Optimal Correlators for Detection and Estimation in Optical Receivers

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What is it About?

Classical signal detection/estimation problems are optimally solved using a matched filter, or matched correlator:

Signal detection:

\( \mathcal{H}_0 : y(t) = n(t), \; 0 \leq t \leq T, \; \{n(t)\} = \text{AWGN} \)

\( \mathcal{H}_1 : y(t) = \lambda(t) + n(t), \; 0 \leq t \leq T \)

LRT: Compare \( \int_0^T y(t) \lambda(t) dt \) to a threshold.
What is it About? (Cont’d)

Estimation of delay:

\[ y(t) = \lambda(t - \tau) + n(t), \ 0 \leq t \leq T. \]

ML estimator:

\[ \hat{\tau} = \arg \max_{\theta} \int_{0}^{T} y(t) \lambda(t - \theta) dt. \]

Difficulty: In optical receivers the noise is not Gaussian.

Approach: Find optimal waveforms for correlation accordingly:

Detection: compare \( \int_{0}^{T} y(t) w_d^*(t) dt \) to a threshold.

Estimation: \( \hat{\tau} = \arg \max_{\theta} \int_{0}^{T} y(t) w_e^*(t - \theta) dt. \)

Motivation: relatively easy to implement.
Background – General

- Modern LIDAR applications → renewed interest in optical det./est.
- Classical optical direct detection: photodiode + TIA.
- Complicated signal+noise model: signal + AWGN + shot-noise.
- In the case of APD: also excess noise (multiplicative noise).
- Difficult/impossible to derive the LR in closed form.
- Researchers have proposed many types of approximations.
Background – Signal Model

The received signal model:

\[ y(t) = \sum_{k=1}^{K} g_k h(t - t_k) + n(t), \quad 0 \leq t \leq T, \]

where:

- \( K \) = total number of photo–electrons.
- \( h(t) \) = current pulse contributed by a photo–electron generated at \( t = 0 \).
- \( \{t_k\} \) = Poissonian arrival times of photo–electrons, induced by \( \lambda(t) \).
- \( \{g_k\} \) = avalanche gains (geom. distributed: \( P(g) \propto e^{-\zeta g} \), \( g = 1, 2, \ldots \)).
- \( \{n(t)\} \) = Gaussian white noise.
Background – Likelihood Ratio

Had $K$, $\{g_k\}$ and $\{t_k\}$ been known,

$$
LR = \exp \left\{ - \frac{1}{N_0} \int_0^T \left[ y(t) - \sum_{k=1}^K g_k h(t - t_k) \right]^2 dt \right\} \exp \left\{ - \frac{1}{N_0} \int_0^T y^2(t) dt \right\}
$$

$$
= \exp \left\{ \frac{2}{N_0} \sum_{k=1}^K g_k \int_0^T y(t) h(t - t_k) dt - \frac{1}{N_0} \sum_{k=1}^K \sum_{l=1}^K g_k g_l R(t_k - t_l) \right\},
$$

where $R(\tau) = \int_0^T h(t) h(t - \tau) dt$.

Main difficulty: averaging over the randomness of $K$, $\{g_k\}$ and $\{t_k\}$.
A Sample of Some Earlier Approaches

♠ Foscini, Gilbert & Salz (‘75): problematic term = charac. func. of a Gaussian process.

♠ Kadota (‘88), Hero (‘91): neglecting the problematic term.

♠ Einarsson (‘96), El–Hadidi et al. (‘81), Geraiotis et al. (‘87): Gaussian approximations – optical matched filter.
The Proposed Approach

Find optimal waveforms for correlation:

**Detection:** compare $\int_0^T y(t)w_d^*(t)dt$ to a threshold.

**Estimation:** $\hat{\tau} = \arg \max_\theta \int_0^T y(t)w_e^*(t - \theta)dt$.

Design $\{w_d^*(t)\}$ and $\{w_e^*(t)\}$ by incorporating the full signal model.

Optimal in what sense?

**Detection:** Max. exponential decay rate of $P_{MD}$ for a given $P_{FA}$.

**Estimation:** minimum MSE (under high SNR).
Signal Detection – No Dark Current

The FA probability.

Although $y(t) = n(t)$, the correlation is above the threshold.

$$P_{FA} = \Pr \left\{ \int_0^T n(t)w(t)dt \geq \theta T \right\}$$

$$= Q \left( \frac{\theta T}{\sqrt{N_0 E/2}} \right)$$

$$\approx \exp \left\{ -\frac{\theta^2 T}{N_0 P} \right\},$$

where $P$ is the power of $\{w(t), 0 \leq t \leq T\}$.

Maximizing the exponent of $P_{MD}$ for a given exponent of $P_{FA}$ is equivalent to maximizing it for a given $P$. 
The MD probability.

Although \( y(t) = \sum_k g_k h(t - t_k) + n(t) \), the correlation is below the threshold.

\[
P_{\text{MD}} = \text{Pr}\left\{ \int_0^T \left[ \sum_k g_k h(t - t_k) + n(t) \right] w(t) \, dt < \theta T \right\}
\]

\[
\approx e^{-ET},
\]

where

\[
E = \sup_{s \geq 0} \left[ \frac{e^\zeta}{T} \int_0^T \lambda(t) \cdot \frac{\exp\{sqe w(t)\} - 1}{\exp\{sqe w(t) + \zeta\} - 1} \, dt - s\theta - s^2 \frac{N_0 P}{4} \right],
\]

where \( q_e \) is the electric charge of the electron.
Consider first the case of deterministic gain, \( g_k \equiv 1 \ (\zeta \to \infty) \):

\[
E = \sup_{s \geq 0} \left[ \frac{1}{T} \int_0^T \lambda(t) \left[ 1 - \exp\{-sq \epsilon w(t)\} \right] dt - s\theta - s^2 \frac{N_0 P}{4} \right].
\]

We wish to maximize \( E \) over \( \{w(t), \ 0 \leq t \leq T\} \) s.t. \( \int_0^T w^2(t) dt \leq PT \).

**Solution:** Let \( p[y] \) be the inverse of the function \( b[x] = xe^x \). Then

\[
w^*_d(t) = \frac{1}{sq \epsilon a} \cdot p[c\lambda(t)],
\]

where \( c \) is chosen so that the power would be exactly \( P \).

For \( N_0 \) and/or \( \theta \) large, \( p[x] \approx x \) and \( w^*_d(t) \propto \lambda(t) \).

For \( N_0 \) and \( \theta \) small, \( p[x] \approx \ln x \), and \( w^*_d(t) \propto \ln \lambda(t) \).
Some Numerical Results

Comparison of MD error exponents of the optical matched filter

\[ w_{\text{omf}}(t) = \frac{\lambda(t)}{\lambda(t) + N_0/(2q_e^2g^2)} \]

and the optimal correlator \( w_d^* \) as functions of \( \theta \), for a two–level signal at levels, \( \lambda_1 = 1, \lambda_2 = 10 \) with duty cycle of 50%.

\[ P = 10, \frac{N_0}{q_e^2} = 0.0001 \]
Here, the optimal correlator is

\[ w_d^*(t) = \frac{1}{sqe} \cdot p_\zeta[c \cdot \lambda(t)], \]

where \( p_\zeta \) is the inverse of the function

\[ b_\zeta[x] = \frac{x(e^x+\zeta - 1)^2}{e^{x+2\zeta} - e^{x+\zeta}}. \]

Once again,

For \( N_0 \) and/or \( \theta \) large, \( p_\zeta[x] \approx x \) and \( w_d^*(t) \propto \lambda(t) \).

For \( N_0 \) and \( \theta \) small, \( p_\zeta[x] \approx \ln x \), and \( w_d^*(t) \propto \ln \lambda(t) \).
The underlying optical signal received is $\lambda(t - \tau)$.

The estimator is $\hat{\tau} = \arg \max_{\theta} \int_{0}^{T} y(t)w(t - \theta)dt$.

Here, $\tau = \arg \max_{\theta} \int_{0}^{T} \lambda(t - \tau)w(t - \theta)dt$.

What is the MSE, $E(\hat{\tau} - \tau)^2$ in terms of $\{w(t), 0 \leq t \leq T\}$?

What is waveform $\{w_e^*(t), 0 \leq t \leq T\}$ minimizes the MSE?
Mean Square Error

The high–SNR MSE is approximated by

\[
\mathbb{E}\{(\hat{\tau} - \tau)^2\} \approx \frac{\int_0^T \left[ \frac{N_0}{2} + g^2 q e^2 \lambda(t) \right] \dot{w}(t)^2 dt}{\bar{g}^2 q e^2 \left[ \int_0^T \lambda(t) \ddot{w}(t) dt \right]^2},
\]

where \(\dot{w}(t)\) and \(\ddot{w}(t)\) are the first two derivatives of \(w(t)\).

Finding the optimal \(w(t)\) is a problem of calculus of variations.
The Optimal Waveform

The solution to the above problem is:

\[ w^*_e(t) = \ln \left[ 1 + \frac{\lambda(t)}{\lambda_0} \right], \]

where

\[ \lambda_0 = \frac{N_0}{2g^2q_0^2}. \]

For large \( N_0 \), \( w^*_e(t) \propto \lambda(t) \).

For small \( N_0 \), \( w^*_e(t) \propto \ln[\lambda(t)] \).

The MMSE is:

\[ \text{MMSE} \approx \frac{1}{\lambda_0} \cdot \frac{1}{\int_0^T \frac{\lambda^2(t) dt}{1 + \lambda(t)/\lambda_0}}. \]
Conclusion

♣ Exact optimal solutions for both problems are hard.
♣ Practical considerations motivate simple correlators.
♣ We found optimal correlator waveforms for both problems.
♣ Solutions are different but their limiting behavior is the same.