# Analogy Between Gambling and Measurement-Based Work Extraction

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***Abstract*—TO BE CONSIDERED FOR AN IEEE JACK KEIL WOLF ISIT STUDENT PAPER AWARD. In information theory, mutual information is a known bound on the gain in the growth rate due to knowledge of side information on a gambling result; the betting strategy that reaches that bound is named the Kelly criterion. In physics, it was recently shown that mutual information is also a bound on the amount of work that can be extracted from a single heat bath using measurement-based control protocols; extraction that is done using "Information Engines". However, to the best of our knowledge, no relation between these two fields has been presented before. In this paper, we briefly review the two fields and then show an analogy between gambling, where bits are converted to wealth, and information engines, where bits representing measurement results are converted to energy. This enables us to use well-known methods and results from one field to solve problems in the other. We present three such cases: maximal work extraction when the joint distribution of $X$ and $Y$ is unknown, work extraction when some energy is lost in each cycle, e.g., due to friction, and an analysis of systems with memory. In all three cases, the analogy enables us to use known results to reach new ones.**

## I. Introduction

While both work extraction from feedback controlled systems and information theoretic analysis of gambling are old concepts, to the best of our knowledge the relation between them has not been highlighted before. This relation includes a straightforward mapping of concepts from one field to the other, e.g., measurements are analogous to side information and control protocols - to betting strategies. Fundamental formulas in either field apply to the other after simple replacement of variables according to the mapping found. This allows us to gain insights on one field from known results in the other one.

The relationship between work extraction and information was first suggested by Maxwell [1] in a thought experiment consisting of an intelligent agent, later named Maxwell's demon; the agent measures the velocity of ideal gas molecules in a box that is divided into two parts by a barrier. Although the box is attached to a heat bath and thus has a constant temperature, $T$, the molecules inside the box have different velocities. The demon opens a small hole in the barrier only when a faster-than-average molecule arrives from the left part of the box, allowing it to pass to the right part, and when slower-than-average molecules arrive from the right part of the box. By doing this, the demon causes molecules of higher energy to concentrate in the right part of the box and those of lower energy - to concentrate in the left part. This causes the right part to heat up and the left part to cool down, thus enabling work extraction when the system returns to equilibrium, in apparent contradiction to the second law of thermodynamics. This experiment shows how information on the speed and location of individual molecules can be transformed to extracted energy, setting the basis for what is now known as "Information Engines".

Extensive research and debate has centered around Maxwell's demon since its inception, expanding the concept to more general cases of feedback control based on measurements where work is extracted at the price of writing bits [2]–[6]. However, it was not until recently that Sagawa *et al.* reached an upper bound on the amount of work that can be extracted [7], [8], owing to the development of fluctuation theorems. That upper bound was found to be closely related to Shannon's mutual information, hinting at a possible relation to problems in information theory; a relation that was not yet explored in full.

Another field where bits of information were given concrete value is gambling, through the analysis of optimal gambling strategies using tools from information theory; an analysis that was first done by Kelly [9]. The setting consisted of consecutive bets on some random variable, where all the money won in the previous bet is invested in the current one. Kelly showed that maximizing over the expectation of the gambler's capital would lead to the loss of all capital with high probability after sufficiently many rounds. However, this problem is solved when maximization is done over the expectation of the capital's logarithm. Moreover, the logarithm of the capital is additive in consecutive bets, which means that the law of large numbers applies. Under these assumptions, the optimal betting strategy is to place bets proportional to the probability of each result, a strategy dubbed the "Kelly criterion". Kelly also showed that given some side information on the event, the profit that can be made compared to someone with no side information is given by Shannon's mutual information. This serves as another hint at a possible relation between information engines and gambling, as the amount of work that can be extracted using measurements, compared to that which can be extracted without measurements, is also given by mutual information.

In this paper, we present an analogy between the analysis of feedback controlled systems and the analysis of gambling in information theory. We show that finding the optimal control protocol in various systems is analogous to finding the optimal betting strategy using the Kelly criterion. Furthermore, the amount of work extracted after $n$ cycles of an information engine is shown to be analogous to the capital gained after $n$ rounds of gambling. The analogy is then shown on two models: the Szilard engine, where the particle's location is discrete, and a particle in some potential field, where the location is continuous.

This analogy allows us to generalize the models presented here to more elaborate cases, such as gambling on continuous-valued random variables. Moreover, it enables us to develop a simple criterion to determine the best control protocol in cases where an optimal protocol is inapplicable, and an optimal protocol when the probabilities governing the system are not known. Finally, well known results for gambling with memory and causal knowledge of side information are transferred to the field of physical systems with memory, yielding the bounds on extracted work in such systems.

Due to space limitations we omit proofs of Lemmas, which will appear in the full paper [10].

## II. The Horse Race Gambling

The problem of gambling, as presented in [9] and [11], consists of $n$ experiments whose results are marked by the random vector $X^n$, e.g., the winning horse in $n$ horse races. We will assume that the gambler has some side information, $Y^n$, about the races, and that the experiments and side information are i.i.d. The following notation is used:

- $P_{X|y}$ - the probability vector of $X$, the winning horse, given an observation $y$ of the side information.
- $b_{X|y}$ - a vector describing the amount of money invested in each result given $y$.
- $o_X$ - a vector describing the amount of money earned for each dollar invested on each horse, if that horse wins.
- $S_n$ - the gambler's capital after $n$ experiments.

$P_{X|y}(x|y)$ (which we will abbreviate as $P(x|y)$) marks the probability that $X = x$, given $y$. Similarly, $b_{X|y}(x|y)$ and $o_X(x)$ (abbreviated $b(x|y)$ and $o(x)$, respectively) mark the amount of money invested and earned, respectively, when $X = x$. Each round, the gambler invests all of his capital.

Without loss of generality, we will set $S_0 = 1$, namely, the gambling starts with 1 dollar. $S_n$ is then given by:

$$S_n = \prod_{i=1}^{n} b(X_i|y_i)o(X_i), \tag{1}$$

and maximization will be done on $\log S_n$. We define the profit in round $i$ as

$$\log S_i - \log S_{i-1} = \log\left[b(X_i|y_i)o(X_i)\right]. \tag{2}$$

Since the experiments are i.i.d., the same betting strategy $b_{X|y}$ will be used in every round. As shown in [11, Chapter 6], the optimal betting strategy is then given by:

$$b^*_{X|y} = \arg\max_{b_{X|y}} E[\log S_n|y^n] = P_{X|y}. \tag{3}$$

Substituting $b^*_{X|y}$ into eq. (1), the following maximum is derived:

$$\max_{b_{X|y}} E[\log S_n|y^n] = n\sum_{x} P(x|y)\log\left[P(x|y)o(x)\right]. \tag{4}$$

The bet is said to be fair if $o(x) = 1/P(x)$, and it can be seen from eq. (4) that without side information no money can be earned in that case. In this paper, we only consider fair bets.

For a fair bet, the expected value of $\log S_n$ with respect to $P(x^n, y^n)$ is

$$\max_{b_{X|y}} E[\log S_n] = nI(X;Y). \tag{5}$$

In a constrained bet, meaning a fair bet where the betting strategy is limited to some set $\mathcal{B}$ of possible strategies, the maximum gain will be given by

$$\max_{b_{X|y}\in\mathcal{B}} E[\log S_n] = nI(X;Y) - n\sum_{y\in\mathcal{Y}} P(y)D(P_{X|y}||b^*_{X|y}), \tag{6}$$

where the optimal betting strategy $b^*_{X|y} \in \mathcal{B}$ is the one that minimizes $D(P_{X|y}||b_{X|y})$.

## III. The Szilard Engine

We now examine the Szilard engine [12], which involves a single particle of an ideal gas enclosed in a box of volume $V$ and attached to a heat bath of temperature $T$. The engine's cycle consists of the following stages (see Fig. 1):

1) The particle is moving freely in equilibrium with the heat bath.
2) A divider is inserted, dividing the box into two parts of volumes $V_0^L$ and $V_0^R$. The part of the box that contains the particle is denoted by $X$, with the alphabet $\mathcal{X} = \{L, R\}$.
3) A noisy measurement of the particle's location is made; the result is denoted $Y$ with $\mathcal{Y} = \{L, R\}$.
4) The divider is moved quasi-statically[1], until the volumes of the parts are $V_f^L$ and $V_f^R$.
5) The divider is removed from the box.

Without loss of generality, we will set $V = 1$. Denote $V_0(x)$ as $V_0^L$ for $x = L$ and $V_0^R$ otherwise. Similarly, $V_f(x|y)$ is $V_f^L$ for $x = L$ and $V_f^R$ otherwise, and these values depend on the measurement $y$. Since the particle starts each cycle in equilibrium with its environment, different cycles of the engine are independent of each other. Assuming $V_0$ is the same for each cycle, $X^n$ are i.i.d. with $P_X = V_0$. Following the analysis in [13], the work extracted in round $i$ given $Y_i = y_i$ is

$$W_i = k_B T \ln \frac{V_f(X_i|y_i)}{P(X_i)}, \tag{7}$$

[1] Infinitesimally slowly, keeping the system close to equilibrium.

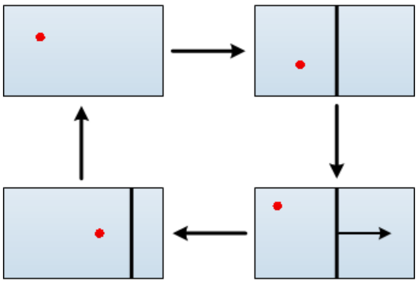

Fig. 1. The cycle of Szilard's engine.

TABLE I
ANALOGY OF GAMBLING AND SZILARD'S ENGINE

| **Gambling** | **Szilard's engine** |
|---|---|
| $X_i$ - result of horse race in round $i$. | $X_i$ - location of the particle in cycle $i$. Namely, left or right. |
| Side information. | Measurements results, possibly with noise. |
| $Y_i$ - some side information on round $i$. | $Y_i$ - noisy measurement of the particle's location in cycle $i$. |
| $P_X$ - PMF of the result. | $P_X$ - PMF of the particle's location. |
| $P_{X\|y}$ - PMF of the result given side information $y$. | $P_{X\|y}$ - PMF of the particle's location given measurement $y$. |
| $o_X$ - amount of money earned for every dollar gambled. | $1/V_0$ - the reciprocal of the initial volume of the box's parts. |
| Placing bets on different horses. | Moving the dividers to their final positions. |
| Choosing the optimal race to bet on. | Choosing the optimal initial location for the divider. |
| $b_{X\|y}$ - amount of money gambled on each result, given $y$. | $V_f(X\|y)$ - the final volume of the box's parts, given $y$. |
| Log of the capital. | Extracted work. |
| $\log S_n$ - log of the acquired money after $n$ rounds of gambling. | $W_n/(k_BT)$ - total work extracted after $n$ cycles of the engine. |
| Transforming bits to wealth. | Transforming bits to energy. |
| Eqs. (2), (3), (5) - Profit in round $i$, optimal betting strategy and maximum profit. | Eqs. (7), (8), (9) - Work extracted in round $i$, optimal control protocol and maximum work extraction. |

where $k_B$ is the Boltzmann constant. The optimal $V_f$ is

$$V_f^*(x|y) = \arg\max_{V_f} E[W|y] = P(x|y), \tag{8}$$

and the maximal amount of work extracted after $n$ cycles is

$$\begin{aligned}\max_{V_f} E[W_n] &= nk_BTE\left[\ln\frac{P_{X|Y}}{P_X}\right]\\ &= nk_BTI(X;Y).\end{aligned} \tag{9}$$

Note that the initial location of the barrier $V_0(x)$ can also be optimized, leading to the following formula

$$\max_{V_f,V_0} E[W_n] = nk_BT\max_{P(x)} I(X;Y). \tag{10}$$

An analogy with gambling arises from this analysis, as presented in Table I. The equations defining both problems, eqs. (2) and (7), are the same when renaming $b(X|y)$ as $V_f(X|y)$ and $o(X)$ as $1/P(X)$. The analogy also holds for the optimal strategy in both problems, presented in eqs. (3) and (8), and maximum gain, presented in eqs. (5) and (9), where $\log S_n$ is renamed $W_n/k_BT$.

Specifically, the Szilard engine is analogous to a fair bet, since $V_0 = P_X$ and this is analogous to $o(x) = 1/P(x)$. As stated previously, in a fair bet no money can be earned without side information. Equivalently, no work can be extracted from the Szilard engine without measurements, which conforms with the second law of thermodynamics. Moreover, the option to maximize over $P(x)$ prompts us to consider an extension to horse race gambling, where the gambler can choose between several different races and thus maximize eq. (5) over all distributions $P(x)$ in some set of possible distributions.

## IV. A Particle in an External Potential and Continuous-Valued Gambling

We now consider a system of one particle that has the Hamiltonian (energy function):

$$H(X,p) = \frac{p^2}{2M} + \mathcal{E}_0(X), \tag{11}$$

where $p$ is the particle's momentum, $M$ its mass, $X$ its location, and $\mathcal{E}_0(X)$ is some potential energy. Again, the particle is kept at constant temperature $T$. The optimal control protocol for this system was presented in [14] and [15] to be as follows:

- Given $y$, change the external potential immediately to be $\mathcal{E}_f^*(X,y)$ such that the induced Boltzmann distribution of $X$ will be $P_f^*(x|y) = P(x|y)$, i.e., equal to the conditional distribution of $X$ given $y$.
- Change the potential quasi-statically back to $\mathcal{E}_0(X)$.

Noting that in eq. (8) $V_f^*$ equals $P_f^*$, one notices that both in this case and in the Szilard engine the control protocol is defined by $P(x|y)$. Furthermore, eq. (7) is also valid for this case. If $X$ is a continuous random variable, $P(x), P(x|y)$ will be the particle's PDF and conditional PDF, respectively.

The protocol presented above is optimal in the sense that it reaches the upper bound on extracted work, i.e.,

$$E[W_n(P_f^*)] = nk_BTI(X;Y), \tag{12}$$

where $W_n$ is the extracted work after $n$ cycles of the engine. If $\mathcal{E}_0$ is under our control, we can maximize over all $P_0$ as well. However, it is important to note that there will always be some constraint over $P_0$, due to the finite volume of the system or to the method of creating the external potential or both. Thus, denoting by $\mathcal{P}$ the set of allowed initial distributions $P_0$, the maximal amount of extracted work is given by

$$\max_{P_0\in\mathcal{P},P_f} E[W_n] = nk_BT\max_{P(x)\in\mathcal{P}} I(X;Y). \tag{13}$$

Another point of interest is that setting $P_f^* = P_{X|y}$ will not necessarily be possible. This gives rise to the following, more general, formula

$$\begin{aligned}&\max_{P_0\in\mathcal{P},P_f} E[W_n]\\ &\quad= k_BT\max_{\substack{P(x)\in\mathcal{P}\\ P_f\in\mathcal{P}_B}}\{I(X;Y) - E_Y[D(P_{X|y}||P_f)]\},\end{aligned} \tag{14}$$

where $\mathcal{P}_B$ is the set of all possible distributions $P_f$, which stems from the set of all possible potentials. Thus, the optimal

$P_f$ is the one that minimizes $E_Y[D(P_{X|y}||P_f)]$. Notice that this analysis holds both for continuous and discrete $X$.

It follows that the analogy presented in Table I can be extended to work extraction from a particle in an external potential. Again, this system is analogous to a fair bet, in conformance with the second law of thermodynamics. This system is also analogous to a constrained bet, as can be seen from eq. (14) and its analogy with eq. (6). If $X$ is continuous, an interesting extension to the gambling problem arises where the bet is on continuous random variables. We will now present this extension in detail.

### A. Continuous-Valued Gambling

We consider a bet on some continuous-valued random variable, where the gambler has knowledge of side information. The gambler's wealth is still given by eq. (1), where the betting strategy, $b(X|y)$, and the odds, $o(X)$, are functions instead of vectors. In the case of stocks or currency exchange rates, for instance, such betting strategy and odds can be implemented using options. The constraint that the gambler invests all his capital on each round is translated in this case to the constraint $\int_{\mathcal{X}} b(x|y)dx = 1$. The optimal betting strategy is then given by $b^*(x|y) = f(x|y)$, where $f(x|y)$ is the conditional probability mass function (PMF) of $X$ given $y$, and the bet is said to be fair if $o(x) = 1/f(x)$, where $f(x)$ is the PMF of $X$. For a fair bet, eq. (5) holds and eq. (6) holds with the sum replaced by an integral and each PDF replaced by the appropriate PMF.

We conclude that two often discussed schemes of work extraction are analogous to the well-known problem of horse race gambling or to the extension of that problem to the continuous-valued case, an extension that actually arose from the analogy. We will now discuss some of the possible benefits from this analogy.

## V. Analogy Consequences

The analogy that was shown in this paper enables us to use well-known methods and results from horse race gambling to solve problems regarding measurement-based work extraction, and vice versa. We present three such cases: maximal work extraction when the joint distribution of $X$ and $Y$ is unknown, work extraction when some energy is lost in each cycle, e.g., due to friction, and an analysis of systems with memory. In all three cases the analogy enables us to use known results to gain new insight.

### A. Universal Work Extraction

The control protocols presented so far consisted of a change to the system that changed the distribution of $X$ from the vector $\mathbf{P_0}$ to some measurement-dependent vector $\mathbf{P_f}(\cdot|y)$, and then a return back to $\mathbf{P_0}$. However, in order to achieve the upper bound of $E[W] = k_B T I(X;Y)$, it was necessary to know $\mathbf{P_{X|y}}$ in advance. The question then arises whether this bound could be achieved when the conditional probability is not known, e.g., a system with an unknown measurement error. For portfolio management, which is a generalization of horse race gambling, the problem of investment with unknown probability distributions was solved by Cover and Ordentlich [16]. They devised the $\mu$-weighted universal portfolio with side information, which was shown to asymptotically achieve the same wealth as the best constant betting strategy for any pair of sequences $x^n, y^n$. Namely, it was shown that

$$\lim_{n\to\infty} \max_{x^n,y^n} \frac{1}{n} \log \frac{S_n^*(\mathbf{x}^n|y^n)}{\hat{S}_n(\mathbf{x}^n|y^n)} = 0, \tag{15}$$

where $\hat{S}_n$ is the wealth achieved by the universal portfolio and $S_n^*$ is the maximal wealth that can be achieved by a portfolio where $\mathbf{b}_i(y_i) = \mathbf{b}^*(y_i)$ for all $i$. Furthermore, choosing $\mu$ to be the uniform (Dirichlet$(1,\ldots,1)$) distribution, it was shown that the wealth achieved by the portfolio can be bounded by

$$\log \hat{S}_n(x^n|y^n) \geq \log S_n^*(x^n|y^n) - k(m-1)\log(n+1), \tag{16}$$

where $m$ is the cardinality of $\mathcal{X}$ and $k$ is the cardinality of $\mathcal{Y}$. For this $\mu$, the universal portfolio can be reduced to the following betting strategy for the horse race gamble:

$$\hat{\mathbf{b}}_\mathbf{i}(y^i, x^{i-1}) = \left(\frac{n_i(1,y_i)+1}{n_i(y_i)+m}, \ldots, \frac{n_i(m,y_i)+1}{n_i(y_i)+m}\right), \tag{17}$$

where $n_i(j,y_i)$ is the number of times $X$ was observed to be $j$ and $Y$ was observed to be $y_i$ before the $i$th cycle, i.e., $n_i(j,y) = |\{l : x_l = j, y_l = y, l < i\}|$, and similarly $n_i(y) = |\{l : y_l = y, l < i\}|$.

Using the analogy presented above, this universal portfolio can be adapted straightforwardly into a universal control protocol in cases where $X$ has a finite alphabet. In this control protocol, $\mathbf{P_{f,i}}$ is given by the right-hand-side of eq. (17) and the extracted work can be bounded by

$$\hat{W}_n \geq W_n^* - k_B T k(m-1)\ln(n+1), \tag{18}$$

a bound that follows directly from eq. (16). Namely, the work extracted by this universal control protocol is asymptotically equal to the work extracted by the best constant control protocol, i.e., the best control protocol for which $\mathbf{P_{f,i}}(\cdot|y_i) = \mathbf{P_f^*}(\cdot|y_i)$ for all $i$. However, this derivation is applicable only for cases where $X$ and $Y$ have finite alphabets.

### B. Imperfect Work Extraction

Another result that arises from the analogy shown above is the analysis of an imperfect system of work extraction. Consider a system where some amount of energy $f(x)$ is lost in each cycle, e.g., due to friction. I.e.,

$$W_i = k_B T \ln \frac{P_f(X_i|y_i)}{P_0(X_i)} - f(X_i). \tag{19}$$

This is analogous to an unfair bet with the odds

$$o(x) = \frac{1}{P(x)} e^{-f_T(x)}, \tag{20}$$

where $f_T(x) = f(x)/k_B T$ and $T$ is an "unfairness" parameter.

As shown, if the gambler has to invest all the capital in each round, the optimal $b(x|y)$ is independent of $o(x)$, i.e., for the odds given in eq. (20) the optimal betting strategy is still given by eq. (3). However, it may be the case that for some values of $y$ the gambler should not gamble at all.

In the same manner, the optimal control protocol for imperfect systems of work extraction is still given by $P_f^*(x|y) = P(x|y)$, but for some measurement results it may be preferable not to perform the cycle at all. Substituting $P_f^*$ into eq. (19) and taking the average w.r.t. $P(x|y)$ yields

$$W_i = k_B T D(P_{X|y_i}||P_X) - E[f(X_i)|y_i]. \qquad (21)$$

Thus, the engine's cycle should be performed only in cases where $W > 0$. Equivalently, the cycle should be performed only if $y$ satisfies $k_B T D(P_{X|y_i}||P_X) > E[f(X_i)|y_i]$.

### C. Systems With Memory

Finally, we would like to analyze cases where the different cycles of the engine, or different measurements, are not independent. Again, we would use known results from the analysis of gambling on dependent horse races. If the gambler has only causal knowledge of the side information, the maximum growth rate of wealth is [17]

$$\max_{b(X^n||Y^n)} E[\log S_n] = I(Y^n \to X^n), \qquad (22)$$

where $I(Y^n \to X^n)$ is the directed information from $Y^n$ to $X^n$, as defined by Massey [18], and $b(X^n||Y^n)$ indicates the betting strategy in round $i$ depends causally on previous results $X^{i-1}$ and side-information $Y^i$. The optimal betting strategy in this case is given by $b^*(x^n||y^n) = P(x^n||y^n)$, where $P(x^n||y^n) = \prod_{i=1}^n P(x_i|y^i, x^{i-1})$ is the causal conditioning of $X^n$ by $Y^n$ as defined by Karmer [19].

In the Szilard engine, dependence arises, for instance, if the initial placement of the barrier in each cycle is done before the system has reached equilibrium. In that case, the location of the particle depends on its location on the previous cycle, i.e., $P(x^n) \neq \prod_{i=1}^n P(x_i)$ and the Markov $X_i - X_{i-1} - (X^{i-2}, Y^{i-1})$ holds. This leads to the following formula for maximizing the extracted work

$$\begin{aligned}\arg\max_{V_{f,i}} E[W|y^i, x^{i-1}] &= \arg\min_{V_{f,i}} D(P_{X_i|y^i,x^{i-1}}||V_{f,i}) \\ &= P_{X_i|y^i,x^{i-1}}, \end{aligned} \qquad (23)$$

which means that maximal work extraction is given by

$$\max_{V_f^n} E[W_n] = k_B T I(Y^n \to X^n), \qquad (24)$$

for some $P(x^n)$ induced by the initial location of the barrier. As was done previously, this initial location can be optimized, yielding the optimal $P(x^n)$ in the set of possible distributions, i.e., the distributions for which the Markov property holds. Denoting this set $\mathcal{P}$, maximal extracted work is given by

$$\max_{V_f^n, V_0^n} E[W_n] = k_B T \max_{P(x^n||y^{n-1}) \in \mathcal{P}} I(Y^n \to X^n), \qquad (25)$$

where $P(y^n||x^n)$ is a constant depending on the measuring device. Due to the Markov property, eq. (25) can reduced to

$$\max_{V_f^n, V_0^n} E[W_n] = k_B T \max_{P(x^n||y^{n-1}) \in \mathcal{P}} \sum_{i=1}^n I(X_i; Y_i | X^{i-1}, Y^{i-1}). \qquad (26)$$

It would be beneficial to have a scheme to find the set of probabilities that achieves the maximum in this case. In order to do that, the following two lemmas are first needed.

**Lemma 1** *The rhs of eq. (26) is concave in $P(x^n||y^{n-1})$ with $P(y^n||x^n)$ constant.*

**Lemma 2** *The maximization problem in eq. (26) is a convex optimization problem over the affine set $\mathcal{P}$.*

Using these two lemmas, the alternating maximization procedure can be used to maximize over each term $P(x_i|x^{i-1}, y^{i-1})$ separately while setting all other terms as constant, beginning with $i = n$ and moving backward to $i = 1$, similarly to [20]. Since each term depends only on previous terms and not on the following ones, this procedure will yield the global maximum as needed.


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