Trading Off Weak–Noise Estimation Performance and Outage Exponents in Nonlinear Modulation

Neri Merhav

The Andrew & Erna Viterbi Faculty of Electrical Engineering

Technion—Israel Institute of Technology

Haifa 3200004, Israel

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The Problem

Consider the problem of conveying a parameter $u \in [0,1]$ across an AWGN channel,

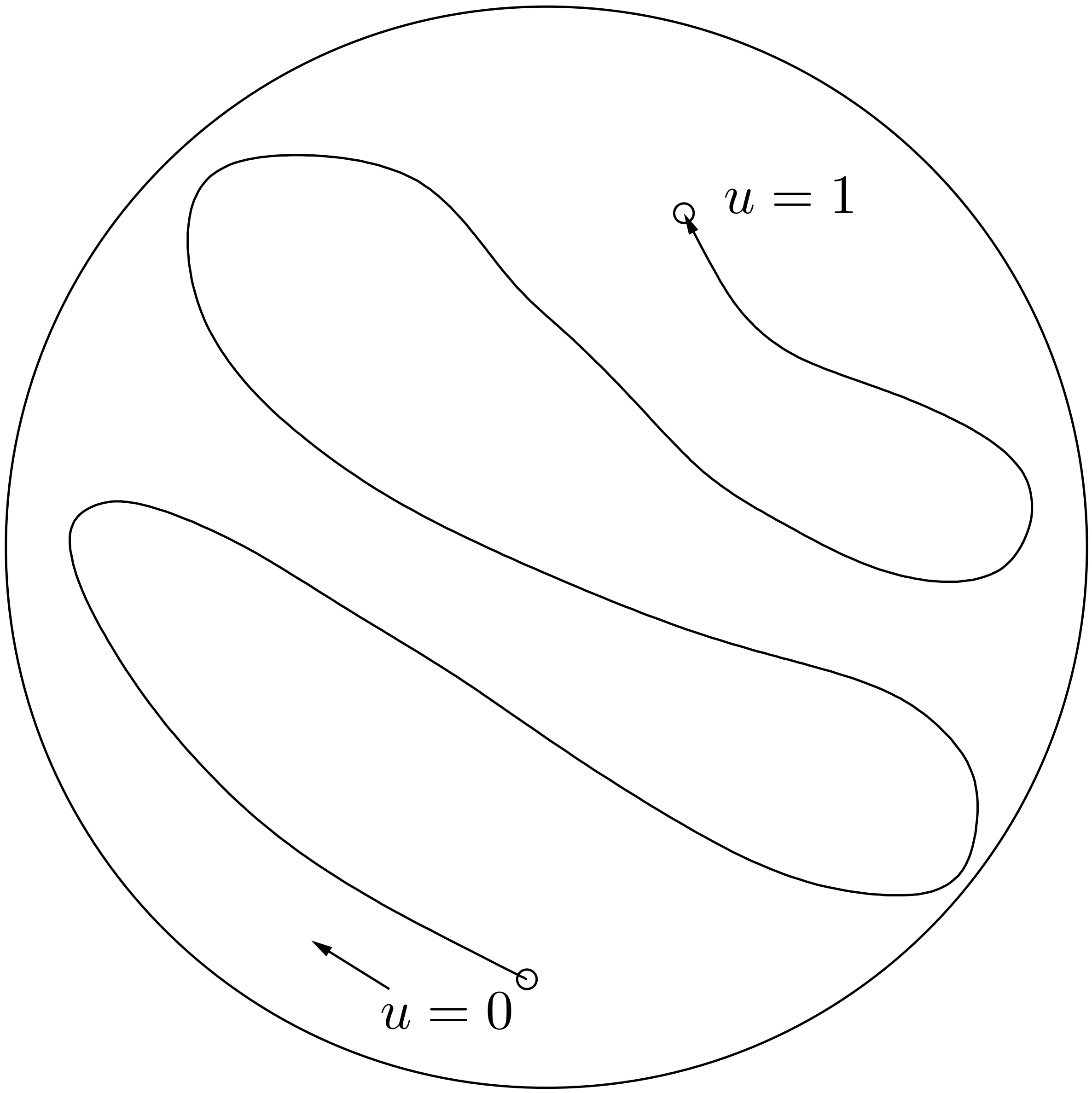
$$y_i = x_i + z_i, \qquad z_i \sim \mathcal{N}(0, \sigma^2) \text{ i.i.d.} \quad i = 1, 2, \ldots, n$$

where $\boldsymbol{x} = (x_1, \ldots, x_n) = f_n(u)$ is subject to a power constraint, $\|\boldsymbol{x}\|^2 \leq nP$.

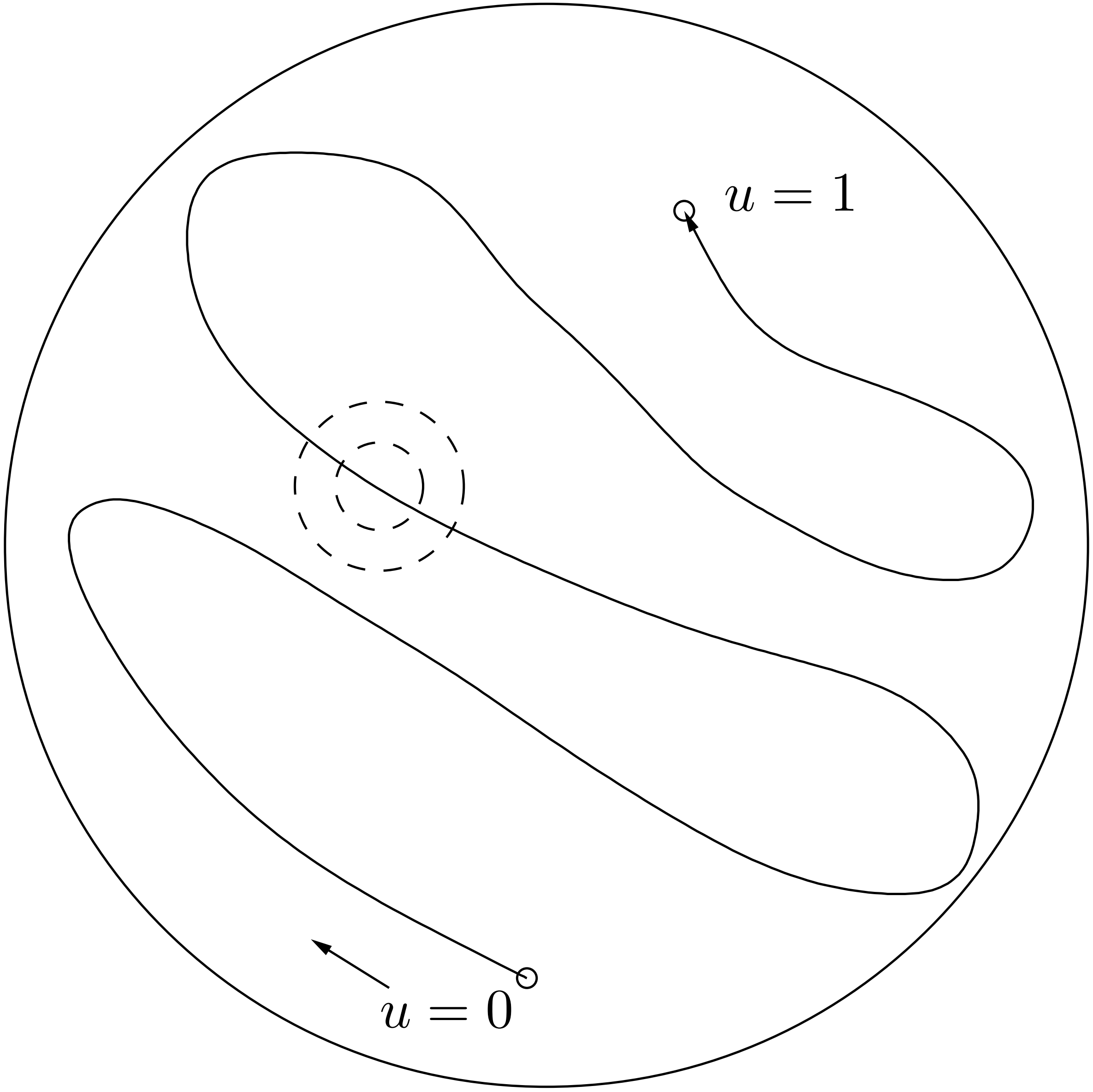
Q: How well can we estimate u if we can choose both f_n and $\hat{u} = g_n[\boldsymbol{y}]$?

- The "waveform communication" problem [Wozencraft & Jacobs 1965].
- Joint source–channel coding problem (Shannon–Kotel'nikov).
- Can be approached from an estimation–theoretic perspective.
- Linear modulation – Fisher–efficient but very limited.
- Nonlinear modulation – flexible but suffers a threshold effect.

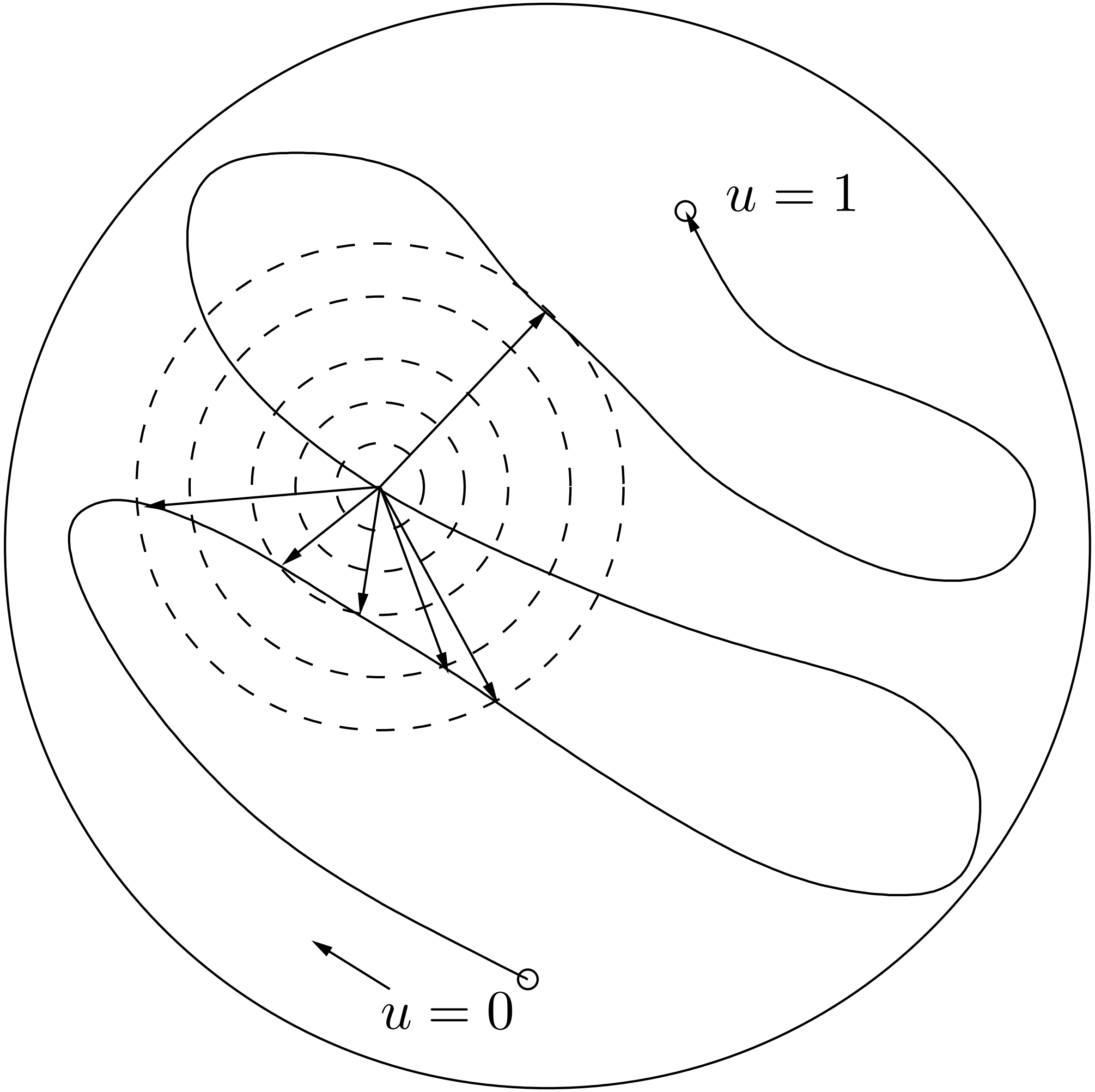
The Threshold Effect



The Threshold Effect



The Threshold Effect



Some Related Work

- Shannon–Kotel'nikov mappings ('49,'59).
- Bounds on the error moments: Cohn ('70); Burnashev ('84,'85).
- Hekland ('07); Floor ('08); Hekland, Floor & Ramstad ('09).
- ...
- Most of the literature: total MSE.
- Köken, Günduz & Tuncel ('17): separating weak–noise MSE and outage.
  - Minimize weak–noise MSE s.t. $\Pr\{\text{Outage}\} \leq \delta$.
  - Converse part – ordinary DPT: does not allow outage.

Formulation

Consider the model,

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{z} = f_n(u) + \boldsymbol{z}, \qquad u \in [0,1],\ \ \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n,\ \boldsymbol{Z} \sim \mathcal{N}(0, \sigma^2 I),$$

Transmitter (modulator): $\boldsymbol{x} = f_n(u)$; $\|\boldsymbol{x}\|^2 \le nP$.

Receiver: $\hat{u} = g_n(\boldsymbol{y})$.

Outage: $\forall\ u \in [0,1]$ define an outage event $\mathcal{O}_n(u) \subseteq \mathbb{R}^n$.

Given a constant $\lambda > 0$,

$$\min_{f_n, g_n, \mathcal{O}_n(\cdot)} \sup_{u \in [0,1]} \mathbf{E}\left\{ (\hat{u} - u)^2 \middle| \boldsymbol{Z} \in \mathcal{O}_n^{\mathsf{c}}(u) \right\}$$

$$\text{s.t.} \sup_{u \in [0,1]} \Pr\{\mathcal{O}_n(u)\} \le e^{-\lambda n}$$

- More precisely: max asymptotic exponential decay rate of the objective.
- In fact, any convex error cost function, $\rho(|\hat{u} - u|)$ can be allowed.

Converse Bound

Let $F = \{f_n(\cdot)\}_{n\geq 1}$, $G = \{g_n(\cdot)\}_{n\geq 1}$, and $\mathcal{O} = \{\mathcal{O}_n(\cdot)\}_{n\geq 1}$ (complying with the outage constraint) be given, and let

$$\mathcal{E}(F, G.\mathcal{O}) = \liminf_{n\geq 1} \left[-\frac{1}{n} \ln \left(\sup_u \mathbf{E} \left\{ (\hat{u} - u)^2 \middle| \boldsymbol{Z} \in \mathcal{O}_n^{\mathsf{c}}(u) \right\} \right) \right].$$

Let

$$q(\theta) \triangleq \frac{\theta - \ln(1+\theta)}{2},$$

and

$$E_{\mathsf{U}}(\lambda, \gamma) = \ln(\gamma) - \ln \left[1 + q^{-1}(\lambda) \right]; \quad \gamma = \frac{P}{\sigma^2}.$$

Then,

$$\limsup_{\gamma\to\infty} [\mathcal{E}(F, G, \mathcal{O}) - E_{\mathsf{U}}(\lambda, \gamma)] \leq 0.$$

Achievability

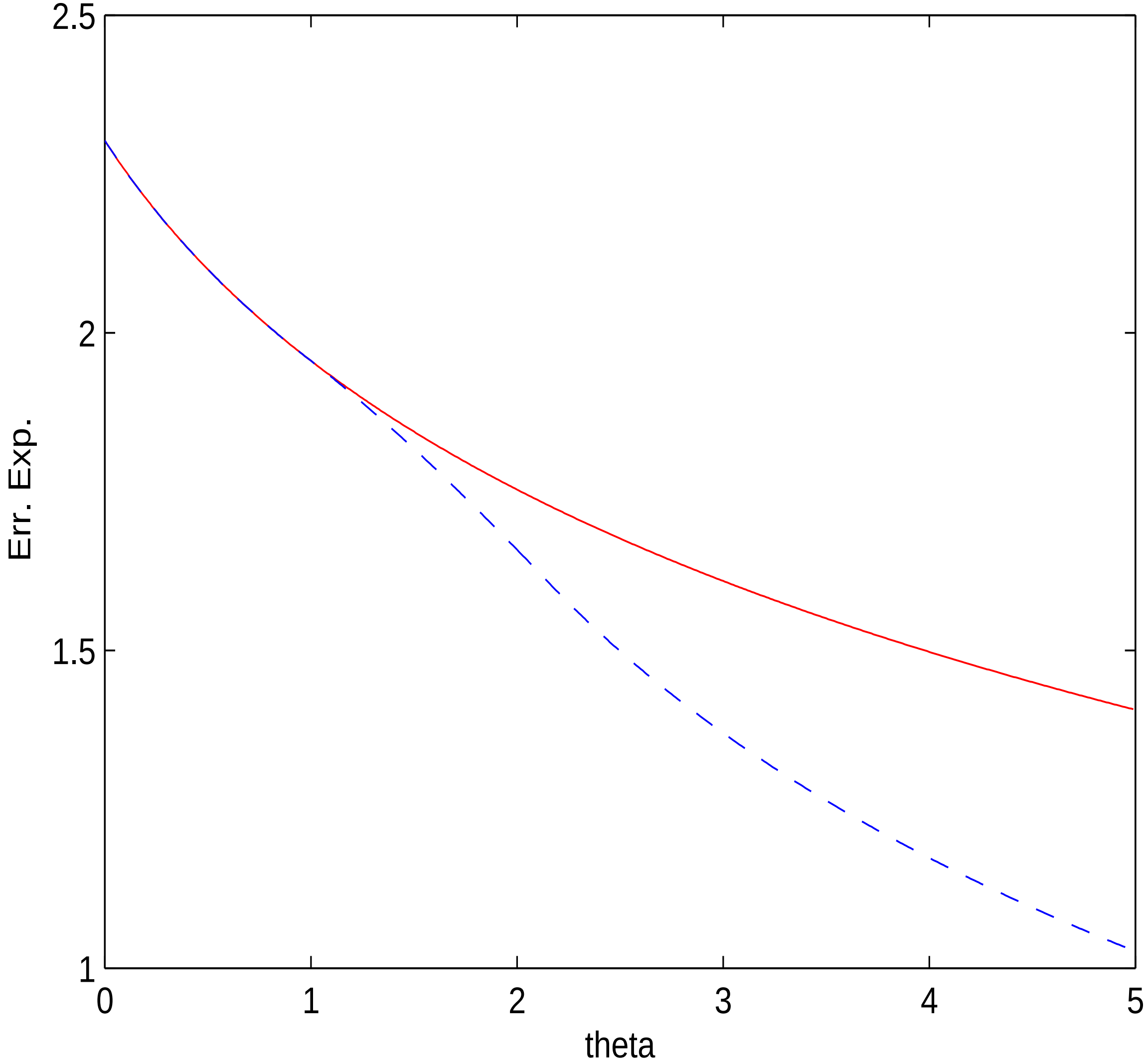
Let

$$E_{\mathsf{L}}(\lambda,\gamma) = \begin{cases} \ln\gamma - \ln\left[1+q^{-1}(\lambda)\right] & 0 \le \lambda \le \frac{1-\ln 2}{2} \\ \ln\left(\frac{e\gamma}{4}\right) - 2\lambda & \frac{1-\ln 2}{2} \le \lambda \le \frac{1}{2} \\ \ln\left(\frac{\gamma}{8}\right) - \ln\lambda & \lambda \ge \frac{1}{2} \end{cases}$$

Then, $\exists F, G, \mathcal{O}$ s.t.

$$\liminf_{\gamma\to\infty}[\mathcal{E}(F,G,\mathcal{O}) - E_{\mathsf{L}}(\lambda,\gamma)] \ge 0.$$

Graphs of $E_{\mathbf{U}}(\cdot, 100)$ and $E_{\mathbf{L}}(\cdot, 100)$



Ideas Behind the Converse

Define the length of the signal locus as

$$L(f_n) = \int_{f_n(0)}^{f_n(1)} \|\mathsf{d} f_n(u)\| = \int_0^1 \|\dot{f}_n(u)\|\mathsf{d}u.$$

We first show that $\forall$ communication system complying with the outage constraint, and $\forall\ M$,

$$\mathbf{E}\left\{(\hat{u}-u)^2 \middle| \boldsymbol{Z} \in \mathcal{O}_n^{\mathsf{c}}(u)\right\} \geq 2 \cdot \left(\frac{1}{2M}\right)^2 \cdot \left(Q\left[\frac{L(f_n)}{2\sigma M}\right] - e^{-\lambda n}\right).$$

We next derive an upper bound to $L(f_n)$ by using a tube–packing argument:

$$\mathsf{Vol}\{\mathcal{S}_n(\sqrt{nP})\} \geq \int \mathsf{d} f_n(u) \cdot \mathsf{Vol}\{\mathcal{O}_n^{\mathsf{c}}(u)\} \geq L(f_n) \cdot \min_u \mathsf{Vol}\{\mathcal{O}_n^{\mathsf{c}}(u)\}.$$

Ideas Behind the Converse (Cont'd)

$$\begin{aligned} L(f_n) &\leq \frac{\text{Vol}\{\mathcal{S}_n(\sqrt{nP})\}}{\min_u \text{Vol}\{\mathcal{O}_n^{\mathsf{c}}(u)\}} \leq \frac{\text{Vol}\{\mathcal{S}_n(\sqrt{nP})\}}{\text{Vol}\{\mathcal{S}_n(\sqrt{n\sigma^2[1+q^{-1}(\lambda)]}\}} \\ &= \left(\frac{\sqrt{nP}}{\sqrt{n\sigma^2[1+q^{-1}(\lambda)]}}\right)^n \\ &= \exp\left\{\frac{n}{2}\ln\frac{\gamma}{1+q^{-1}(\lambda)}\right\} \triangleq L_n^* \end{aligned}$$

as spheres minimize $\text{Vol}\{\mathcal{O}_n^{\mathsf{c}}(u)\}$ for a given $\Pr\{\mathcal{O}_n(u)\}$. The converse is obtained by

$$\mathbf{E}\left\{(\hat{u}-u)^2 \middle| \boldsymbol{Z} \in \mathcal{O}_n^{\mathsf{c}}(u)\right\} \geq 2 \cdot \left(\frac{1}{2M}\right)^2 \cdot \left(Q\left[\frac{L_n^*}{2\sigma M}\right] - e^{-\lambda n}\right),$$

and then setting $M \propto L_n^*$.

Achievability

Quantize u using a grid of L_n^* points mapped to a lattice code with Voronoi cells $\sim \mathcal{S}_n(\sqrt{n\sigma^2[1+q^{-1}(\lambda)]})$.

- Weak–noise estimation $=$ quantization error.
- Outage event $=$ probability of error of the lattice code.

The expression of $E_{\mathsf{L}}(\lambda, \gamma)$ comes from random–coding/expurgated exponents for well–known ensembles of lattice codes.

Future Directions

- Closing/shrinking the gap between $E_{\mathsf{U}}(\lambda, \gamma)$ and $E_{\mathsf{L}}(\lambda, \gamma)$.
- Deriving results for all γ, not just $\gamma \to \infty$.
- Estimation of a vector parameter.
- Colored Gaussian channel.
- Non–Gaussian noise.