Universal Decoding for Asynchronous Slepian–Wolf Encoding

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Asynchronous Slepian–Wolf (ASW) Encoding

ASW encoding – unknown relative delay: X_i – correlated to Y_{i+d}.

Reasoning:

- ♣ No information exchange between the encoders.
- ♣ Different processing delays at the two encoders.
- ♣ Sampling clocks at the two locations are not synchronized.
- ♣ Relative delays between measurements due to different distances.

Earlier Work

♠ Willems ('88): delay unknown to encoders, known to decoder.

♠ Rimoldi & Urbanke ('77); Sun *et al.* ('10): source splitting; decoder waits.

♠ Matsuta & Uyematsu ('20): worst–case approach:

The joint distribution, P_{XY}, is only known to be in $\mathcal{S}$.

Relative delay, d, is only known to be between $-\Delta n$ and Δn, $\Delta \in [0,1)$.

Their main result:

$$\begin{aligned} R_x &\geq \sup_{P_{XY} \in \mathcal{S}} [H(X|Y) + \Delta \cdot I(X;Y)] \\ R_y &\geq \sup_{P_{XY} \in \mathcal{S}} [H(Y|X) + \Delta \cdot I(X;Y)] \\ R_x + R_y &\geq \sup_{P_{XY} \in \mathcal{S}} [H(X,Y) + \Delta \cdot I(X;Y)] \end{aligned}$$

Earlier Work (Cont'd)

The worst case approach is pessimistic:

For example, if $\mathcal{S} =$ the entire simplex of distributions over $\mathcal{X} \times \mathcal{Y}$,

$$
\begin{aligned}
R_x &\geq \log|\mathcal{X}| \\
R_y &\geq \log|\mathcal{Y}| \\
R_x + R_y &\geq \log|\mathcal{X}| + \log|\mathcal{Y}|
\end{aligned}
$$

which is an uninteresting triviality.

Their analysis of the probability of error is also for the worst source in $\mathcal{S}$.

Q1: But what if $\{P_{XY}\}$ is **not** the worst source in $\mathcal{S}$?

Q2: Can one devise a **universal decoder** for unknown P_{XY} and d?

Earlier Work on Universal Decoding

Universal channel decoding:

- Goppa ('75) - the MMI decoder achieves capacity.
- Csiszár & Körner ('81) – MMI decoder achieves $E_{\mathrm{r}}(R)$.
- Csiszár ('82); Ziv ('85); Feder & Lapidoth ('98); Feder & Merhav ('02).
- ...

Universal S–W decoding:

- Csiszár & Körner ('81): minimum entropy decoder.
- Oohama & Han ('94); Draper ('04); Chen *et al.* ('08); Merhav ('16),...

Main Contributions in This Work

- ♣ Proposing a universal decoder for unknown source and delay.
- ♣ Same random coding exponent as the MAP decoder.
- ♣ The lower bound is based on a lower bound to Pr$\{\cup_{i,j}\mathcal{A}_i \cap \mathcal{B}_j\}$.
- ♣ Deriving a Lagrange–dual formula of the error exponent.
- ♣ Outlining a possible extension to sources with memory.

Problem Formulation

- $\{(X_i, Y_{i+d})\}$ are i.i.d. pairs of finite–alphabet RVs $\sim P_{XY}$.
- $P_d(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{d} P_Y(y_i) \cdot \prod_{i=n-d+1}^{n} P_X(x_i) \cdot \prod_{i=1}^{n-d} P_{XY}(x_i, y_{i+d})$.
- Encoder X: $f(\boldsymbol{x}) \sim \text{unif}\{1, 2, \ldots, 2^{nR_{\mathsf{X}}}\}$.
- Encoder Y: $g(\boldsymbol{y}) \sim \text{unif}\{1, 2, \ldots, 2^{nR_{\mathsf{Y}}}\}$.
- Neither P_{XY} nor $d = \delta n$ are known.
- MAP decoder: $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \text{arg}\max_{\boldsymbol{x}', \boldsymbol{y}'} P_d(\boldsymbol{x}', \boldsymbol{y}')$
- General metric decoder: $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \text{arg}\max_{\boldsymbol{x}', \boldsymbol{y}'} q(\boldsymbol{x}', \boldsymbol{y}')$.
- Error probability: $P_{\mathsf{e}} = \Pr\{(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \neq (\boldsymbol{x}, \boldsymbol{y})\}$.
- Error exponent: $E(R_{\mathsf{X}}, R_{\mathsf{Y}}) = \lim_{n\to\infty} -\frac{\log P_{\mathsf{e}}}{n}$.

The Proposed Universal Decoding Metric

For $0 \le k \le n$, define

$$u_k(\boldsymbol{x}, \boldsymbol{y}) = k\hat{H}(y_1^k) + (n-k)\hat{H}(x_1^{n-k}, y_{k+1}^n) + k\hat{H}(x_{n-k+1}^n),$$

$$v_k(\boldsymbol{x}, \boldsymbol{y}) = (n-k)\hat{H}(x_1^{n-k} | y_{k+1}^n) + k\hat{H}(x_{n-k+1}^n),$$

$$w_k(\boldsymbol{x}, \boldsymbol{y}) = k\hat{H}(y_1^k) + (n-k)\hat{H}(y_{k+1}^n | x_1^{n-k}),$$

$$q_k(\boldsymbol{x}, \boldsymbol{y}) = \max\{u_k(\boldsymbol{x}, \boldsymbol{y}) - n(R_x + R_y), v_k(\boldsymbol{x}, \boldsymbol{y}) - nR_x, w_k(\boldsymbol{x}, \boldsymbol{y}) - nR_y\},$$

and finally, the universal decoding metric, q, is defined as

$$q(\boldsymbol{x}, \boldsymbol{y}) = \min_{0 \le k \le n} q_k(\boldsymbol{x}, \boldsymbol{y}).$$

Main Theorem

$$E_{\text{univdec}}(R_x, R_y) = E_{\text{MAP}}(R_x, R_y) = \min\{E_{x|y}(R_x), E_{y|x}(R_y), E_{xy}(R_x, R_y)\},$$

where

$$E_{x|y}(R_x) = \max_{0 \le \rho \le 1} \rho \cdot \left[R_x - \delta H_{1/(1+\rho)}(X) - (1-\delta) H_{1/(1+\rho)}(X|Y)\right]$$

$$E_{y|x}(R_y) = \max_{0 \le \rho \le 1} \rho \cdot \left[R_y - \delta H_{1/(1+\rho)}(Y) - (1-\delta) H_{1/(1+\rho)}(Y|X)\right]$$

$$E_{xy}(R_x, R_y) = \max_{0 \le \rho \le 1} \rho \cdot \Bigg[R_x + R_y -$$

$$\delta H_{1/(1+\rho)}(X) - \delta H_{1/(1+\rho)}(Y) - (1-\delta) H_{1/(1+\rho)}(X, Y)\Bigg].$$

Discussion

- ♡ Three types of errors: X only, Y only, and both X and Y.
- ♡ The decoding metric has three components correspondingly.
- ♡ Uncertaintly in d - treated differently than that of P_{XY}.
- ♡ Unconditional Rényi entropies correspond to independent segments.
- ♡ Achievable rate region for error exponent E:

$$\mathcal{R}(E) = \{(R_x, R_y):\ R_x \geq \boldsymbol{R}_x(E),\ R_y \geq \boldsymbol{R}_y(E),\ R_x + R_y \geq \boldsymbol{R}_{xy}(E)\},$$

$$\boldsymbol{R}_x(E) = \inf_{s \geq 1} \left[sE + \delta H_{s/(1+s)}(X) + (1-\delta) H_{s/(1+s)}(X|Y)\right]$$

and similar expressions for $\boldsymbol{R}_y(E)$ and $\boldsymbol{R}_{xy}(E)$.

- ♡ Proof: upper bound – MoT. Lower bound – de Caen's inequality.
- ♡ Extension to Markov: replace empirical entropies by LZ metrics.

Conclusion

- Replacing the worst–case approach by an “adaptive” approach.
- Achieves not only the rate region, but also the error exponent.
- Non-trivial universality in d.
- Universal metric with three components, one for every type of error.
- Characterization of rate region for a given error exponent, E.
- Lower bound on $P[\cup_{i,j}\mathcal{A}_i \cap \mathcal{B}_j]$: could be useful elsewhere too.