It then follows that the Boltzmann distribution of X changes from P_X to $Q_{X|y}$, which for every y is the probability distribution of X chosen by the

controller. The optimal final distribution, denoted $Q_{X|y}^*$, was shown in [14] and [15] to be equal to the conditional distribution of X given y , i.e., $Q_{X|y}^* = P_{X|y}$.

- The potential is changed back to $\mathcal{E}_0(x)$ quasi-statically.

Noting that v_f^* as presented in (9) is equal to $Q_{X|y}^*$, one notices that both in this case and in the Szilard Engine the optimal control protocol is defined by $P_{X|Y}$. Furthermore, (8) is also valid for this case, with v_f replaced by $Q_{X|y}$. If X is a continuous random variable, $P_X(x)$, $P_{X|Y}(x|y)$ and $Q_{X|y}(x)$ will be the particle's probability density function (PDF), conditional PDF and the PDF chosen by the controller, respectively.

The protocol presented above is optimal in the sense that it attains the upper bound on extracted work, i.e., the extracted work using this protocol with $Q_{X|y} = Q_{X|y}^*$ is given by:

$$E[W_n] = nk_B T I(X; Y). \quad (17)$$

If the controller controls $\mathcal{E}_0(x)$ as well, the expression in (17) can be maximized over all distributions $\{P_X\}$. However, it is important to note that there will always be some constraint over P_X , due to the finite volume of the system or due to the method of creating the external potential, or both. Thus, denoting by \mathcal{P} the set of allowed initial distributions P_X , the maximal amount of extracted work is given by:

$$E[W_n] = nk_B T \max_{P_X \in \mathcal{P}} I(X; Y). \quad (18)$$

Another point of interest is that setting $Q_{X|y}^* = P_{X|y}$ is not necessarily possible, e.g., if for some values of y , $P_{X|y}$ is not of the form

$$P_{X|y}(x) = \frac{1}{Z} e^{-f(x,y)/k_B T}, \quad (19)$$

and thus not a Boltzmann distribution. This gives rise to the following, more general, formula:

$$E[W_n] = nk_B T \max_{P_X \in \mathcal{P}} \left\{ I(X; Y) - \sum_{y \in \mathcal{Y}} P_Y(y) \min_{Q_{X|y} \in \mathcal{P}_B} D(P_{X|y} || Q_{X|y}) \right\}, \quad (20)$$

where \mathcal{P}_B is the set of all possible distributions P_X that stems from the set of all possible potentials. Thus, for every y , the optimal $Q_{X|y} \in \mathcal{P}_B$ is the one that minimizes $D(P_{X|y} || Q_{X|y})$. Notice that this analysis holds for both continuous and discrete random variables X, Y .

It follows that the analogy presented in Table I can be extended to work extraction from a particle in an external potential. Again, this system is analogous to a fair bet, in conformance with the second law of thermodynamics. This system is also analogous to a constrained bet, as can be seen from (20) and its analogy with (7). If X is continuous, an interesting extension

to the gambling problem arises where the bet is on continuous random variables. We will now present this extension in detail.

B. Continuous-Valued Gambling

We consider a bet on some continuous-valued random variable, where the gambler has knowledge of side information. The gambler's wealth is still given by (1), but the betting strategy, $b_{X|y}(x)$, and the odds, $o_X(x)$, are functions of the continuous variable x instead of vectors. In the case of stocks or currency exchange rates, for instance, such a betting strategy and odds can be implemented using options.

The constraint that the gambler invests all his capital in each round is translated in this case to the constraint

$$\int_{\mathcal{X}} b_{X|y}(x) dx = 1 \quad \forall y \in \mathcal{Y}. \quad (21)$$

The optimal betting strategy is then given by $b_{X|Y}^*(x|y) = f_{X|Y}(x|y)$, where $f_{X|Y}(x|y)$ is the conditional PDF of X given Y , and the bet is said to be fair if $o_X(x) = 1/f_X(x)$, where $f_X(x)$ is the PDF of X . For a fair bet, (6) holds and (7) holds with the sum replaced by an integral and each probability mass function replaced by the appropriate PDF.

As an example, consider the price of some stock or currency exchange rates, which are continuous-valued. In order to gamble using this model, *binary put options* are used. A binary put option is defined by its *strike price* - if the price of the stock at the expiry date, denoted X , is below the strike price, the option pays 1 dollar and otherwise it is worthless. The investment strategy will consist of selling a binary put with strike price K and buying a binary put with strike price $K + \Delta$ for some $\Delta > 0$; a combination denoted as *a spread* on $(K, K + \Delta]$, which yields 1 dollar if $x \in (K, K + \Delta]$ and is worthless otherwise. This strategy will be shown to conform with the model of horse race gambling on a continuous-valued random variable.

First, note that the price of a binary option depends on its strike price. We assume it is linear in the interval of a spread, which means that the price of a spread on $(K, K + \Delta]$ is $C_K \Delta$, where C_K is the slope of the option's price in that interval. This assumption is valid for sufficiently small values of Δ . It follows that the gain per dollar for such a spread is

$$o_{K,\Delta} = \frac{1}{C_K \Delta}. \quad (22)$$

In order to invest, divide the x axis into N intervals $(K_j, K_j + \Delta_j]$, $j \in \{1, \dots, N\}$, where $K_{j+1} = K_j + \Delta_j$ and N can be arbitrarily large. For each interval j , denote by $\varphi_j(y)$ the

fraction of the capital invested in buying spreads on that interval, given side information y . The betting strategy $b_{X|Y}$ is then set as

$$b_{X|Y}(x|y) = \sum_{j=1}^N \frac{\varphi_j(y)}{\Delta_j} \mathbb{1}_{(K_j, K_j + \Delta_j]}(x), \quad (23)$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. Then, for all $y \in \mathcal{Y}$, the constraint on $b_{X|y}$ is

$$\int_{-\infty}^{\infty} b_{X|y}(x) dx = \sum_{j=1}^N \varphi_j(y) = 1, \quad (24)$$

which is the same as the constraint in (21). Similarly, $o_X(x)$ is the piecewise constant function

$$o_X(x) = \sum_{j=1}^N \Delta_j \cdot o_{K_j, \Delta_j} \cdot \mathbb{1}_{(K_j, K_j + \Delta_j]}(x). \quad (25)$$

The capital at the end of the i th round is then given by the capital invested in the spread containing x_i times the gain per dollar for this spread, i.e.,

$$\begin{aligned} S_i &= \sum_{j=1}^N \varphi_j(y_i) \cdot S_{i-1} \cdot o_{K_j, \Delta_j} \cdot \mathbb{1}_{(K_j, K_j + \Delta_j]}(X_i) \\ &= b_{X|Y}(X_i|y_i) o_X(X_i) S_{i-1}. \end{aligned} \quad (26)$$

It follows that (1) holds. The analogous case to the maximization on P_X in (18), in this case, is choosing the options to invest in.

For $\Delta \rightarrow 0$, the betting strategy $b_{X|y}(x)$ is not necessarily piecewise constant. Denoting the slope of the option's price as $C(x)$, (25) is then rewritten as

$$o_X(x) = \frac{1}{C(x)}, \quad (27)$$

which is also not piecewise constant. It follows that, in this case, the bet is fair if the price of an option with striking price x is $\Pr(X \leq x)$. Also note that, in this limit, the term $I(X; Y)$ in (6) is generally not bounded, as is the case in (17) when the particle's location is not discrete. This means that the gain from knowledge of a stock's exact price at a future date is unlimited, similar to the unlimited work extracted from knowledge of the particle's exact location.

We conclude that two often-discussed schemes of work extraction are analogous to the well-known problem of horse race gambling or to the extension of that problem to the continuous-valued case, an extension that actually arose from the analogy. We will now discuss some of the possible benefits from this analogy.

VI. CONSEQUENCES OF THE ANALOGY

The analogy that was shown in this paper enables us to use well-known methods and results from horse race gambling to solve problems regarding measurement-based work extraction, and vice versa. Two such cases have already been shown: the Szilard Engine with multiple dividers and continuous-valued gambling. In this section, we present three more problems solved using the analogy: maximum work extraction when the joint distribution of X and Y is unknown, work extraction when some energy is lost in each cycle, e.g., due to friction, and an analysis of systems with memory. In all three cases, the analogy enables us to use known results to gain new insight.

In this section, we assume the control protocol is defined by a probability distribution $Q_{X|y}$, chosen by the controller. Since for every choice of $v_f(x|y)$, which defines the control protocol in the Szilard Engine, the following holds

$$\begin{aligned} v_f(x|y) &\in [0, 1] \quad \forall x \in \mathcal{X} \\ \sum_{x \in \mathcal{X}} v_f(x|y) &= 1, \end{aligned} \quad (28)$$

and since (8) holds for both problems, the analysis done henceforth for $Q_{X|y}$ is applicable for $v_f(x|y)$ as well.

A. Universal Work Extraction

In both control protocols presented so far, in order to achieve the upper bound of $E[W] = k_B TI(X; Y)$, it was necessary to know the conditional distribution $P_{X|Y}$ in advance. The question then arises whether this bound could also be achieved when the conditional probability is not known, e.g., a system with an unknown measurement error. For portfolio management, which is a generalization of horse race gambling, the problem of investment with unknown probability distributions was solved by Cover and Ordentlich [16]. They devised the *μ -weighted universal portfolio with side information*, which was shown to asymptotically achieve the same wealth as the best constant betting strategy for any pair of sequences x^n, y^n . Namely, it was shown that

$$\lim_{n \rightarrow \infty} \max_{\mathbf{x}^n, \mathbf{y}^n} \frac{1}{n} \log \frac{S_n^*(\mathbf{x}^n | \mathbf{y}^n)}{\hat{S}_n(\mathbf{x}^n | \mathbf{y}^n)} = 0, \quad (29)$$

where \hat{S}_n is the wealth achieved by the universal portfolio and S_n^* is the maximal wealth that can be achieved by a constant portfolio, i.e., where $b_{X_i|y_i} = b_{X|y_i}^*$ for all i . Following the

notation in [16], \mathbf{x}_i will denote the price-relative vector at time i and $\hat{\mathbf{b}}_i(y^i, \mathbf{x}^{i-1})$ will denote the portfolio (vector of investments) at time i , given the causal knowledge of the gambler.

The universal portfolio was given by:

$$\hat{\mathbf{b}}_i(y^i, \mathbf{x}^{i-1}) = \frac{\int_{\mathcal{B}} \mathbf{b} S_{i-1}(\mathbf{b}|y_i) d\mu(\mathbf{b})}{\int_{\mathcal{B}} S_{i-1}(\mathbf{b}|y_i) d\mu(\mathbf{b})}, \quad (30)$$

where μ is a measure that can be chosen by the gambler under the constraint $\int_{\mathcal{B}} d\mu = 1$, \mathcal{B} is the set of all possible portfolios \mathbf{b} and $S_{i-1}(\mathbf{b}|y_i)$ is the wealth acquired using portfolio \mathbf{b} along the subsequence $\{j < i : y_j = y_i\}$, i.e.,

$$S_{i-1}(\mathbf{b}|y) = \prod_{j < i: y_j = y} \mathbf{b}^t \cdot \mathbf{x}_j. \quad (31)$$

Choosing μ to be the uniform (Dirichlet(1, ..., 1)) distribution, they also showed that the wealth achieved by the portfolio can be lower bounded by:

$$\log \hat{S}_n(\mathbf{x}^n | y^n) \geq \log S_n^*(\mathbf{x}^n | y^n) - k(m-1) \log(n+1), \quad (32)$$

where m is the length of vector \mathbf{x} and k is the cardinality of \mathcal{Y} .

We will now consider the case of horse race gambling. Denote $o_j = o_X(j)$, i.e., the odds of the j th horse, and, similarly, b_j denotes the j th component of \mathbf{b} , i.e., the fraction of the capital invested in the j th horse by some betting strategy \mathbf{b} . Then, (31) can be rewritten as

$$S_{i-1}(\mathbf{b}|y) = \prod_{j=1}^m (b_j o_j)^{n_i(j, y_i)}, \quad (33)$$

where $n_i(j, y_i)$ is the number of times X was observed to be j and Y was observed to be y_i before the i th cycle, i.e., $n_i(j, y) = |\{l : x_l = j, y_l = y, l < i\}|$. When μ is the uniform distribution, the universal portfolio is then reduced to the following betting strategy for the case of horse race gambling:

$$\begin{aligned} \hat{\mathbf{b}}_i(y^i, \mathbf{x}^{i-1}) &= \frac{\int_0^1 \int_0^{1-b_1} \cdots \int_0^{1-\sum_{j=1}^{m-2} b_j} \mathbf{b} \prod_{j=1}^m (b_j o_j)^{n_i(j, y_i)} db_1 db_2 \cdots db_{m-1}}{\int_0^1 \int_0^{1-b_1} \cdots \int_0^{1-\sum_{j=1}^{m-2} b_j} \prod_{j=1}^m (b_j o_j)^{n_i(j, y_i)} db_1 db_2 \cdots db_{m-1}} \\ &= \frac{\int_0^1 \int_0^{1-b_1} \cdots \int_0^{1-\sum_{j=1}^{m-2} b_j} (b_1, \dots, b_m) \prod_{j=1}^m b_j^{n_i(j, y_i)} db_1 db_2 \cdots db_{m-1}}{\int_0^1 \int_0^{1-b_1} \cdots \int_0^{1-\sum_{j=1}^{m-2} b_j} \prod_{j=1}^m b_j^{n_i(j, y_i)} db_1 db_2 \cdots db_{m-1}} \\ &= \left(\frac{n_i(1, y_i) + 1}{n_i(y_i) + m}, \dots, \frac{n_i(m, y_i) + 1}{n_i(y_i) + m} \right), \end{aligned} \quad (34)$$

where $n_i(y) = |\{l : y_l = y, l < i\}|$ is the number of times Y was observed to be y before the i th cycle and $b_m = 1 - \sum_{j=1}^{m-1} b_j$.

Using the analogy presented above, this universal portfolio can be adapted straightforwardly into a universal control protocol in cases where X has a finite alphabet. In this control protocol, $Q_{X_i|y^i, x^{i-1}}$ is given by the right-hand-side (RHS) of (34) and the extracted work is lower bounded by:

$$\hat{W}_n \geq W_n^* - k_B T k(m-1) \ln(n+1), \quad (35)$$

a bound that follows directly from (32), where m is the cardinality of \mathcal{X} and k is the same as before. Namely, the work extracted by this universal control protocol is asymptotically equal to the work extracted by the best constant control protocol, i.e., the control protocol in which $Q_{X_i|y_i} = Q_{X|y_i}^*$ for all i . However, this derivation is applicable only for finite alphabets.

B. Imperfect Work Extraction

Another outcome that arises from the analogy shown above is the analysis of an imperfect system of work extraction. Consider a system where some amount of energy $f(x)$ is lost in each cycle, e.g., due to friction. The work extracted in each cycle is then given by:

$$W = k_B T \ln \frac{Q_{X|Y}(X|Y)}{P_X(X)} - f(X). \quad (36)$$

This is analogous to an unfair bet with the odds

$$o_X(x) = \frac{1}{P_X(x)} e^{-f_T(x)}, \quad (37)$$

where $f_T(x) = f(x)/k_B T$ and T is an “unfairness” parameter.

As shown in [10, Chapter 6], if the gambler has to invest all the capital in each round, the optimal betting strategy is independent of $o_X(x)$, i.e., for the odds given in (37) the optimal betting strategy is still given by (4), which yields

$$E[\ln S_i - \ln S_{i-1} | Y_i = y_i] = D(P_{X|y_i} || P_X) - E[f_T(X_i) | Y_i = y_i]. \quad (38)$$

However, it may be the case that for some values of y the gambler should not gamble at all. Specifically, in rounds where $Y_i = y_i$ and $D(P_{X|y_i} || P_X) \leq E[f_T(X_i) | Y_i = y_i]$, gambling should be avoided.

In the same manner, the optimal control protocol for imperfect systems of work extraction is still given by:

$$Q_{X|y}^* = P_{X|y}, \quad (39)$$

but for some measurement results it may be preferable not to perform the cycle at all. Substituting (39) into (36) and taking the average w.r.t. $P_{X|y}$ yields

$$E[W|Y = y] = k_B T D(P_{X|y} || P_X) - E[f(X)|Y = y]. \quad (40)$$

Thus, the engine's cycle should be performed only if y_i satisfies $k_B T D(P_{X_i|y_i} || P_{X_i}) > E[f(X_i)|Y_i = y_i]$.

C. Systems With Memory

Finally, we would like to analyze cases where the different cycles of the engine, or different measurements, are not independent. Again, we use known results from the analysis of gambling on dependent horse races. The gain in wealth due to casual knowledge of side information, as shown in [17], is

$$E[\log S_n(X^n || Y^n)] - E[\log S_n(X^n)] = I(Y^n \rightarrow X^n). \quad (41)$$

The term $I(Y^n \rightarrow X^n) \triangleq \sum_{i=1}^n I(X_i; Y^i | X^{i-1})$ is the directed information from Y^n to X^n , as defined by Massey [18], and $S_n(X^n || Y^n)$ indicates the betting strategy at round i depends causally on previous results X^{i-1} and side-information Y^i . The optimal betting strategy in this case is given by $b_{X^n || Y^n}^*(x^n || y^n) = P_{X^n || Y^n}(x^n || y^n)$, where $P_{X^n || Y^n}(x^n || y^n) = \prod_{i=1}^n P_{X_i | Y^i, X^{i-1}}(x_i | y^i, x^{i-1})$ is the causal conditioning of X^n by Y^n , as defined by Kramer [19], [20].

Analogously, in a physical system of work extraction where different cycles are dependent, e.g., when the system does not reach equilibrium between cycles, the formulas presented so far are no longer valid. Instead, the controller's causal knowledge of previous states and measurements can be used, meaning the optimal control protocol is given by

$$\begin{aligned} & \arg \max_{Q_{X_i | y^i, x^{i-1}}} E[W_i | Y^i = y^i, X^{i-1} = x^{i-1}] \\ &= \arg \max_{Q_{X_i | y^i, x^{i-1}}} k_B T \sum_{i=1}^n E \left[\ln \frac{Q_{X_i | y^i, x^{i-1}}(X_i)}{P_X(X_i)} \middle| Y^i = y^i, X^{i-1} = x^{i-1} \right] \\ &= P_{X_i | y^i, x^{i-1}}, \end{aligned} \quad (42)$$

where for each (y^i, x^{i-1}) , $Q_{X_i | y^i, x^{i-1}}$ is some probability distribution of X chosen by the

controller. This means that maximal work extraction is given by:

$$\begin{aligned} E[W_n(X^n||Y^n)] &= k_B T \sum_{i=1}^n E \left[\ln \frac{P_{X_i|Y^i, X^{i-1}}(X_i|Y^i, X^{i-1})}{o_i(X^i)} \right] \\ &= -k_B T \sum_{i=1}^n [H(X_i|Y^i, X^{i-1}) - O_i], \end{aligned} \quad (43)$$

where $o_i(X^i)$ is some function of the current and previous states, $O_i = E[\ln o_i(X^i)]$ and the notation $W_n(X^n||Y^n)$ indicates that the control protocol at round i depends causally on previous states X^{i-1} and side-information Y^i . Without access to measurement results, which is equivalent to setting $Y_i = \emptyset$ for all i , the maximal work extraction is

$$\begin{aligned} E[W_n(X^n)] &= k_B T \sum_{i=1}^n E \left[\ln \frac{P_{X_i|X^{i-1}}(X_i|X^{i-1})}{o_i(X^i)} \right] \\ &= -k_B T \sum_{i=1}^n [H(X_i|X^{i-1}) - O_i]. \end{aligned} \quad (44)$$

Subtracting (44) from (43), the gain in work extraction due to causal knowledge of measurement results is

$$E[W_n(X^n||Y^n)] - E[W_n(X^n)] = k_B T I(Y^n \rightarrow X^n), \quad (45)$$

analogously to the horse race gamble, where the gain in wealth growth rate due to causal knowledge of side information is $I(Y^n \rightarrow X^n)$.

Example 1 Consider the Szilard Engine where the initial placement of the barrier in each cycle is done before the system reaches equilibrium. As a result, in each cycle the particle has a higher probability to be in the same part of the box that it was in in the previous one. Denote by p the probability that the particle moved from one part of the box to the other between cycles and q the probability of measurement error, i.e., $\Pr(X_i \neq X_{i-1}) = p$ and $\Pr(X_i \neq Y_i) = q$ for each i . Since the measurement device has no memory and only the previous location affects the current one, the system has the Markov properties $X_i - (X_{i-1}, Y_i) - (Y^{i-1}, X^{i-2})$, $X_i - X_{i-1} - X^{i-2}$ and $Y_i - X_i - (X^{i-1}, Y^{i-1})$. Equation

(45) can then be written explicitly as

$$\begin{aligned}
E[W_n(X^n||Y^n)] - E[W_n(X^n)] &= k_B T \sum_{i=1}^n I(X_i; Y^i | X^{i-1}) \\
&= k_B T \sum_{i=1}^n [H(X_i | X_{i-1}) - H(X_i | X_{i-1}, Y_i)] \\
&= k_B T \sum_{i=1}^n I(X_i; Y_i | X_{i-1}) \\
&= k_B T \sum_{i=1}^n [H(Y_i | X_{i-1}) - H(Y_i | X_i)] \\
&= k_B T n [H_b(p * q) - H_b(q)], \tag{46}
\end{aligned}$$

where $p * q = pq + (1 - p)(1 - q)$ and $H_b(x) = -x \ln x - (1 - x) \ln(1 - x)$. For this example, the analogous case is a horse race with two horses, where the probability that a horse would win two consecutive races is $1 - p$, and the gambler has side information on the outcome of the race with error probability q . In that case, the gain in wealth growth rate due to the side information is given by

$$\begin{aligned}
E[\log(S(X^n||Y^n))] - E[\log S(X^n)] &= I(Y^n \rightarrow X^n) \\
&= \sum_{i=1}^n I(X_i; Y^i | X^{i-1}) \\
&= n [H_b(p * q) - H_b(q)]. \tag{47}
\end{aligned}$$

Specifically, the particle remaining in the same part of the box is analogous to a horse winning two consecutive races, and the error in side information is analogous to a measurement error.

Example 2 Consider a system of work extraction where the position of the particle in each cycle is independent of previous cycles, but each measurement depends on previous ones, e.g., due to hysteresis. However, both for the Szilard Engine and for a particle in an external potential, the probability distribution at the i th cycle can be chosen by the controller. Thus, the controller can introduce a dependence of X_i on X^{i-1}, Y^{i-1} through the choice of P_{X_i} . Maximal work extraction is then given by

$$\begin{aligned}
E[W_n] &= k_B T \max_{\{P_{X_i|X^{i-1}, Y^{i-1}}\}_{i=1}^n \in \mathcal{P}} \sum_{i=1}^n E \left[\ln \frac{P_{X_i|Y^i, X^{i-1}}(X_i | Y^i, X^{i-1})}{P_{X_i|Y^{i-1}, X^{i-1}}(X_i | Y^{i-1}, X^{i-1})} \right] \\
&= k_B T \max_{\{P_{X_i|X^{i-1}, Y^{i-1}}\}_{i=1}^n \in \mathcal{P}} \sum_{i=1}^n I(X_i; Y_i | X^{i-1}, Y^{i-1}), \tag{48}
\end{aligned}$$

where \mathcal{P} is the set of possible distributions and $\{P_{Y_i|Y^{i-1}, X^i}\}_{i=1}^n$ is a constant of the measuring device.

If the same probability distribution of X is chosen for every i , X_i depends on X^{i-1}, Y^{i-1} only given Y_i . As an example, consider a case where the measurement device can either be in a "good" state, yielding error-free measurements, or a "bad" state, yielding measurements which are independent of X , and in each cycle it is likely to remain in the same state as it was in the previous cycle. Thus, even though X_i is independent of X^{i-1} and Y^{i-1} , previous states and measurements contain information on the state of the measurement device given Y_i . In such a case, X_i are i.i.d. and (48) is reduced to

$$\begin{aligned} E[W_n] &= k_B T \sum_{i=1}^n E \left[\ln \frac{P_{X_i|Y^i, X^{i-1}}(X_i|Y^i, X^{i-1})}{P_X(X_i)} \right] \\ &= k_B T I(Y^n \rightarrow X^n). \end{aligned} \quad (49)$$

This example is analogous to gambling where the gambler can choose in each round between several different race tracks, with different tracks independent of each other. In that case, in each round the probability distribution of the horses depends on the chosen track, which itself depends on the causal knowledge the gambler has. Thus, X_i depends on X^{i-1}, Y^{i-1} through the gambler's choice of P_{X_i} , i.e., if in round i track j was chosen, and the bet is fair, the odds at round i are

$$o_i(x_i) = \frac{1}{P_j(x_i)} = \frac{1}{P_{X_i|X^{i-1}, Y^{i-1}}(x_i|x^{i-1}, y^{i-1})}, \quad (50)$$

where $P_j(x_i)$ is the PMF corresponding to the j th track. The maximal wealth growth rate is then given by

$$\begin{aligned} \mathcal{W}^* &= \max_{\{P_{X_i|X^{i-1}, Y^{i-1}}\}_{i=1}^n \in \mathcal{P}} \frac{1}{n} \sum_{i=1}^n E \left[\ln \frac{P_{X_i|Y^i, X^{i-1}}(X_i|Y^i, X^{i-1})}{P_{X_i|Y^{i-1}, X^{i-1}}(X_i|Y^{i-1}, X^{i-1})} \right] \\ &= \max_{\{P_{X_i|X^{i-1}, Y^{i-1}}\}_{i=1}^n \in \mathcal{P}} \frac{1}{n} \sum_{i=1}^n I(X_i; Y_i|X^{i-1}, Y^{i-1}), \end{aligned} \quad (51)$$

where \mathcal{P} is the set of distributions of the different tracks. This equation is analogous to (48), and in both cases no gain is possible without knowledge of Y_i .

In Example 2, the amount of work that can be extracted is given by (48), which is a solution to a maximization problem. Hence, it would be beneficial if the maximized expression was concave.

Lemma 1 *Let $f(P_{X^n||Y^{n-1}}, P_{Y^n||X^n}) = \sum_{i=1}^n I(X_i; Y_i|X^{i-1}, Y^{i-1})$. Then f is concave in $P_{X^n||Y^{n-1}}$ with $P_{Y^n||X^n}$ constant.*

Proof: See the appendix. ■

Notice that a one-to-one mapping exists between $P_{X^n||Y^{n-1}}$ and $\{P_{X_i|X^{i-1}, Y^{i-1}}\}_{i=1}^n$ [21, Lemma 3]. It then follows from Lemma 1 that the maximization problem in (48) can be solved using the tools of convex optimization, if \mathcal{P} is convex. Alternatively, the alternating maximization procedure can be used to maximize over each term $P_{X_i|X^{i-1}, Y^{i-1}}$ separately while setting all other terms to be constant, beginning with $i = n$ and moving backward to $i = 1$, similarly to [22]. Since each term depends only on previous terms and not on the following ones, this procedure will yield the global maximum as needed.

VII. CONCLUSIONS

In this paper we have shown an analogy between the field of gambling in information theory and the analysis of information engines in statistical mechanics. This analogy consisted of a one-to-one mapping of concepts and equations between those two fields, which enabled us to use methods and results from one field to gain new insights in the other. Such insights included universal work extraction, continuous-valued gambling and information engines with memory, among others.

While in this paper we reviewed only two information engines, the analogy is valid for every engine where the optimal control protocol dictates a change of the Boltzmann distribution so that it is equal to $P_{X|y}$ for every measurement y . Analysis of other systems could yield further insight into this analogy and, through it, into gambling.

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APPENDIX

In this appendix we prove the concavity of $\sum_{i=1}^n I(X_i; Y_i|X^{i-1}, Y^{i-1})$ in $P_{X^n||Y^{n-1}}$ with $P_{Y^n||X^n}$ constant, where $P_{X^n||Y^{n-d}}$ is the causal conditioning given by:

$$P_{X^n||Y^{n-d}}(x^n||y^{n-d}) = \prod_{i=1}^n P_{X_i|X^{i-1}, Y^{i-d}}(x_i|x^{i-1}, y^{i-d}). \quad (52)$$

Namely, we would like to show that for any $\lambda \in [0, 1]$ and causal conditioning measures

$P_{X^n||Y^{n-1}}^1$ and $P_{X^n||Y^{n-1}}^2$,

$$\begin{aligned} f(\lambda P_{X^n||Y^{n-1}}^1 + \bar{\lambda} P_{X^n||Y^{n-1}}^2, P_{Y^n||X^n}) &\geq \lambda f(P_1(x^n||y^{n-1}), P_{Y^n||X^n}) \\ &\quad + \bar{\lambda} f(P_2(x^n||y^{n-1}), P_{Y^n||X^n}), \end{aligned} \quad (53)$$

where $\bar{\lambda} = 1 - \lambda$ and

$$f(P_{X^n||Y^{n-1}}^j, P_{Y^n||X^n}) = \sum_{i=1}^n I_j(X_i; Y_i | X^{i-1}, Y^{i-1}), \quad (54)$$

where $I_j(X_i; Y_i | X^{i-1}, Y^{i-1})$ is the mutual information induced by $P_{X^n||Y^{n-1}}^j$ for $j \in \{1, 2\}$.

Let $S \sim \mathcal{B}(\lambda)$. Denote

$$\begin{aligned} P_{X^n||Y^{n-1}}^1(x^n||y^{n-1}) &= \prod_{i=1}^n P_{X_i|X^{i-1}, Y^{i-1}, S}(x_i|x^{i-1}, y^{i-1}, 0) \\ P_{X^n||Y^{n-1}}^2(x^n||y^{n-1}) &= \prod_{i=1}^n P_{X_i|X^{i-1}, Y^{i-1}, S}(x_i|x^{i-1}, y^{i-1}, 1). \end{aligned} \quad (55)$$

It follows that for all i

$$\begin{aligned} I_1(X_i; Y_i | X^{i-1}, Y^{i-1}) &= I(X_i; Y_i | X^{i-1}, Y^{i-1}, S = 0) \\ I_2(X_i; Y_i | X^{i-1}, Y^{i-1}) &= I(X_i; Y_i | X^{i-1}, Y^{i-1}, S = 1). \end{aligned} \quad (56)$$

The RHS of (53) emerges from the following derivation:

$$\begin{aligned} \sum_{i=1}^n I(S, X_i; Y_i | X^{i-1}, Y^{i-1}) &\geq \sum_{i=1}^n I(X_i; Y_i | X^{i-1}, Y^{i-1}, S) \\ &= \sum_{i=1}^n P(S = 0) I(X_i; Y_i | X^{i-1}, Y^{i-1}, S = 0) \\ &\quad + \sum_{i=1}^n P(S = 1) I(X_i; Y_i | X^{i-1}, Y^{i-1}, S = 1) \\ &= \lambda \sum_{i=1}^n I_1(X_i; Y_i | X^{i-1}, Y^{i-1}) \\ &\quad + \bar{\lambda} \sum_{i=1}^n I_2(X_i; Y_i | X^{i-1}, Y^{i-1}) \\ &= \lambda f(P_{X^n||Y^{n-1}}^1, P_{Y^n||X^n}) \\ &\quad + \bar{\lambda} f(P_{X^n||Y^{n-1}}^2, P_{Y^n||X^n}). \end{aligned} \quad (57)$$

As for the LHS, notice that

$$\begin{aligned} \sum_{i=1}^n I(S, X_i; Y_i | X^{i-1}, Y^{i-1}) &= \sum_{i=1}^n I(X_i; Y_i | X^{i-1}, Y^{i-1}) + \sum_{i=1}^n I(S; Y_i | X^i, Y^{i-1}) \\ &\stackrel{(a)}{=} \sum_{i=1}^n I(X_i; Y_i | X^{i-1}, Y^{i-1}), \end{aligned} \quad (58)$$

where (a) follows from the fact that $P_{Y^n|X^n}$ is constant and thus the Markov property $Y_i - (X^i, Y^{i-1}) - S$ holds for all i . From (57) and (58), it follows that for any $\lambda \in [0, 1]$

$$\begin{aligned} \sum_{i=1}^n I(X_i; Y_i | X^{i-1}, Y^{i-1}) &\geq \lambda f(P_{X^n|Y^{n-1}}^1, P_{Y^n|X^n}) \\ &\quad + \bar{\lambda} f(P_{X^n|Y^{n-1}}^2, P_{Y^n|X^n}). \end{aligned} \quad (59)$$

In order to complete the proof of (53), it is necessary to show that the RHS of (58) is the LHS of (53), i.e., it is needed to show that

$$P_{X^n|Y^{n-1}} = \lambda P_{X^n|Y^{n-1}}^1 + \bar{\lambda} P_{X^n|Y^{n-1}}^2. \quad (60)$$

Lemma 2 For every pair of r.v. vectors $\{X^n, Y^n\}$ and r.v. S that satisfy the Markov property $Y_i - (X^i, Y^{i-1}) - S$,

$$P_{X^n|Y^{n-1}}(x^n || y^{n-1}) = \sum_s P_{S, X^n|Y^{n-1}}(s, x^n || y^{n-1}), \quad (61)$$

where $P_{S, X^n|Y^{n-1}} = P_{Z^{n+1}|Y^{n-1}}$ for $Z^{n+1} = \{S, X^n\}$.

Proof:

$$\begin{aligned} P_{X^n|Y^{n-1}}(x^n || y^{n-1}) &= \frac{P_{X^n, Y^n}(x^n, y^n)}{P_{Y^n|X^n}(y^n || x^n)} \\ &= \frac{\sum_s P_{S, X^n, Y^n}(s, x^n, y^n)}{P_{Y^n|X^n}(y^n || x^n)} \\ &\stackrel{(a)}{=} \frac{\sum_s P_{S, X^n|Y^{n-1}}(s, x^n || y^{n-1}) P_{Y^n|X^n}(y^n || x^n)}{P_{Y^n|X^n}(y^n || x^n)}, \end{aligned} \quad (62)$$

where (a) follows from the definition of $P_{S, X^n|Y^{n-1}}$, the probability chain rule and the Markov property. ■

Lemma 3 For every pair of r.v. vectors $\{X^n, Y^n\}$ and r.v. S ,

$$P_{S, X^n, Y^{n-1}}(s, x^n || y^{n-1}) = P_S(s) P_{X^n|S, Y^{n-1}}(x^n || s, y^{n-1}), \quad (63)$$

where $P_{X^n|S, Y^{n-1}} = P_{X^n|Z^n}$ for $Z^n = \{S, Y^{n-1}\}$.

Proof: The proof follows directly from the definitions of $P_{S, X^n || Y^{n-1}}$ and $P_{X^n || S, Y^{n-1}}$. ■

Equation (60) emerges from the previous lemmas as follows:

$$\begin{aligned}
 P_{X^n || Y^{n-1}}(x^n || y^{n-1}) &= \sum_s P_S(s) P_{X^n || S, Y^{n-1}}(x^n || s, y^{n-1}) \\
 &= \lambda \prod_{i=1}^n P_{X_i | X^{i-1}, Y^{i-1}, S}(x_i | x^{i-1}, y^{i-1}, 0) \\
 &\quad + \bar{\lambda} \prod_{i=1}^n P_{X_i | X^{i-1}, Y^{i-1}, S}(x_i | x^{i-1}, y^{i-1}, 1) \\
 &= \lambda P_{X^n || Y^{n-1}}^1(x^n || y^{n-1}) + \bar{\lambda} P_{X^n || Y^{n-1}}^2(x^n || y^{n-1}). \tag{64}
 \end{aligned}$$

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