Error Exponents of Typical Random Trellis Codes

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Typical Random Codes

Traditional random coding error exponents are defined as

$$E_{\mathsf{r}}(R) = \lim_{n\to\infty}\left[-\frac{\ln \mathbf{E} P_{\mathsf{e}}(\mathcal{C}_n)}{n}\right].$$

We define typical–code error exponents as

$$E_{\mathsf{typ}}(R) = \lim_{n\to\infty}\left[-\frac{\mathbf{E} \ln P_{\mathsf{e}}(\mathcal{C}_n)}{n}\right].$$

- By Jensen's inequality, $E_{\mathsf{typ}}(R) \geq E_{\mathsf{r}}(R)$.
- $E_{\mathsf{r}}(R)$ – dominated by bad codes; $E_{\mathsf{typ}}(R)$ – dominated by typical codes.

Let $\mathcal{G}_E = \{\mathcal{C}_n : \quad P_{\mathsf{e}}(\mathcal{C}_n) \doteq e^{-nE}\}$.

$$\overline{P_{\mathsf{e}}(\mathcal{C}_n)} \doteq \sum_E P(\mathcal{G}_E)\cdot e^{-nE} \doteq P(\mathcal{G}_E^*)\cdot e^{-nE^*}.$$

Otoh, $E_{\mathsf{typ}}(R) = \sum_E P(\mathcal{G}_E)\cdot E = E_0$, where $P[\mathcal{G}_{E_0}] \to 1$.

Motivation

- $E_{\text{typ}}(R)$ is never worse than $E_{\text{r}}(R)$.
- Code selected once and for all: no LLN to support $\mathbf{E}P_{\text{e}}(\mathcal{C}_n)$.
- Once selected, w.h.p. $P_{\text{e}}(\mathcal{C}_n) \sim e^{-nE_0}$, forever.
- Theoretical framework for random–like codes (Battail, 1995).
- Analogy: physics of disordered sys. – quenched vs. annealed average.

Q: With all these motivations, why wasn't it explored much more before?

A: Not so easy to analyze (also in physics) ....

Related Work

- Barg & Forney ('02): i.i.d. random coding, BSC:

  At low rates: $E_{\text{typ}}(R) = E_{\text{ex}}(2R) + R$.

- Nazari ('11); Nazari, Anastasopoulos & Pradhan ('14):

  upper and lower bounds for the α–decoder.

- Stat. phys. literature: Kabashima ('08), Mora & Riviore ('06), ...:

  LDPC codes - replica analysis and cavity method.

- Battail ('95):

  random–like codes.

- Merhav ('18): fixed–composition rand. coding, DMC, gen. likelihood dec.

  Exact error exponent of the typical random code (TRC).

Contributions

Deriving the error exponent of the typical random trellis code for:

- A general DMC (not just MBIOS).
- A general rational coding rate (not only $1/n$).
- A general random selection of a time–varying code.

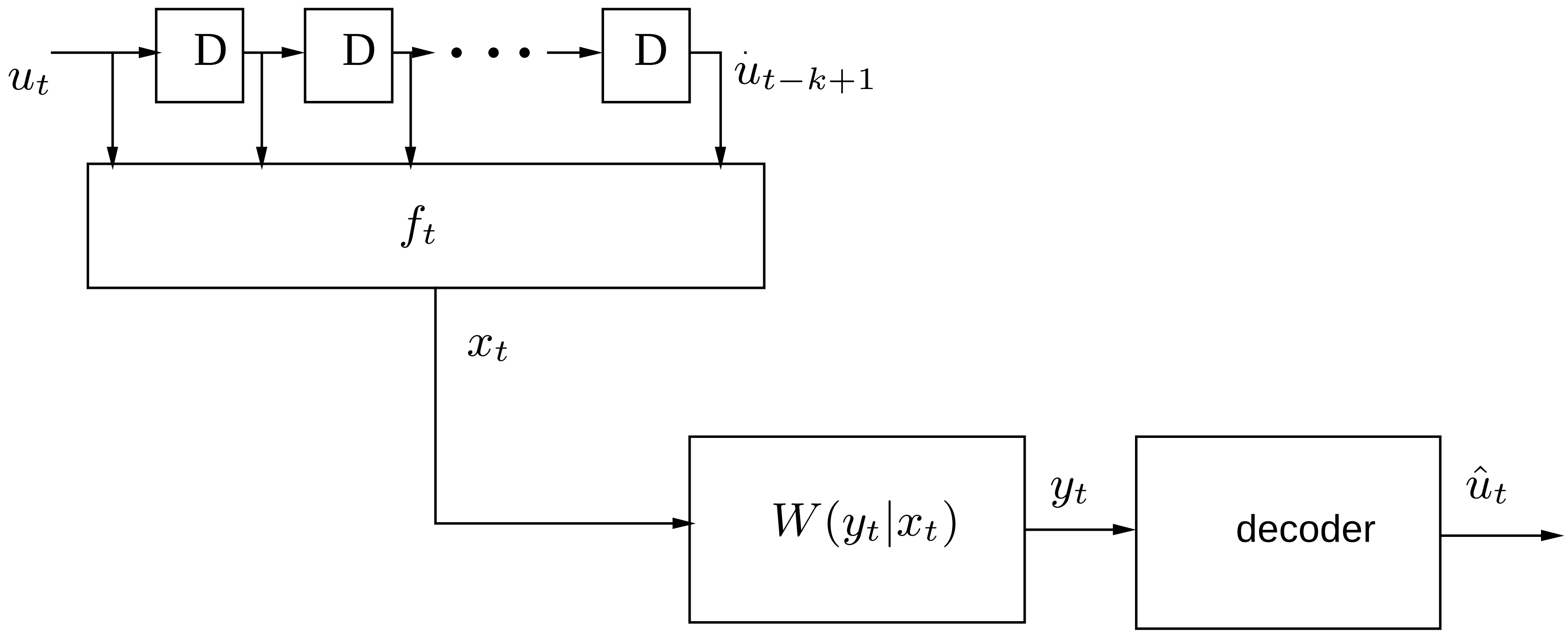
The analysis method (based on the MoT) provides insights on:

- Structure of the typical random trellis code.
- Dominant error events.

Additional extensions:

- Channels with finite input memory (ISI).
- Mismatched decoding metric.

Problem Setting



- $\{\boldsymbol{u}_t\}$ – m–vectors of purely random bits.
- $f_t : \{0,1\}^{mk} \to \mathcal{X}^n$ randomly selected according to Q^n.
- Coding rate: $R = m/n$; constraint length: $K = mk$.
- For convolutional codes, $\{f_t\}$ are linear.
- $W =$ DMC.
- Asymptotic regime: $k \to \infty$ while m and n are held fixed.

Background

Traditional performance metric: $E(R,Q) = \liminf_{K\to\infty}[-\log \mathbf{E}P_{\mathsf{e}}]/K$.

$$E(R,Q) = \begin{cases} R_0(Q)/R & R < R_0(Q) \\ E_0(\rho,Q)/R & R > R_0(Q) \end{cases}$$

where ρ satifies: $\rho R = E_0(R,Q)$.
For $R > R_0(Q)$: $\exists$ matching converse.
For $R < R_0(Q)$: improvement by an expurgated bound,

$$R_0(Q)/R \to E_{\mathsf{cex}}(R,Q) \stackrel{\triangle}{=} E_{\mathsf{x}}(\rho,Q)/R$$

with $\rho R = E_{\mathsf{x}}(\rho,Q)$.
In [Viterbi & Odenwalder, 1969]: for at least half of the convolutional codes

$$P_{\mathsf{e}} \leq \left(\frac{2L}{1-2^{-\epsilon/\rho R}}\right)^{\rho} \cdot \exp\{-KE_{\mathsf{cex}}(R,Q)\}.$$

If $2L$ is replaced by $100L$, the bound applies to 99% of the codes.

Objectives

Studying the typical ensemble performance,

$$\mathcal{E}_{\text{trtc}}(R,Q) = \liminf_{K\to\infty} \frac{-\mathbf{E}\log P_{\text{e}}}{K}$$

as well as $\mathcal{E}_{\text{trcc}}(R,Q)$ defined similarly for convolutional codes.

- Deriving both “Csiszár–style” and “Gallager–style” expressions.
- Comparing to the random coding exponent and expurgated exponent.
- Comparing to typical random block codes of the same complexity.

Main Result

For $R < R_0(Q)$:

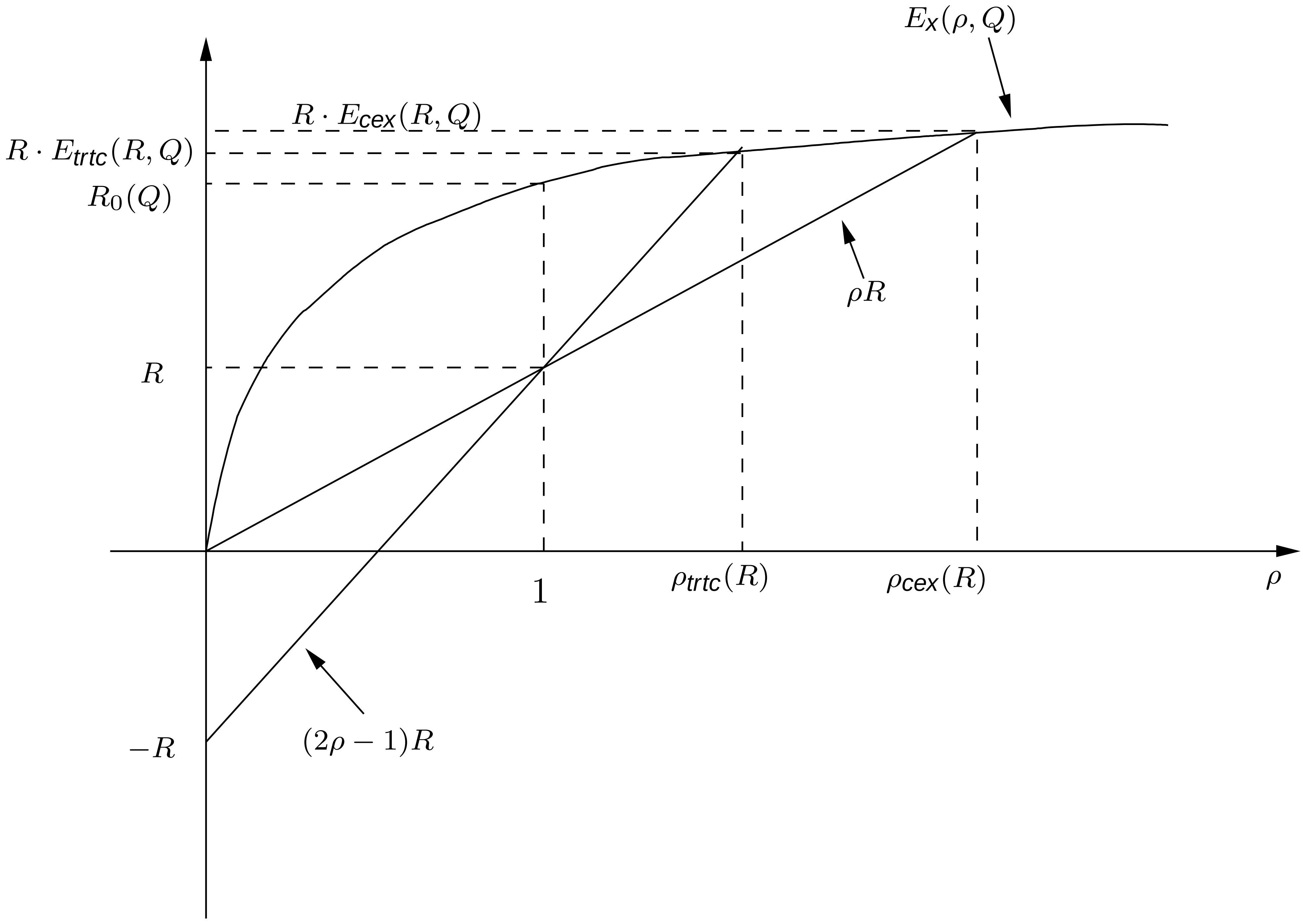
$$\mathcal{E}_{\mathsf{trtc}}(R,Q) \geq E_{\mathsf{trtc}}(R,Q) \triangleq \frac{E_{\mathsf{X}}(\rho,Q)}{R},$$

where ρ satisfies

$$\frac{E_{\mathsf{X}}(\rho,Q)}{2\rho-1} = R.$$

Also,

$$\mathcal{E}_{\mathsf{trcc}}(R,Q) \geq E_{\mathsf{cex}}(R,Q).$$


$E_x(\rho, Q)$
$R \cdot E_{cex}(R, Q)$
$R \cdot E_{trtc}(R, Q)$
$R_0(Q)$
ρR
R
1
$\rho_{trtc}(R)$
$\rho_{cex}(R)$
ρ
$(2\rho - 1)R$
$-R$

Characterizing the Typical Codes

The probability of error

$$P_{\mathsf{e}}(\mathcal{C}) \leq \sum_{\ell \geq 1} 2^{-m\ell} \sum_{\{P_{XX'}\}} N_\ell(P_{XX'}) \exp\{-n(k+\ell)\Delta(P_{XX'})\}.$$

For typical codes, all $\{N_\ell(P_{XX'})\}$ that have a small expectation, vanish simultaneously w.h.p.

This amounts to the condition

$$2\ell R < (k+\ell)D(P_{XX'}\|Q \times Q).$$

Joint types that are "too far" from $Q \times Q$ are not populated.

For populated joint types,

$$N_\ell(P_{XX'}) \sim \exp\{m[2\ell - (k+\ell)D((P_{XX'}\|Q \times Q)/R]\}.$$

An Alternative Expression

$$E_{\mathsf{trtc}}(R,Q) = \inf_{\hat{R}<R} \inf_{\{P_{XX'}:\ D(P_{XX'}\|Q\times Q)<2\hat{R}\}} \frac{\mathbf{E}_P d_{\mathsf{B}}(X,X') + \hat{R}}{R-\hat{R}}.$$

Dominant error events:

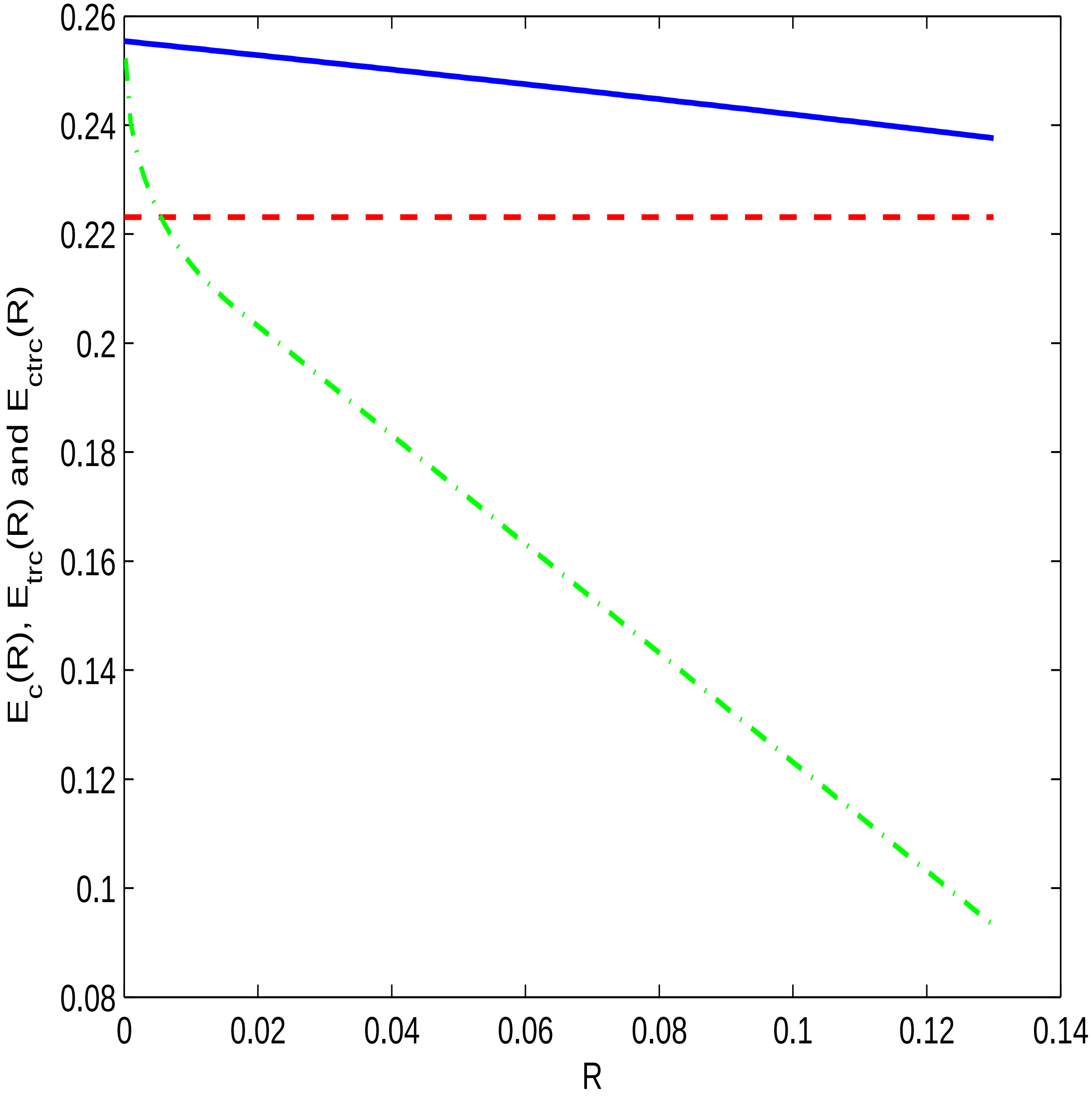
A sub–exponential number of paths with joint type,

$$P_{XX'}(x,x') = \frac{Q(x)Q(x')\exp\{-d_{\mathsf{B}}(x,x')/\rho\}}{Z}$$

and critical length of

$$k+\ell = \frac{kR}{2R - D(P_{XX'}\|Q\times Q)}.$$

A Numerical Example: BSC with $p = 0.1$



Two Words Regarding an Extension

The paper contains also an extension to a channel with mismatch and input memory (ISI),

$$W(\boldsymbol{y}|\boldsymbol{x}) = \prod_t W(y_t|x_t, x_{t-1}).$$

$E_{\mathsf{X}}(\rho, Q)$ is now replaced by $-\rho \log \lambda$, where λ is the Perron-Frobenius eigenvalue of a certain matrix that depends on the Q and on the channel single–letterly (details – in the paper).