Optimal Correlators for Detection and Estimation in Optical Receivers

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What is it About?

Classical signal detection/estimation problems are optimally solved using a matched filter, or matched correlator:

Signal detection:

$$\mathcal{H}_0 : \; y(t) = n(t), \; 0 \leq t \leq T, \;\; \{n(t)\} = \text{AWGN}$$

$$\mathcal{H}_1 : \; y(t) = \lambda(t) + n(t), \; 0 \leq t \leq T$$

LRT: Compare $\int_0^T y(t)\lambda(t)\mathrm{d}t$ to a threshold.

What is it About? (Cont'd)

Estimation of delay:

$$y(t) = \lambda(t-\tau) + n(t),\ 0 \le t \le T.$$

ML estimator:

$$\hat{\tau} = \text{arg}\max_{\theta} \int_0^T y(t)\lambda(t-\theta)\text{d}t.$$

Difficulty: In optical receivers the noise is not Gaussian.

Approach: Find optimal waveforms for correlation accordingly:

Detection: compare $\int_0^T y(t) w_{\text{d}}^*(t)\text{d}t$ to a threshold.

Estimation: $\hat{\tau} =$ arg $\max_\theta \int_0^T y(t) w_{\text{e}}^*(t-\theta)\text{d}t$.

Motivation: relatively easy to implement.

Background – General

- Modern LIDAR applications $\rightarrow$ renewed interest in optical det./est.
- Classical optical direct detection: photodiode + TIA.
- Complicated signal+noise model: signal + AWGN + shot-noise.
- In the case of APD: also excess noise (multiplicative noise).
- Difficult/impossible to derive the LR in closed form.
- Researchers have proposed many types of approximations.

Background – Signal Model

The received signal model:

$$y(t) = \sum_{k=1}^{K} g_k h(t - t_k) + n(t), \quad 0 \leq t \leq T,$$

where:

$K =$ total number of photo–electrons.

$h(t) =$ current pulse contributed by a photo–electron generated at $t = 0$.

$\{t_k\} =$ Poissonian arrival times of photo–electrons, induced by $\lambda(t)$.

$\{g_k\} =$ avalanche gains (geom. distributed: $P(g) \propto e^{-\zeta g}$, $g = 1, 2, \ldots$).

$\{n(t)\} =$ Gaussian white noise.

Background – Likelihood Ratio

Had K, $\{g_k\}$ and $\{t_k\}$ been known,

$$\text{LR} = \frac{\exp\left\{-\frac{1}{N_0}\int_0^T \left[y(t) - \sum_{k=1}^K g_k h(t-t_k)\right]^2 \mathrm{d}t\right\}}{\exp\left\{-\frac{1}{N_0}\int_0^T y^2(t)\mathrm{d}t\right\}}$$

$$= \exp\left\{\frac{2}{N_0}\sum_{k=1}^K \int_0^T y(t)h(t-t_k)\mathrm{d}t - \frac{1}{N_0}\sum_{k=1}^K\sum_{l=1}^K g_k g_l R(t_k - t_l)\right\},$$

where $R(\tau) = \int_0^T h(t)h(t-\tau)\mathrm{d}t$.

Main difficulty: averaging over the randomness of K, $\{g_k\}$ and $\{t_k\}$.

A Sample of Some Earlier Approaches

- Foscini, Gilbert & Salz ('75): problematic term = charac. func. of a Gaussian process.
- Kadota ('88), Hero ('91): neglecting the problematic term.
- Einarsson ('96), El–Hadidi *et al.* ('81), Geraiotis *et al.* ('87): Gaussian approximations – optical matched filter.

The Proposed Approach

Find optimal waveforms for correlation:

Detection: compare $\int_0^T y(t) w_{\mathsf{d}}^*(t) \mathrm{d}t$ to a threshold.

Estimation: $\hat{\tau} = \arg\max_\theta \int_0^T y(t) w_{\mathsf{e}}^*(t-\theta) \mathrm{d}t$.

Design $\{w_{\mathsf{d}}^*(t)\}$ and $\{w_{\mathsf{e}}^*(t)\}$ by incoporating the full signal model.

Optimal in what sense?

Detection: Max. exponential decay rate of P_{MD} for a given P_{FA}.

Estimation: minimum MSE (under high SNR).

Signal Detection – No Dark Current

The FA probability.

Although $y(t) = n(t)$, the correlation is above the threshold.

$$
\begin{aligned}
P_{\text{FA}} &= \Pr\left\{\int_0^T n(t)w(t)\mathrm{d}t \geq \theta T\right\} \\
&= Q\left(\frac{\theta T}{\sqrt{N_0 E/2}}\right) \\
&\approx \exp\left\{-\frac{\theta^2 T}{N_0 P}\right\},
\end{aligned}
$$

where P is the power of $\{w(t), 0 \leq t \leq T\}$.

Maximizing the exponent of P_{MD} for a given exponent of P_{FA} is equivalent to maximizing it for a given P.

Signal Detection – No Dark Current (Cont'd)

The MD probability.

Although $y(t) = \sum_k g_k h(t - t_k) + n(t)$, the correlation is below the threshold.

$$\begin{aligned} P_{\mathrm{MD}} &= \Pr\left\{ \int_0^T \left[\sum_k g_k h(t - t_k) + n(t) \right] w(t)\mathrm{d}t < \theta T \right\} \\ &\approx e^{-ET}, \end{aligned}$$

where

$$E = \sup_{s \geq 0} \left[\frac{e^{\zeta}}{T} \int_0^T \lambda(t) \cdot \frac{\exp\{s q_{\mathrm{e}} w(t)\} - 1}{\exp\{s q_{\mathrm{e}} w(t) + \zeta\} - 1} \mathrm{d}t - s\theta - s^2 \frac{N_0 P}{4} \right],$$

where q_{e} is the electric charge of the electron.

Optimizing the Correlator (Deterministic Gain)

Consider first the case of deterministic gain, $g_k \equiv 1$ ($\zeta \to \infty$):

$$E = \sup_{s \geq 0} \left[\frac{1}{T} \int_0^T \lambda(t)[1 - \exp\{-sq_{\mathsf{e}} w(t)\}] \mathsf{d}t - s\theta - s^2 \frac{N_0 P}{4} \right].$$

We wish to maximize E over $\{w(t),\ 0 \leq t \leq T\}$ s.t. $\int_0^T w^2(t) \mathsf{d}t \leq PT$.

Solution: Let $p[y]$ be the inverse of the function $b[x] = xe^x$. Then

$$w_{\mathsf{d}}^*(t) = \frac{1}{sq_{\mathsf{e}} a} \cdot p[c\lambda(t)],$$

where c is chosen so that the power would be exactly P.

For N_0 and/or θ large, $p[x] \approx x$ and $w_{\mathsf{d}}^*(t) \propto \lambda(t)$.

For N_0 and θ small, $p[x] \approx \ln x$, and $w_{\mathsf{d}}^*(t) \propto \ln \lambda(t)$.

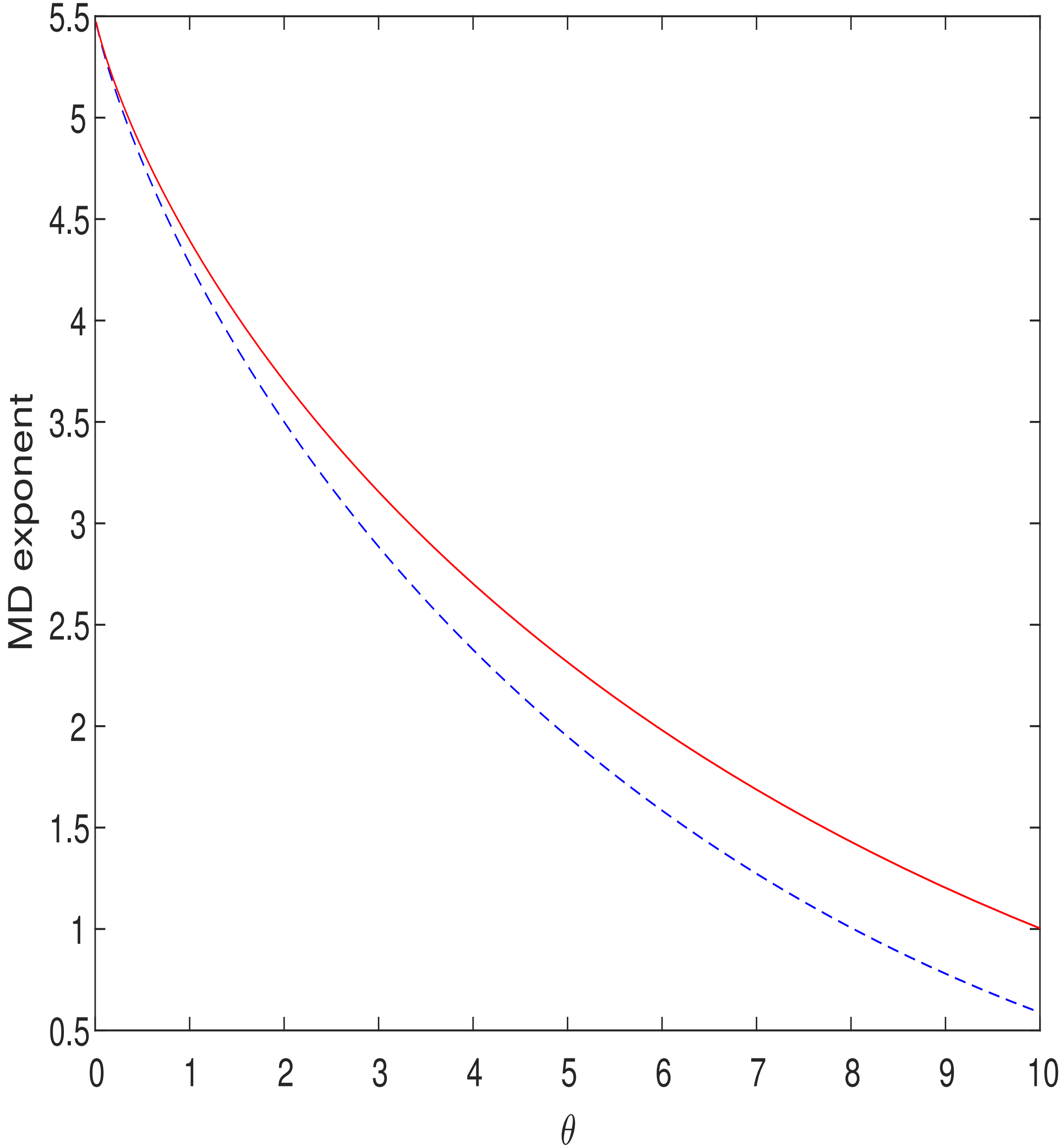
Some Numerical Results

Comparison of MD error exponents of the optical matched filter

$$w_{\mathsf{omf}}(t) = \frac{\lambda(t)}{\lambda(t) + N_0/(2q_{\mathsf{e}}^2\overline{g^2})}$$

and the optimal correlator w_{d}^* as functions of θ, for a two–level signal at levels, $\lambda_1 = 1,\ \lambda_2 = 10$ with duty cycle of 50%.

$$P = 10,\ N_0/q_{\mathsf{e}}^2 = 0.0001$$


MD exponent
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
1
2
3
4
5
6
7
8
9
10
θ

Optimizing the Correlator (Random Gain)

Here, the optimal correlator is

$$w_{\mathsf{d}}^{*}(t) = \frac{1}{sq_{\mathsf{e}}} \cdot p_{\zeta}[c \cdot \lambda(t)],$$

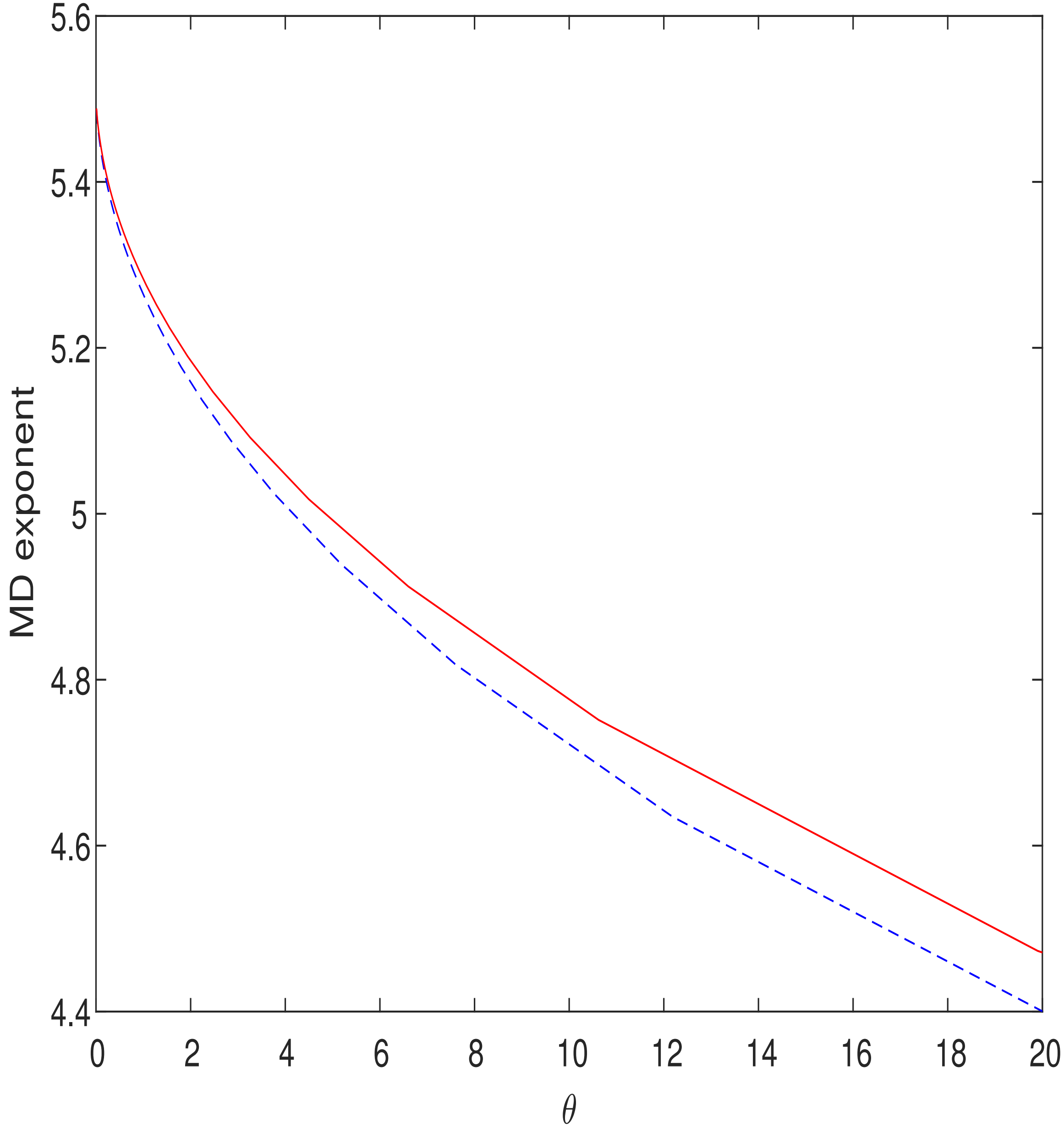
where p_{ζ} is the inverse of the function

$$b_{\zeta}[x] = \frac{x(e^{x+\zeta} - 1)^2}{e^{x+2\zeta} - e^{x+\zeta}}.$$

Once again,

For N_0 and/or θ large, $p_{\zeta}[x] \approx x$ and $w_{\mathsf{d}}^{*}(t) \propto \lambda(t)$.

For N_0 and θ small, $p_{\zeta}[x] \approx \ln x$, and $w_{\mathsf{d}}^{*}(t) \propto \ln \lambda(t)$.



5.6
5.4
5.2
5
4.8
4.6
4.4
MD exponent
0
2
4
6
8
10
12
14
16
18
20
θ

Time Delay Estimation (high SNR)

- The underlying optical signal received is $\lambda(t-\tau)$.
- The estimator is $\hat{\tau} = \arg\max_\theta \int_0^T y(t)w(t-\theta)\mathrm{d}t$.
- Here, $\tau = \arg\max_\theta \int_0^T \lambda(t-\tau)w(t-\theta)\mathrm{d}t$
- What is the MSE, $\mathbf{E}(\hat{\tau}-\tau)^2$ in terms of $\{w(t),\ 0 \le t \le T\}$?
- What is waveform $\{w_{\mathsf{e}}^*(t),\ 0 \le t \le T\}$ minimizes the MSE?

Mean Square Error

The high–SNR MSE is approximated by

$$\mathbf{E}\{(\hat{\tau}-\tau)^2\} \approx \frac{\int_0^T \left[\frac{N_0}{2} + \overline{g^2} q_{\mathsf{e}}^2 \lambda(t)\right] \dot{w}^2(t) \mathsf{d}t}{\overline{g}^2 q_{\mathsf{e}}^2 \left[\int_0^T \lambda(t) \ddot{w}(t) \mathsf{d}t\right]^2},$$

where $\dot{w}(t)$ and $\ddot{w}(t)$ are the first two derivatives of $w(t)$.

Finding the optimal $w(t)$ is a problem of calculus of variations.

The Optimal Waveform

The solution to the above problem is:

$$w_{\mathsf{e}}^*(t) = \ln\left[1 + \frac{\lambda(t)}{\lambda_0}\right],$$

where

$$\lambda_0 = \frac{N_0}{2\overline{g^2}q_{\mathsf{e}}^2}.$$

For large N_0, $w_{\mathsf{e}}^*(t) \propto \lambda(t)$.

For small N_0, $w_{\mathsf{e}}^*(t) \propto \ln[\lambda(t)]$.

The MMSE is:

$$\text{MMSE} \approx \frac{1}{\lambda_0} \cdot \frac{1}{\int_0^T \frac{\dot{\lambda}^2(t)\mathsf{d}t}{1+\lambda(t)/\lambda_0}}.$$

Conclusion

- Exact optimal solutions for both problems are hard.
- Practical considerations motivate simple correlators.
- We found optimal correlator waveforms for both problems.
- Solutions are different but their limiting behavior is the same.