# On Error Exponents of Encoder–Assisted Communication Systems

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ISIT 2021, Melbourne, Victoria, Australia, July 2021

# Encoder–Assisted Communication

- Encoder $\Longrightarrow$ additive noise channel $\Longrightarrow$ decoder
- Helper $\Longrightarrow$ rate–limited description of the noise $\Longrightarrow$ encoder
- Similar, but not identical, to Gel'fand–Pinsker ('80).
- Motivation: helper $=$ interfering transmitter; noise $=$ his codeword.
- Lapidoth & Marti ('20): AWGN capacity $=$ ord. capacity $+$ help rate.
- Achievability – flash help: high res. compression of a small segment.
- This work: error exponents

# Formulation

AWGN channel:

$$Y_i = X_i + Z_i, \qquad Z_i \sim \mathcal{N}(0, \sigma^2), \quad i = 1, 2, \ldots, n$$

Helper:

$$T = T(Z^n), \qquad T : \mathbb{R}^n \to \mathcal{T} = \{0, 1, \ldots, \exp\{nR_{\mathsf{h}}\} - 1\}.$$

Encoder:

$$X^n = \phi(m, T(Z^n)), \qquad \phi : \mathcal{M} \times \mathcal{T} \to \mathcal{C} \subset \mathbb{R}^n, \quad \mathcal{M} = \{0, 1, \ldots, e^{nR} - 1\}.$$

$$\|X^n\|^2 \leq nP$$

Decoder:

$$\hat{m} = \psi(Y^n), \qquad \psi : \mathbb{R}^n \to \mathcal{M}.$$

Error probability: $P_{\mathsf{e}}(\phi, T, \psi) = \mathsf{Pr}\{\psi(\phi(m, T(Z^n)) + Z^n) \neq m\}.$

# Formulation (Cont'd)

$R =$ achievable–rate:

$\forall\ \epsilon > 0\ \exists$ suff. large $n$ s.t. $\min_{\phi,T,\psi} P_{\mathsf{e}}(\phi,T,\psi) \leq \epsilon$.

Capacity:

$C \stackrel{\triangle}{=}$ sup. of achievable rates $= C_0 + R_{\mathsf{h}}$ (Lapidoth & Marti, 2020).

$$\text{where } C_0 = \frac{1}{2}\log(1+\gamma), \quad \gamma \stackrel{\triangle}{=} \frac{P}{\sigma^2}.$$

Reliability function:

$$E(R) = \lim_{n\to\infty}\left[-\frac{1}{n}\log\min_{\phi,T,\psi} P_{\mathsf{e}}(\phi,T,\psi)\right].$$

# Main Theorem

Lower bound (achievability):

$$E(R) \geq E_L(R) \triangleq \begin{cases} \infty & R < R_{\mathsf{h}} \\ E_{\mathsf{a}}(R - R_{\mathsf{h}}) & R_{\mathsf{h}} < R \leq R_{\mathsf{h}} + C_0 \\ 0 & R \geq R_{\mathsf{h}} + C_0 \end{cases}$$

Upper bound ("converse"):

$$E(R) \leq E_U(R) \triangleq \begin{cases} \infty & R < R_{\mathsf{h}} \\ E_{\mathsf{wsp}}(R - R_{\mathsf{h}}) & R_{\mathsf{h}} < R \leq R_{\mathsf{h}} + C_0 \\ 0 & R \geq R_{\mathsf{h}} + C_0 \end{cases}$$

where

$$E_{\mathsf{wsp}}(R) = \frac{1}{2}\left[\frac{e^{2C_0} - 1}{e^{2R} - 1} - \ln\left(\frac{e^{2C_0} - 1}{e^{2R} - 1}\right) - 1\right].$$

# Discussion

- Both bounds: $= \infty$ below $R_{\mathsf{h}}$; $= 0$ beyond $C_0 + R_{\mathsf{h}}$; finite in between.
- The bounds are “compatible” from a qualitative viewpoint ...
- Inherent phase transition at $R = R_{\mathsf{h}}$.
- Helper - equivalent to a noiseless bit pipe of capacity $R_{\mathsf{h}}$.
- Achievability: using the flash–help approach.
- Fixed–rate helper: arbitrarily large exponent for $R < R_{\mathsf{h}}$.
- Variable–rate helper: $P_{\mathsf{e}} = 0$ for $R < R_{\mathsf{h}}$.

# The Achievability Scheme

Helper uses all $nR_{\mathsf{h}}$ bits to describe only $Z^t$, $t = n\tau$, $0 < \tau \ll 1$:

If $Z^t \in \mathcal{B}(\sqrt{t\sigma^2(1+s)})$, quantize uniformly with step size $\Delta$ s.t.

$$nR_{\mathsf{h}} = \log \left( \frac{\mathsf{Vol}\{\mathcal{B}(\sqrt{t\sigma^2(1+s)})\}}{\Delta^t} \right)$$

which yields

$$\Delta \approx \sqrt{2\pi e \sigma^2(1+s)} \cdot \exp\{-R_{\mathsf{h}}/\tau\}.$$

During this segment, the encoder transmits $x^t(m) - [z^t]_Q$.

The corresponding received signal is

$$y^t = x^t(m) + z^t - [z^t]_Q \triangleq x^t(m) + \tilde{z}^t, \quad \tilde{z}^t \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]^t$$

# The Achievability Scheme (Cont'd)

where $\{x^t(m)\} \in$ cubic lattice code with step-size $\Delta$,

supporting rates up to

$$nR' = \log\left(\frac{\text{Vol}\{\mathcal{B}(\sqrt{nP})\}}{\Delta^t}\right) \approx nR_{\text{h}} + \frac{n\tau}{2}\log\frac{2\pi eP}{\sigma^2(1+s)} \approx nR_{\text{h}}$$

An error occurs in the short segment only if $Z^t \geq t\sigma^2(1+s)$,

which happens with probability of about

$$\exp\left\{-n\cdot\frac{\tau}{2}[s-\ln(1+s)]\right\}.$$

For $\tau$ however small, $s$ can always be taken sufficiently large

to make this exponential rate arbitrarily fast.

The remaining rate, $R - R_{\text{h}}$, is encoded in the complementary segment

of $n(1-\tau)$ and error exponent $\approx E_{\text{a}}(R - R_{\text{h}})$.

# The Converse Bound

Difficuly: Due to the helper, $X^n$ depends on $Z^n$ in an arbitrary manner.

Consequence: sphere–packing bound is based on a

change of measures:

$$Z_i \sim \mathcal{N}(0, \sigma^2) \to \mathcal{N}(0, \tilde{\sigma}^2) \text{ s.t. } \frac{1}{2} \log \left(1 + \frac{P}{\tilde{\sigma}^2}\right) + R_{\mathsf{h}} = R$$

rather than

$$Z_i \sim \mathcal{N}(0, \sigma^2) \to \mathcal{N}(\theta x_i, \tilde{\sigma}^2)$$

for the worst $(\theta, \tilde{\sigma}^2)$ over channels of capacity $\leq R$.

The result is the weaker version of the sphere–packing bound.

# More in the Full Article:

- General continuous–alphabet additive channels.
- The modulo–additive channel:
  - Fixed–rate helper
  - Variable–rate helper
- The Gaussian MAC with a given total help rate for both encoders.