Error Exponents of Typical Random Codes for the Colored Gaussian Channel

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Typical Random Codes

Traditional random coding error exponents are defined as

$$E_{\mathsf{r}}(R) = \lim_{n\to\infty}\left[-\frac{\ln \mathbf{E} P_{\mathsf{e}}(\mathcal{C}_n)}{n}\right].$$

We define typical–code error exponents as

$$E_{\mathsf{typ}}(R) = \lim_{n\to\infty}\left[-\frac{\mathbf{E}\ln P_{\mathsf{e}}(\mathcal{C}_n)}{n}\right].$$

- By Jensen's inequality, $E_{\mathsf{typ}}(R) \geq E_{\mathsf{r}}(R)$.
- $E_{\mathsf{r}}(R)$ – dominated by bad codes; $E_{\mathsf{typ}}(R)$ – dominated by typical codes.

Let $\mathcal{G}_E = \{\mathcal{C}_n : \quad P_{\mathsf{e}}(\mathcal{C}_n) \doteq e^{-nE}\}$.

$$\overline{P_{\mathsf{e}}(\mathcal{C}_n)} \doteq \sum_E P(\mathcal{G}_E)\cdot e^{-nE} \doteq P(\mathcal{G}_E^*)\cdot e^{-nE^*}.$$

Otoh, $E_{\mathsf{typ}}(R) = \sum_E P(\mathcal{G}_E)\cdot E = E_0$, where $P[\mathcal{G}_{E_0}] \to 1$.

Motivation

- $E_{\text{typ}}(R)$ is never worse than $E_{\text{r}}(R)$.
- Code selected once and for all: no LLN to support $\mathbf{E}P_{\text{e}}(\mathcal{C}_n)$.
- Once selected, w.h.p. $P_{\text{e}}(\mathcal{C}_n) \sim e^{-nE_0}$, forever.
- Theoretical framework for random–like codes (Battail, 1995).
- Analogy: physics of disordered sys. – quenched vs. annealed average.

Q: With all these motivations, why wasn't it explored much more before?

A: Not so easy to analyze (also in physics) ....

Related Work

- Barg & Forney ('02): i.i.d. random coding, BSC:

  At low rates: $E_{\text{typ}}(R) = E_{\text{ex}}(2R) + R$.

- Nazari ('11); Nazari, Anastasopoulos & Pradhan ('14):

  upper and lower bounds for the α–decoder.

- Stat. phys. literature: Kabashima ('08), Mora & Riviore ('06), ...:

  LDPC codes - replica analysis and cavity method.

- Battail ('95):

  random–like codes.

- Merhav ('18): fixed–composition rand. coding, DMC, gen. likelihood dec.

  Exact error exponent of the typical random code (TRC).

Contributions and Incentives

Contributions:

Deriving the error exponent of the TRC of the colored Gaussian channel, for:

- a class of generalized likelihood decoders;
- a general channel input spectrum;
- a general channel transfer function;
- a general noise spectrum.

Also: 0–rate exponent, range of tightness; comparison to RC; water–pouring.

Incentives:

- The great importance of the colored Gaussian channel.
- The generality of the above.
- Availability of closed–form expressions.

The AWGN

Random coding distribution:

$$Q(\boldsymbol{x}) = \begin{cases} \frac{1}{\text{Surf}(\sqrt{nP})} & \|\boldsymbol{x}\|^2 = nP \\ 0 & \text{elsewhere} \end{cases}$$

Likelihood decoder

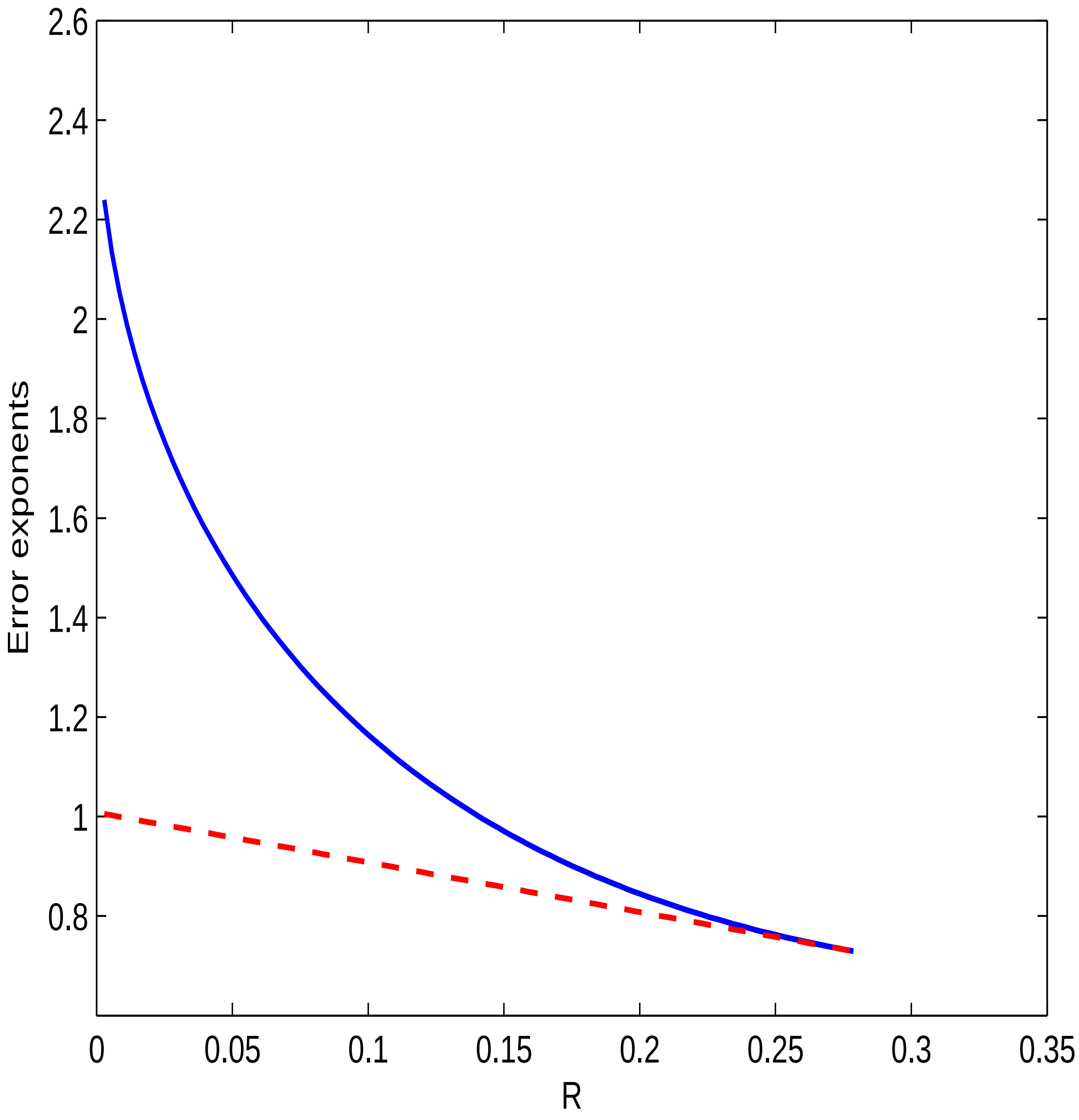
$$P_\beta(\hat{m} = m|\boldsymbol{y}) = \frac{\exp\{\beta \boldsymbol{x}^T[m] \cdot \boldsymbol{y}/\sigma^2\}}{\sum_{m'=0}^{M-1} \exp\{\beta \boldsymbol{x}^T[m'] \cdot \boldsymbol{y}/\sigma^2\}}.$$

$$E_{\text{trc}}(R) \geq \begin{cases} \frac{\text{snr}}{4}(1 - \sqrt{1 - e^{-4R}}) + R & R \leq R_* \\ \frac{\text{snr}}{4} - \frac{\text{snr}^2/8}{1+\sqrt{1+\text{snr}^2/4}} + 2R_* - R & R \geq R_* \end{cases}$$

where $\text{snr} = P/\sigma^2$ and

$$R_* = \frac{1}{4} \ln\left(\frac{1 + \sqrt{1 + \text{snr}^2/4}}{2}\right).$$

Graphs for snr $= 10$



Discussion

- For $R < R_*$, curvy part, starting at slope $-\infty$, ending at slope -1.
- $R = R_*$ designates a phase transition:
  - Below R_*: error dominated by subexponentially few codewords at distance $2nP(1 - \sqrt{1 - e^{-4R}})$.
  - Above R_* error dominated by $e^{2n(R-R_*)}$ codewords at distance $2nP(1 - \rho^*)$, where ρ^* depends only on snr.
- Straight–line part $=$ random coding exponent below the critical rate.
- TRC exponent exceeds RC exponent at least for $R < R_*$.
- There is some R_{t} below which the TRC exponent bound is tight.

The Colored Gaussian Channel

- Random coding: uniform on spheres in freq. bins according to $S_X(e^{j\omega})$.
- The channel: $Y_t = X_t + Z_t$, where the noise spectrum is $S_Z(e^{j\omega})$.
- Decoder: GLD that assumes the wrong noise spectrum, $\tilde{S}_Z(e^{j\omega})$.

Defining

$$\text{snr}(\omega) = \frac{S_X(e^{j\omega})}{S_Z(e^{j\omega})}; \qquad \mu(\omega) = \frac{S_Z(e^{j\omega})}{\tilde{S}_Z(e^{j\omega})}.$$

$$L(\omega) = \lambda\mu(\omega)[1 - \lambda\mu(\omega)]\text{snr}(\omega).$$

$$\begin{aligned} A(\omega, \theta, \lambda) \quad &= \quad L(\omega)\left(1 - \frac{2L(\omega)}{\sqrt{\theta^2 + 4L^2(\omega)} + \theta}\right) - \\ &\quad \frac{\theta}{2}\ln\left(\frac{2\theta}{\sqrt{\theta^2 + 4L^2(\omega)} + \theta}\right). \end{aligned}$$

Main Result

For the colored Gaussian channel,

$$E_{\text{trc}}(R) \geq \sup_{\theta \geq 1} \sup_{0 \lambda \leq \beta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} A(\omega, \theta, \lambda) \mathrm{d}\omega - (2\theta - 1)R \right\}.$$

Some comments:

- The matched case ($\mu(\omega) \equiv 1$): optimal $\lambda = \min\{\beta, 1/2\}$.
- TRC and random coding exponent coincide above

$$R_* = \frac{1}{8\pi} \int_0^{2\pi} \ln\left(\frac{1 + \sqrt{1 + 4L^2(\omega)}}{2} \right) \mathrm{d}\omega.$$

- There is an interval of small rates where the lower bound is tight.
- Zero–rate exponent

$$E_{\text{trc}}(0) = \frac{1}{8\pi} \cdot \frac{\left[\int_0^{2\pi} \mu(\omega) \cdot \mathrm{snr}(\omega) \mathrm{d}\omega \right]^2}{\int_0^{2\pi} \mu^2(\omega) \cdot \mathrm{snr}(\omega) \mathrm{d}\omega}.$$

Optimal Input Spectrum

Defining $K(\omega) = \lambda\mu(\omega)[1 - \lambda\mu(\omega)]$, the optimal spectrum is

$$S_X(\omega) = \frac{4\theta B[4BK(\omega) - S_Z(\omega)]_+}{4BK(\omega) + [4BK(\omega) - S_Z(\omega)]_+},$$

where B is chosen such that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{4B[4BK(\omega) - S_Z(\omega)]_+}{4BK(\omega) + [4BK(\omega) - S_Z(\omega)]_+} \cdot \mathrm{d}\omega = \frac{P}{\theta}.$$

Comments:

- The dependence on the rate is via the optimal θ.
- For $R \to 0$, $S_X(\omega)$ puts all the power on one frequency.
- In the matched case,

$$S_X(\omega) = \frac{4\theta B[B - S_Z(\omega)]_+}{B + [B - S_Z(\omega)]_+}.$$

  similarly to the solution for the expurgated exponent.

Some Final Remarks

- Main technique used: a Gaussian analogue of the method of types.
- There is a range of rates where the TRC error exponent bound is tight.
- Beyond R_* the TRC exponent coincides with the ordinary RC exponent.
- All results extend easily to continuous–time colored Gaussian channels.