Conveying Individual Source Sequences over Memoryless Channels using Finite–State Decoders with Source Side Information

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Abstract

We study the following semi–deterministic setting of the joint source–channel coding problem: a deterministic source sequence (a.k.a. individual sequence) is transmitted via a memoryless channel, and decoded by a delay–limited, finite–state decoder with access to side information, which is a noisy version of the source sequence. We first prove a separation theorem with regard to the expected distortion between the source sequence and its reconstruction, which is an extension of earlier work of the same flavor (Ziv 1980, Merhav 2014), but without decoder side information. We then derive a lower bound to the best achievable excess–distortion probability and discuss situations where it is achievable. Here, of course, source coding and channel coding cannot be completely separated if one wishes to meet the bound. Finally, we outline a few variations and extensions of the model considered, such as: (i) incorporating a common reconstruction constraint, (ii) availability of side information at both ends, and (iii) extension to the Gel'fand-Pinsker channel.

Index Terms: Wyner–Ziv problem, Gel’fand–Pinsker channel, individual sequences, separation theorem, joint source–channel coding, finite–state machine, delay, excess–distortion exponent.
1 Introduction

In a collection of works that appeared during the late seventies and eighties of the previous century, Ziv [22], [23], [24], and Ziv and Lempel [11], [26], have established a fascinating theory of universal source coding for deterministic sequences (a.k.a. individual sequences) by means of encoders/decoders that are implementable using finite-state machines. Specifically, in [22] Ziv addressed the issue of fixed-rate, universal (nearly) lossless compression of deterministic source sequences using finite-state encoders and decoders, which was later further developed to the celebrated Lempel–Ziv algorithm [11], [26]. In [23], the model setting of [22] was broadened to lossy transmission over both clean and noisy channels (subsections II.A and II.B therein, respectively), where in the noisy case, the channel was modeled as an ordinary, probabilistic memoryless channel, as opposed to the source sequence, that was still assumed deterministic. Henceforth, we will refer to this type of setting as a semi-deterministic setting, similarly as in [14].

Subsequently, the results of the first part of [23] (clean channels) were further elaborated in other directions, such as exploiting side information in a scenario of almost lossless source coding, where the side information is modeled too as being deterministic [24], i.e., a deterministic analogue of Slepian–Wolf coding was investigated in [24]. More than two decades later, this setup was extended to the lossy case [16], that is, a semi–deterministic counterpart of Wyner–Ziv coding, where the source to be compressed is deterministic, but the side information available to the decoder is generated from the source sequence via a discrete memoryless channel (DMC). In [14], a few inaccuracies in the coding theorem for noisy channels in [23, Subsection II.B] were corrected, and it was strengthened and refined from several aspects. Among other things, in [14], only the decoder was assumed to be a finite–state machine, while the encoder was allowed to be rather general (as opposed to [23], where the encoder was assumed to be a finite–state machine too). Also, the finite–state decoder of [14] was allowed to be periodically time–varying with a given period length, \( \ell \), along with a modulo–\( \ell \) time counter (clock).

In this work, we further develop the findings of [14] and [16] in a few directions, and at the same time, we also take the opportunity to correct some (minor) imprecisions in [14] and improve the rigor of the derivation. In our more general model, we still allow a general encoder and a periodically time–varying, delay–limited, finite–state decoder, but we also allow the decoder to
access side information, which is a noisy version (corrupted by a DMC) of the deterministic source sequence to be conveyed – see Fig. 1. In other words, it is a semi-deterministic setting of a joint source–channel coding problem that combines the semi-deterministic version of the Wyner–Ziv (W-Z) source model with the DMC, in analogy to the purely stochastic version of this model [17]. The W-Z channel (see Fig. 1) can be motivated by the uncoded transmission of the systematic part of a systematic code (see also [15], [17]).

At first glance, one might wonder about this asymmetric modeling approach of the semi-deterministic setting (both here and in the earlier works, [14], [16], [23]), where the source is regarded completely deterministically, without any statistical assumptions, whereas the channels (namely, both the main channel and the Wyner–Ziv channel) are modeled probabilistically, exactly as in the classical tradition of the information theory literature. The motivation for this distinction, is that in many frequently encountered situations, the channels obey some relatively well-understood physical laws that can be reasonably well be modeled probabilistically, whereas the source to be conveyed is very different in nature. Indeed, in many applications, the source is a man–made data file (or a group of files), generated using artificial means. These include computer–generated images and video streams, texts of various types, audio signals (such as music), sequences of output results from computer calculations, and any combinations of those. It is simply inconceivable to use ordinary probabilistic models for such sources.

For the above described semi-deterministic model, we first derive a lower bound to the minimum achievable expected distortion between the source and its reconstruction at the decoder. This lower bound depends only the $\ell$–th order empirical statistics of the source ($\ell$ being the period of the decoder, as described above), the allowed delay, $d$, the number of states, $s$, and the capacity, $C$, of the channel. It can be nearly achieved by separate W-Z source coding and channel coding, using long block codes, provided that both $d$ and $\log s$ are small compared to $\ell$.

In addition to the the expected distortion, we also address a related, but different figure of merit – the probability of excess distortion, similarly as in [4] and [12]. We first derive a lower bound to this probability for the simpler model setting without side information, as in [23] and [14]. We relate it to the probability of excess to distortion in the purely probabilistic setting [3], [4], and then discuss when this bound is asymptotically achievable. Subsequently, we extend the scope to
the above-described model with decoder side information. Achievability is discussed too, but it should be pointed out that our emphasis, in this work, is on fundamental limits and lower bounds, more than on achievability.

In the last part of this work, we discuss a few variants of the model, where more explicit results can be stated, such as the case where the source side information is available at the encoder too. The case where a common reconstruction constraint is imposed, following [19], is also presented. Finally, we discuss an extension where the ordinary DMC is replaced by the Gel’fand-Pinsker (G-P) channel [7].

The outline of the remaining part of this article is as follows. In Section 2, we establish notation conventions and formalize the problem setting and the objectives. In Section 3, we provide a few additional definitions in order to establish the preparatory background needed to state the main results, and we also provide a preliminary result, for the case of an ordinary DMC and without side information. In Section 4, we provide the extension that incorporates source side information, and finally, in Section 5, we outline a few modifications and extensions of our setting as described in the previous paragraph.

2 Notation Conventions and Problem Formulation

2.1 Notation Conventions

Throughout the paper, random variables will be denoted by capital letters, specific values they may take will be denoted by the corresponding lower case letters, and their alphabets will be denoted by calligraphic letters. Similarly, random vectors, their realizations, and their alphabets, will be denoted, respectively, by capital letters, the corresponding lower case letters, and calligraphic letters, all superscripted by their dimensions. For example, the random vector $Y^n = (Y_1, \ldots, Y_n)$, ($n$ – positive integer) may take a specific vector value $y^n = (y_1, \ldots, y_n)$ in $\mathcal{Y}^n$, the $n$–th order Cartesian power of $\mathcal{Y}$, which is the alphabet of each component of this vector. An infinite sequence will be denoted by the bold face font, for example, $u = (u_1, u_2, \ldots)$. The notation $u_i$, on the other hand, will be used to denote the $i$–th $\ell$–block $(u_{i\ell+1}, u_{i\ell+2}, \ldots, u_{i\ell+\ell})$. For $i \leq j$, $(i, j$ – positive integers), $x^j_i$ will denote the segment $(x_i, \ldots, x_j)$, where for $i = 1$ the subscript will be omitted. If, in addition, $j = 1$, the superscript will be omitted too, and the notation will be simply $x$. 

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Owing to the semi-deterministic modeling approach, we distinguish between two kinds of random variables: ordinary random variables (or vectors), governed by certain given probability distributions (like the channel output vector), and auxiliary random variables that emerge from empirical distributions associated with certain sequences. Random variables of the second kind will be denoted using ‘hats’. For example, consider a deterministic sequence \( u^n = (u_1, \ldots, u_n) \). Then, \( \hat{U}^\ell = (\hat{U}_1, \ldots, \hat{U}_\ell) \) designates an auxiliary random vector, ‘governed’ by the empirical distribution extracted from the non-overlapping \( \ell \)-blocks of \( u^n \), provided that \( \ell \) divides \( n \) (this empirical distribution will be defined precisely in the sequel). We denote this empirical distribution by \( P_{\hat{U}^\ell} = \{ P_{\hat{U}^\ell}(u^\ell), \ u^\ell \in U^\ell \} \). The use of empirical distributions, however, will not be limited to deterministic sequences only. It could be defined also for (realizations of) random sequences. For example, if \( Y^n \) is a random sequence, \( \hat{Y}^\ell \) would designate the auxiliary random \( \ell \)-vector associated with the (random) realization, \( y^n \). In this case, the empirical distribution, \( P_{Y^\ell}, \) is of course, itself random, but it may converge to the true \( \ell \)-th order distribution of \( Y^\ell, \) \( P_{Y^\ell}, \) under certain conditions. Information measures, like entropies, conditional entropies, divergences, and mutual informations, will be denoted according to the conventional rules of the information theory literature, where it should be kept in mind that these measures may involve both ordinary and auxiliary random variables. For example, \( I(\hat{U}^\ell; Y^\ell) \) and \( H(Y^\ell|\hat{U}^\ell) \) are, respectively, the mutual information and the conditional entropy induced by the empirical distribution of \( \hat{U}^\ell, \) \( P_{\hat{U}^\ell}, \) and the conditional distribution, \( P_{Y^\ell|\hat{U}^\ell}, \) of the ordinary random vector, \( Y^\ell, \) given the auxiliary random vector, \( \hat{U}^\ell. \) The conditional divergence, \( D(Q_{Y|\hat{X}}||P_{Y|\hat{X}}|P_{\hat{X}}), \) will be understood to be given by

\[
D(Q_{Y|\hat{X}}||P_{Y|\hat{X}}|P_{\hat{X}}) = \sum_{x \in \hat{X}} P_{\hat{X}}(x) \sum_{y \in \hat{Y}} Q_{Y|\hat{X}}(y|x) \log \frac{Q_{Y|\hat{X}}(y|x)}{P_{Y|\hat{X}}(y|x)},
\]

where logarithms, here and throughout the sequel, will be understood to be taken to the base 2, unless specified otherwise.

### 2.2 Problem Formulation

Referring to Fig. 1, let \( \mathbf{u} = (u_1, u_2, \ldots) \) be a deterministic source sequence of symbols in a finite alphabet \( \mathcal{U} \) of cardinality \( |\mathcal{U}| = \alpha \). The sequence \( \mathbf{u} \) is encoded using a general block encoder of length \( n \), whose output at time \( i \) is \( x_i \in \mathcal{X} \), where \( \mathcal{X} \) is another finite alphabet\(^1\) of size \( |\mathcal{X}| = \beta \).

\(^1\)In general, the source and the channel may work at different rates, so that source vectors are mapped to channel vectors of different dimensions. This option is, in principle, available in this model too, by defining \( \mathcal{U} \) and \( \mathcal{X} \) to be
Figure 1: Source–channel coding for the Wyner–Ziv source. The decoder is assumed to be a delay–limited, finite–state machine, as can be seen on the right.

The sequence $\mathbf{x} = (x_1, x_2, \ldots)$ is fed into a DMC, characterized by the single-letter transition probabilities $\{P_{Y|X}(y|x), x \in \mathcal{X}, y \in \mathcal{Y}\}$, where $\mathcal{Y}$ is a finite alphabet of size $|\mathcal{Y}| = \gamma$. The channel output $\mathbf{y} = (y_1, y_2, \ldots)$ is fed into a delay–limited, periodically time–varying finite–state decoder, which is defined by

$$
t = i \mod \ell \quad (2)
$$

$$
v_{i-d} = f_t(w_i, y_i, z_i), \quad i = d + 1, d + 2, \ldots \quad (3)
$$

$$
z_{i+1} = g_t(w_i, y_i, z_i), \quad i = 1, 2, \ldots \quad (4)
$$

where $z_i \in \mathcal{Z}$ is the decoder state at time $i$, $\mathcal{Z}$ being a finite set of states of size $s$, $w_i \in \mathcal{W}$ is the side information at time $i$, and $v_{i-d}$ is the reconstructed version of the source sequence, delayed by $d$ time units ($d$ – positive integer). The reconstruction alphabet is $\mathcal{V}$ of size $\delta$. The side information sequence, $\mathbf{w} = (w_1, w_2, \ldots)$, is generated from $\mathbf{u}$ by means of a DMC, characterized by a matrix of transition probabilities, $P_{W|U} = \{P_{W|U}(w|u), u \in \mathcal{U}, w \in \mathcal{W}\}$. The function $f_t$ is called the output function of the decoder and the function $g_t$ is called the next–state function. As can be seen, both functions are periodically time–varying with a period of length $\ell$.

Let $V^n = (V_1, \ldots, V_n)$, $W^n = (W_1, \ldots, W_n)$, and $Y^n = (Y_1, \ldots, Y_n)$, designate random vectors pertaining to the variables $v^n = (v_1, \ldots, v_n)$, $w^n = (w_1, \ldots, w_n)$, and $y^n = (y_1, \ldots, y_n)$, respectively, where the randomness stems from the main channel and the W–Z channel. The vector $u^n$ clearly superalphabets of the appropriate sizes, for the given bandwidth expansion factor.
does not have a stochastic counterpart. The same comment applies also to \( x^n \) since the encoder is assumed deterministic (without loss of optimality).

The objectives of the paper are the following: given the source sequence \( u^n \), the channel, \( P_{Y|X} \), and the W–Z channel \( P_{W|U} \), the number of decoder states, \( s \), the decoder period, \( \ell \), and the allowed delay, \( d \), and given a single–letter distortion function \( \rho : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^+ \), we wish to find non–trivial lower bounds to:

1. The expected distortion, \( \frac{1}{n} \sum_{i=1}^{n} E\{\rho(u_i, V_i)\} \), and
2. The probability of excess distortion, \( \Pr\{\sum_{i=1}^{n} \rho(u_i, V_i) \geq nD\} \), where \( D > 0 \) is a constant larger than the best achievable normalized expected distortion.

In some instances of the problem, we will also discuss asymptotic achievability.

3 Background and Preliminary Results

Before moving forward to our main results, we need a few more definitions, as well as some background on the relevant results from [14]. We conclude this section with a preliminary result on the excess distortion probability in the case of an ordinary DMC and without source–related side information at the decoder.

Let \( \ell \) divide \( n \) and consider the segmentation of all relevant sequences into \( n/\ell \) non–overlapping blocks of length \( \ell \), that is,

\[
u^n = (u_0, u_1, \ldots, u_{n/\ell - 1}), \quad u_i = (u_{i\ell+1}, u_{i\ell+2}, \ldots, u_{i\ell+\ell}), \quad i = 0, 1, \ldots, n/\ell - 1,\]

and similar definitions for \( v^n, w^n, x^n, \) and \( y^n \), where \( v_{n-d+1}, v_{n-d+2}, \ldots, v_n \) (which are not yet reconstructed at time \( t = n \)) are defined as arbitrary symbols in \( \mathcal{V} \). Let us define the empirical joint probability mass function

\[
P_{\hat{U}^{\ell} \hat{V}^{\ell} \hat{W}^{\ell} \hat{X}^{\ell} \hat{Y}^{\ell} \hat{Z}}(u^{\ell}, v^{\ell}, w^{\ell}, x^{\ell}, y^{\ell}, z) = \frac{1}{n} \sum_{i=0}^{n/\ell - 1} \delta(u_i = u^{\ell}, v_i = v^{\ell}, w_i = w^{\ell}, x_i = x^{\ell}, y_i = y^{\ell}, z_{i\ell+1} = z),\]

where \( \delta(\cdot, \cdot, \ldots, \cdot) \) is the indicator function of the combination of events indicated in its argument. Clearly, since \( P_{\hat{U}^{\ell} \hat{V}^{\ell} \hat{W}^{\ell} \hat{X}^{\ell} \hat{Y}^{\ell} \hat{Z}} \) is a legitimate probability distribution, all the rules of manipulating
information measures (the chain rule, conditioning reduces entropy, etc.) hold as usual. Marginal and conditional marginal distributions associated with subsets of the set of random variables, \((\hat{U}^\ell, \hat{V}^\ell, \hat{W}^\ell, \hat{X}^\ell, \hat{Y}^\ell, \hat{Z})\), which are derived from \(P_{\hat{U}^\ell\hat{V}^\ell\hat{W}^\ell\hat{X}^\ell\hat{Y}^\ell\hat{Z}}\), will be denoted using the conventional notation, for example, \(P_{\hat{U}^\ell\hat{V}^\ell}\) is the joint empirical distribution of \((\hat{U}^\ell, \hat{V}^\ell)\), \(P_{Y^\ell|\hat{X}^\ell\hat{Z}}\) is the conditional empirical distribution of \(\hat{Y}^\ell\) given \((\hat{X}^\ell, \hat{Z})\), and so on.

We now define the W-Z rate-distortion function \([20]\) of the source \(P_{\hat{U}^\ell}\) with respect to the real side-information channel \(P_{W^\ell|\hat{U}^\ell}\) (as opposed to the empirical side-information channel, \(P_{\hat{W}^\ell|\hat{U}^\ell}\)) according to

\[
R_{\hat{U}^\ell|W^\ell}(D) = \frac{1}{\ell} \min \{ I(\hat{U}^\ell; A) - I(W^\ell; A) \} \equiv \min \frac{1}{\ell} I(\hat{U}^\ell; A|W^\ell),
\]

(7)

where both minima are taken over all conditional distributions, \(\{P_{A|\hat{U}^\ell}\}\), such that \(A \to \hat{U}^\ell \to W^\ell\) is a Markov chain and \(\min_{\{G: A \to W^\ell \to Y^\ell\}} \mathbb{E}\{\rho(\hat{U}^\ell, G(A, W^\ell))\} \leq \ell \cdot D\), where \(\rho(u^\ell, v^\ell)\) is defined additively as \(\sum_{i=1}^{\ell} \rho(u_i, v_i)\) and \(A\) is an auxiliary RV whose alphabet size is \(|A| = \alpha + 1\). It follows from these definitions that if \(W^\ell \to \hat{U}^\ell \to A\) is a Markov chain, then

\[
I(\hat{U}^\ell; A) - I(W^\ell; A) \equiv I(\hat{U}^\ell; A|W^\ell) \geq \ell \cdot R_{\hat{U}^\ell|W^\ell}[\Delta(\hat{U}^\ell|W^\ell, A)/\ell],
\]

(8)

where

\[
\Delta(\hat{U}^\ell|W^\ell, A) \equiv \min_{G: W^\ell \times A \to U^\ell} \mathbb{E}\{\rho(\hat{U}^\ell, G(W^\ell, A))\},
\]

(9)

where \(A\) is the alphabet of \(A\) and the expectation is taken with respect to \((w.r.t.)\) \(P_{\hat{U}^\ell\hat{V}^\ell\hat{W}^\ell} = P_{\hat{U}^\ell\hat{V}^\ell\hat{W}^\ell} \times P_{W^\ell|\hat{U}^\ell}\). Clearly, if \(W^\ell\) is independent of \(\hat{U}^\ell\), that is, \(P_{W^\ell|\hat{U}^\ell}(\cdot | u^\ell)\) is the same for all \(u^\ell \in \hat{U}^\ell\), then \(R_{\hat{U}^\ell|W^\ell}(D)\) degenerates to the ordinary rate-distortion function, which will be denoted by \(R_{\hat{U}^\ell}(D)\). In the sequel, we will also refer to the conditional rate-distortion, of \(\hat{U}^\ell\) given \(W^\ell\), which will be denoted by \(R_{\hat{U}^\ell|W^\ell}(D)\), where \(W^\ell\) is available to both encoder and decoder. This function is also given by the minimum of \(I(\hat{U}^\ell; A|W^\ell)/\ell\), except that the Markov condition is dropped \([21]\).

The corresponding distortion-rate functions, \(D_{\hat{U}^\ell|W^\ell}(R), D_{\hat{U}^\ell}(R),\) and \(D_{\hat{U}^\ell|W^\ell}(R)\), are the inverse functions of \(R_{\hat{U}^\ell|W^\ell}(D), R_{\hat{U}^\ell}(D)\), and \(R_{\hat{U}^\ell|W^\ell}(D)\), respectively.

For the given channel, \(P_{Y^\ell|X^\ell}\), we denote by \(C_{P_{Y^\ell|X^\ell}}(\Gamma)\) the channel capacity with a transmission cost constraint, \(\sum_{i=1}^{n} \mathbb{E}\{\phi(X_i)\} \leq n\Gamma\) (\(\phi(\cdot)\) being the single-letter transmission cost function), that is

\[
C_{P_{Y^\ell|X^\ell}}(\Gamma) = \max_{P_X: \mathbb{E}\{\phi(X)\} \leq \Gamma} I(X; Y),
\]

(10)
When the channel, $P_{Y|X}$, is clear from the context, the subscript “$P_{Y|X}$” will be omitted, and the notation will be simplified to $C(\Gamma)$.

In [14], we considered the simpler case without decoder side information, $w$, related to the source. One of the main results of [14] (in particular, Theorem 1 therein) is a lower bound to the expected distortion, which has the following form:

$$\frac{1}{n} \sum_{i=1}^{n} E\{\rho(u_i, V_i)\} \geq D_{\hat{U}^\ell}(C(\Gamma) + \zeta(s, d, \ell) + \epsilon(\ell, n)),$$

where

$$\epsilon(\ell, n) \triangleq \frac{(\alpha \beta)^\ell \log \gamma}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

and $\zeta(s, d, \ell)$ is a certain function\(^2\) with the property $\lim_{\ell \to \infty} \zeta(s, d, \ell) = 0$. As discussed in [14], it is interesting that the distortion bound depends on $u^n$ only via its $\ell$–th order empirical distribution, $P_{\hat{U}^\ell}$, where, as defined above, $\ell$ is duration of the decoder’s period. It is also discussed in that work that the term $\zeta(s, d\ell)$, on the right–hand side, can be thought of as an “extra capacity” term, that is induced by the memory of the decoder (encapsulated in the decoder state, $z_i$) and the allowed delay, but its effect is diminished when $\ell$ is chosen large. The bound can then be asymptotically approached by separate source– and channel coding, using long block codes. On the other hand, by letting $\ell$ grow, one also affects the distortion–rate function, $D_{\hat{U}^\ell}(\cdot)$, and so, the overall effect of $\ell$ is not trivial to assess in general.

We now state our preliminary result on the excess distortion probability for the case of DMC and without any side information.

**Theorem 1** Assume that $\rho_{\text{max}} = \max_{u,v} \rho(u,v) < \infty$. For any given $u^n$, any block encoder of length $n$, and any finite–state decoder with $s$ states and delay $d$,

$$\Pr\left\{\sum_{i=1}^{n} \rho(u_i, V_i) \geq nD\right\} \geq \sup_{\Delta > 0} \left[\frac{\Delta}{\rho_{\text{max}} - D} - o(n)\right] \times \exp\left\{-(n + d) E_{sp} \left[R_{\hat{U}^\ell}(D + \Delta) - \zeta(s, d, \ell) - \epsilon(\ell, n)\right]\right\},$$

where $E_{sp}(R)$ is the sphere–packing exponent of the channel, i.e.,

$$E_{sp}(R) = \sup_{Q_X} \inf_{\{Q_{Y|X}: I_Q(X;Y) \leq R\}} D(Q_{Y|X}||P_{Y|X|Q_X}),$$

\(^2\)The exact form of this function is immaterial for the purpose of this discussion. In fact, the formula of $\zeta(s, d, \ell)$, given in [14] is slightly imprecise, and so, we both correct and extend it in this work. Nevertheless, the property $\lim_{\ell \to \infty} \zeta(s, d, \ell) = 0$ remains valid.
with \( I_Q(X;Y) \) denoting the mutual information induced by \( Q_X \times Q_{Y|X} \).

As a simple conclusion from this theorem, we have that

\[
\lim_{n \to \infty} \frac{1}{n} \log \left[ \Pr \left\{ \sum_{i=1}^{n} p(u_i, V_i) \geq nD \right\} \right] \geq -E_{\text{sp}} \left[ R_{\hat{U}_t}(D + 0) - \zeta(s, d, \ell) \right], \quad (15)
\]

where

\[
\hat{R}_{\hat{U}_t}(D + 0) \triangleq \lim_{\Delta \to 0} \hat{R}_{\hat{U}_t}(D + \Delta). \quad (16)
\]

**Discussion.**

First, observe that the “extra capacity” term, \( \zeta(s, d, \ell) \), plays a role here too. This time, it appears in the form of an effective rate reduction in the argument of the sphere–packing error exponent. But once again, if \( \ell \) is very large while \( s \) and \( d \) are fixed, this term becomes insignificant. In this case, as long as \( D \) is smaller than \( D_{\hat{U}_t}(R_{\text{crit}}) \), \( R_{\text{crit}} \) being the critical rate [6] of the channel (and assuming \( D_{\hat{U}_t}(R_{\text{crit}}) > 0 \)), the bound is asymptotically achievable using long blocks (of size \( n \gg \ell \)), by rate–distortion coding (based on the type covering lemma) in the superalphabet of \( \ell \)-vectors, followed by channel coding, as in Csiszár’s works, [3] and [4, Theorem 2]. As in those references, strictly speaking, this is not quite considered separate source– and channel coding, because there is a certain linkage between the channel code design and the source: Each type class of source sequences (in the level of \( \ell \)-blocks) is mapped into a channel sub–code at rate \( \hat{R}_{\hat{U}_t}(D) \) (approximately), and the corresponding channel codewords are of the type, \( P_{\hat{X}} \), that achieves the maximum sphere–packing exponent at that particular rate.

Speaking of Csiszár’s source–channel error exponents, [3] and [4], it is interesting to relate Theorem 1 above to its purely probabilistic counterpart. The proof of Theorem 1 above is based on a change–of–measure argument. Since only the channel is probabilistic in our setting, the upper bound on the exponent includes only a channel–related term, which is the channel’s sphere–packing exponent. Applying a similar line of thought in the purely probabilistic case, we have to change measures for both the source and the channel, and so, we end up minimizing the sum of two divergence terms, i.e.,

\[
\min_{\{ (Q_U, Q_{Y|X}): R_{Q_U}(D) \geq I_Q(X;Y) \}} \left\{ D(Q_U || P_U) + D(Q_{Y|X} || P_{Y|X} | Q_X) \right\}, \quad (17)
\]
where \( P_U \) is the memoryless source and \( P_{Y|X} \) is the memoryless channel. This upper bound on the joint source–channel exponent can be further upper bounded by arbitrarily selecting a positive real \( R \) and arguing that

\[
\min_{\{Q_U, Q_{Y|X}: R_{Q_U}(D) \geq R \geq I_{Q}(X;Y)\}} \left\{ D(Q_U \| P_U) + D(Q_{Y|X} \| P_{Y|X}|Q_X) \right\}
\leq \min_{\{Q_U : R_{Q_U}(D) \geq R \}} D(Q_U \| P_U) + \min_{\{Q_{Y|X} : I_{Q}(X;Y) \leq R \}} D(Q_{Y|X} \| P_{Y|X}|Q_X)
\leq F(R, D, U) + E_{sp}(R),
\]

where \( F(R, D, U) \) is Marton’s source coding exponent \([12]\) for the memoryless source \( U \sim P_U \). Since this argument is applicable to any value of \( R \), the tightest upper bound is obtained by minimizing over \( R \), namely, the resulting upper bound on the exponent is

\[
\min_R [F(R, D, U) + E_{sp}(R)],
\]

which coincides with Csiszár’s upper bound \([4, \text{Theorem 4}]\) to the best achievable excess–distortion exponent. This argument, however, is quite different from the one used in \([4]\), which in turn is based on the list–decoding argument of Shannon, Gallager and Berlekamp \([18]\), that originally, sets the stage for the straight–line bound \([6, \text{Theorem 5.8.2}]\).

The remaining part of this section is devoted to the proof of Theorem 1.

**Proof of Theorem 1.** Let \( \Delta > 0 \) be arbitrarily small and let \( Q_{Y^n|X^n}(y^n|x^n) = \prod_{i=1}^n Q_{Y|X}(y_i|x_i) \) be an auxiliary DMC such that

\[
\ell \cdot R_{U^\ell}(D + \Delta) \geq \ell \cdot I_Q(\hat{X};Y) + \ell \zeta(s, d, \ell) + \ell \epsilon(\ell, n) \triangleq \ell [I_Q(\hat{X};Y) + \lambda],
\]

where the empirical channel input distribution \( P_{\hat{X}} \) is induced by \( u^n \) and the encoder. Since \( Q_{Y|X} \) is assumed memoryless, we have the following relationship between \( I_Q(\hat{X};Y) \) and \( I_Q(\hat{X}^{\ell};Y^{\ell}) \):

\[
\frac{I_Q(\hat{X}^{\ell};Y^{\ell})}{\ell} \leq \frac{1}{\ell} \sum_{j=1}^{\ell} I_Q(\hat{X}_j;Y_j) \leq I(\hat{X}_j;Y_j|J)
\]

\[
= H(Y_j|J) - H(Y_j|\hat{X}_j, J)
\]

\[
\leq H(Y_j) - H(Y_j|\hat{X}_j, J)
\]

\[
(21)
\]

\[
(22)
\]

\[
(23)
\]

\[
(24)
\]
\[ H(Y_J) = H(Y_J|\hat{X}_J) \]
\[ = I_Q(\hat{X}_J; Y_J) \]
\[ = I_Q(\hat{X}; Y), \]

where \( \hat{X}_J \) is the random variable derived from the \( j \)-th marginal of \( P_{\hat{X}_J} \), \( j = 1, 2, \ldots, \ell \), \( J \) is an integer random variable, uniformly distributed over \( \{1, 2, \ldots, \ell\} \), and where we have used the Markovity of the chain \( J \leftrightarrow \hat{X}_J \leftrightarrow Y_J \), and the identities \( \hat{X}_J = \hat{X}, Y_J = Y \), which follow from the fact that \( P_{\hat{X}} = \frac{1}{\ell} \sum_{j=1}^{\ell} P_{\hat{X}_J} = P_{\hat{X}_J} \). Therefore, if \( Q \) satisfies (20), it must also satisfy

\[ \ell \cdot R_{U_{\ell}}(D + \Delta) \geq I_Q(\hat{X}_J; Y_J) + \ell \lambda. \]  

According to the above cited Theorem 1 of [14], for such a channel, the expected distortion under \( Q \), denoted \( D_Q \), must be lower bounded by

\[ D_Q = \frac{1}{n} \mathbb{E}_Q \left\{ \rho(u^n, V^n) \right\} \]
\[ = \frac{1}{n} \mathbb{E}_Q \{ \rho(u^n, V^n) \cdot 1(\mathcal{E}) \} + \frac{1}{n} \mathbb{E}_Q \{ \rho(u^n, V^n) \cdot 1(\mathcal{E}^c) \} \]
\[ \leq \frac{1}{n} \cdot Q(\mathcal{E}) \cdot n \rho_{\text{max}} + \frac{1}{n} \left[ 1 - Q(\mathcal{E}) \right] \cdot n D \]
\[ = [1 - Q(\mathcal{E})] \cdot D + Q(\mathcal{E}) \cdot \rho_{\text{max}}, \]

which implies that

\[ Q(\mathcal{E}) \geq \frac{D_Q - D}{\rho_{\text{max}} - D} \geq \frac{D + \Delta - D}{\rho_{\text{max}} - D} = \frac{\Delta}{\rho_{\text{max}} - D}. \]

Now, for a given, arbitrarily small \( \epsilon_0 > 0 \), let us define

\[ \mathcal{T} = \left\{ y^{n+d} : \sum_{i=1}^{n+d} \ln \frac{Q_{Y|X}(y_i|x_i)}{P_{Y|X}(y_i|x_i)} \leq (n + d)[D(Q_{Y|X}||P_{Y|X}P_X) + \epsilon_0] \right\}, \]

where here \( P_X \) is the single-symbol empirical distribution extracted from \( x^{n+d} \). Now,

\[ \Pr \{ \rho(u^n, V^n) \geq nD \} = \sum_{y^{n+d} \in \mathcal{E}} P_{Y^{n+d}|X^{n+d}}(y^{n+d}|x^{n+d}) \]
\[ \geq \sum_{y^{n+d} \in \mathcal{E} \cap \mathcal{T}} P_{Y^{n+d}|X^{n+d}}(y^{n+d}|x^{n+d}) \]

12
Theorem 2
result is the following:
Consider again the more general setting described in Section 2 and depicted in Fig. 1. Our first
length
allowed delay,
which completes the proof of Theorem 1 by the arbitrariness of
Since
Q
only via its empirical distribution from the order that corresponds to the period of the decoder,
Discussion.
where we have used the fact that \(Q(T^c) = o(n)\) for every \(\epsilon_0 > 0\), by the weak law of large numbers.
Since \(Q_{Y|X}\) is an arbitrary channel that satisfies (20), we have
\[
\Pr \{\rho(u^n, V^n) \geq nD\} \geq \left[ \frac{\Delta}{\rho_{\text{max}} - D} - o(n) \right] \times \\
\exp \left\{ -(n + d) \sum \inf_{\{Q_{Y|X} : I_Q(X; Y) \leq R_{U|T}(D + \Delta) - \lambda\}} D(Y|X; X_{\hat{P}}) + \epsilon_0 \right\} \\
\geq \left[ \frac{\Delta}{\rho_{\text{max}} - D} - o(n) \right] \cdot \exp \left\{ -(n + d) \left( E_{\lambda}[R_{U|T}(D + \Delta) - \lambda] + \epsilon_0 \right) \right\},
\]
which completes the proof of Theorem 1 by the arbitrariness of \(\epsilon_0 > 0\).

4 Side Information at the Decoder

Consider again the more general setting described in Section 2 and depicted in Fig. 1. Our first result is the following:

**Theorem 2** Assume that \(\rho_{\text{max}} = \max_{u,v} \rho(u,v) < \infty\). For any given \(u^n\), any block encoder of length \(n\), and any finite-state decoder with \(s\) states and delay \(d\),
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \{\rho(u_i, V_i)\} \geq D_{U|T}^{\text{HY}} \left( C(T) + \frac{\log s}{\ell} + \epsilon(\ell, n) \right) - \frac{\rho_{\text{max}} d}{\ell}.
\]

Discussion.

Similarly as in Theorem 1, the distortion lower bound depends on the source sequence, \(u^n\), only via its empirical distribution from the order that corresponds to the period of the decoder, \(\ell\).
We also observe that the two resources of the decoder, namely, the number of states, \(s\), and the allowed delay, \(d\), take parts in the lower bound in two different ways. The former plays the role
of ‘excess capacity’, whereas the latter serves in a term of distortion reduction. But this difference is not really crucial, because excess rate and reduced distortion are two faces of the same coin. Indeed, one of technical issues in the proof of Theorem 2 below (which was not handled perfectly rigorously in the parallel derivation in [14]), evolves around the following question: how can one assess the effect of the decoder state contribution in estimating the source (and thereby reducing the distortion relative to the absence of the state), in order to bound the distortion in terms of W-Z block code performance. In other words, the question is: how to obtain a lower bound in terms of block codes, where no state carries over from block to block? As will be seen in the proof below, the idea is that since the state cannot carry more than log s information bits about the source, its effect cannot be better than that of adding an excess rate of \( \Delta R = (\log s)/\ell \) to the corresponding W-Z encoder. This demonstrates clearly the point that distortion reduction can be traded with excess rate. Note that the resulting W-Z rate–distortion function here is different from the one in an earlier work on the same problem [16], where not only the decoder was assumed to be a finite-state machine, but also the encoder.

Once again, for very large \( \ell \) (compared to both \( d \) and \( \log s \)), the lower bound can be asymptotically approached by separation: W-Z coding in the superalphabet of \( \ell \)–vectors, followed by channel coding. This is in spite of the fact the W-Z channel violates the Markov structure of traditional \( u \to x \to y \to v \). This is coherent with the analogous behavior in the purely probabilistic setting [17], and moreover, even if the DMC is replaced by the Gel’fand–Pinsker (G-P) channel [15]. In the next section, we will refer to the case of the G-P channel, which is not trivial in our setting.

Finally, on the basis of Theorem 2, and similarly as in Theorem 1, one can easily derive a lower bound on the excess distortion probability for the case of decoder side information, considered here. This time, however, the change of measures should involve, not only the main channel, \( P_{Y|X} \), as before, but also the W-Z channel, \( P_{W|\hat{U}} \). The resulting exponential lower bound would be of the form,

\[
\Pr \left\{ \sum_{i=1}^{n} \rho(u_i, V_i) \geq nD \right\} \geq \sup_{\Delta > 0} \left[ \frac{\Delta}{\rho_{\text{max}} - D - o(n)} \times \exp \left\{ - (n + d) \max_{Q_{\hat{X}}} \min_{Q_{\hat{X}}} \left[ D(Q_{W|\hat{U}} \| P_{W|\hat{U}} P_{\hat{U}}) + D(Q_{Y|\hat{X}} \| P_{Y|\hat{X}} Q_{\hat{X}}) + 2\epsilon_0 \right] \right\} \right].
\]

(36)
where the minimum is over all pairs \( \{(Q_{W|U}, Q_{Y|X})\} \) such that
\[
R_{U^\ell, Q_{W|U}}^{WZ} \left( D + \frac{\rho_{\max} d}{\ell} + \Delta \right) \geq I_Q(\hat{X}; Y) + \log \frac{\log s}{\ell} + \epsilon(\ell, n),
\] (37)
where \( R_{U^\ell, Q_{W|U}}^{WZ}(\cdot) \) is the W-Z rate–distortion function of \( P_{U^\ell} \) with the W-Z channel, \( Q_{W|U} \). Similarly as before, the bound is asymptotically achievable following the same line of thought as in [3] and [4], for \( D < D_{U^\ell|W^\ell}(R_{\text{crit}}) \), where \( R_{\text{crit}} \) is the critical rate of the channel, provided that \( D_{U^\ell|W^\ell}(R_{\text{crit}}) > 0 \).

The remaining part of this section is devoted to the proof of Theorem 2.

Proof of Theorem 2. Owing to the decoder model (2), it is clear that \( v_{i\ell+\ell-d} \) is a deterministic function of \( z_{i\ell+1}, y_{i\ell+1} \) and \( w_{i\ell+1} \). Accordingly, we denote
\[
v_{i\ell+\ell-d} = m(z_{i\ell+1}, y_{i\ell+1}, w_{i\ell+1})
\] (38)
where \( m : Z \times W \times Y \rightarrow V^{\ell-d} \) is the function induced by the given finite–state decoder. The proof of the theorem is based on deriving both a lower bound and an upper bound to the expected empirical mutual information, \( E\{I(\hat{U}^\ell; \hat{Y}^\ell)\} \), which applies to any block encoder and any \( s \)-state decoder with delay \( d \).

As for an upper bound, we have following.
\[
E\{I(\hat{U}^\ell; \hat{Y}^\ell)\} \leq E\{I(\hat{U}^\ell, \hat{X}^\ell; \hat{Y}^\ell)\} = E\{H(\hat{Y}^\ell)\} - E\{H(\hat{Y}^\ell|\hat{U}^\ell, \hat{X}^\ell)\} \leq H(Y^\ell) - E\{H(\hat{Y}^\ell|\hat{U}^\ell, \hat{X}^\ell)\} \leq H(Y^\ell) - H(Y^\ell|\hat{X}^\ell) + \ell \epsilon(\ell, n) = I(\hat{X}^\ell; Y^\ell) + \ell \epsilon(\ell, n) \leq \ell \cdot [C(\Gamma) + \epsilon(\ell, n)].
\] (39)

The second and the third inequalities of this chain are explained as follows. The second inequality is due to the the concavity of the entropy as a functional of the underlying distribution, where \( P_{Y^\ell} \) is understood to be induced by \( P_{\hat{X}^\ell} \times P_{Y^\ell|\hat{X}^\ell} \). The third inequality is obtained, similarly as in [14, eqs. (23)–(32)], by invoking a known lower bound on the expectation of the empirical entropy
under a memoryless source (see [1]), and using it to obtain a lower bound the expectation of the empirical conditional entropy.

Note in passing that in the last step of (39) one could use a tighter upper bound: instead of maximizing over all \( \{ P_{X^\ell} \} \) that comply with the transmission cost constraint, we could have also maximized over all \( \{ P_{X^\ell} \} \) that maintain the same empirical single–letter marginal, \( P_{\hat{X}} \) (refer to eq. (21)). While this fact is immaterial for the expected distortion lower bound, it will be important when it comes to the lower bound on the excess–distortion probability.

To derive a lower bound to \( E\{ I(\hat{U}^\ell; \hat{Y}^\ell) \} \), we first underestimate \( I(\hat{U}^\ell; \hat{Y}^\ell) \) without taking the expectation.

\[
I(\hat{U}^\ell, \hat{Y}^\ell) = I(\hat{U}^\ell, W^\ell, \hat{Y}^\ell) \\
\geq I(\hat{U}^\ell, W^\ell; \hat{Y}^\ell) - I(W^\ell; \hat{Y}^\ell) \\
= I(\hat{U}^\ell; \hat{Y}^\ell|W^\ell) \\
\geq \ell \cdot R_W^{\hat{U\hat{Y}}}(\Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell)/\ell). \tag{43}
\]

where the underlying joint distribution of \( (\hat{U}^\ell, \hat{Y}^\ell, W^\ell) \) is assumed \( P_{\hat{U}\hat{Y}W} \times P_{W^\ell|\hat{U}\hat{Y}} \). The first equality follows from the fact that \( W^\ell \leftrightarrow \hat{U}^\ell \leftrightarrow \hat{Y}^\ell \) forms a Markov chain.

Consider now the joint distribution \( P_{\hat{U}\hat{Y}W\hat{Z}} \) is assumed \( P_{\hat{U}\hat{Y}W} \times P_{W^\ell|\hat{U}\hat{Y}} \), where \( \hat{Z} \) designates the state variable whose alphabet size is \( s \). We argue that no matter what this distribution may be, the best resulting distortion in estimating \( \hat{U}^\ell \) from \( (W^\ell, \hat{Y}^\ell, \hat{Z}) \), that is, \( \Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, \hat{Z}) \), cannot be better than the minimum achievable distortion when \( \hat{Z} \) is replaced by another random variable, \( Z^* \), of the same alphabet size \( s \), that is given by a deterministic function of \( \hat{U}^\ell \). Indeed,

\[
\Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, \hat{Z}) = \min_G \sum_{u^\ell} P_{\hat{U}^\ell}(u^\ell) \sum_z P_{\hat{Z}|\hat{U}^\ell}(z|u^\ell) \times
\sum_{w^\ell, y^\ell} P_{W^\ell\hat{Y}^\ell|\hat{U}^\ell\hat{Z}}(w^\ell, y^\ell|u^\ell, z) \rho(u^\ell, G(w^\ell, y^\ell, z)) \tag{44}
\]

is minimized by the conditional distribution, \( P_{\hat{Z}|\hat{U}^\ell} \), that puts all its mass on

\[
z^*(u^\ell) = \arg \min_z c(u^\ell, z) \tag{45}
\]

where

\[
c(u^\ell, z) \triangleq \sum_{w^\ell, y^\ell} P_{W^\ell\hat{Y}^\ell|\hat{U}^\ell\hat{Z}}(w^\ell, y^\ell|u^\ell, z) \rho(u^\ell, G(w^\ell, y^\ell, z)). \tag{46}
\]
Since $Z^*$ is a deterministic function of $\hat{U}^\ell$, it is available to the encoder. Consider now a coding scheme that transmits $I(\hat{U}^\ell; \hat{Y}^\ell|W^\ell)$ bits per $\ell$-vector using a Wyner–Ziv code (with $W^\ell$ serving as side information at the decoder), plus additional $\log s$ bits to transmit $Z^*$ as additional information on $\hat{U}^\ell$. The overall code-length of $I(\hat{U}^\ell; \hat{Y}^\ell|W^\ell) + \log s$ bits cannot be smaller than that of the best Wyner–Ziv code that makes $(W^\ell, \hat{Y}^\ell, Z^*)$ available to the decoder, with an extra rate of $\log s$ bits, as the latter can potentially exploit the statistical dependence between $\hat{U}^\ell$ and $Z^*$. In other words, the point $(I(\hat{U}^\ell; \hat{Y}^\ell|W^\ell) + \log s)/\ell, \Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, Z^*)/\ell)$ is an achievable point on the rate–distortion plane, which implies that

$$I(\hat{U}^\ell; \hat{Y}^\ell|W^\ell) + \log s \geq \ell \cdot R_{U^\ell|W^\ell}^{WZ}[\Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, Z^*)/\ell] \geq \ell \cdot R_{U^\ell|W^\ell}^{WZ}[\Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, \hat{Z})/\ell],$$

where the last equality follows from $\Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, Z^*) \leq \Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, \hat{Z})$ and the non-increasing monotonicity of the Wyner–Ziv rate–distortion function.

Let us now focus on the quantity $\Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, \hat{Z})$: \[
\frac{1}{\ell} \Delta(\hat{U}^\ell|W^\ell, \hat{Y}^\ell, \hat{Z}) = \min_G \frac{1}{\ell} \sum_{u^\ell, w^\ell, y^\ell, z} P_{U^\ell W^\ell \hat{Y}^\ell Z}(u^\ell, w^\ell, y^\ell, z) \rho(u^\ell, G(w^\ell, y^\ell, z)) \\
\leq \frac{1}{\ell} \sum_{u^\ell, w^\ell, y^\ell, z} P_{U^\ell W^\ell \hat{Y}^\ell Z}(u^\ell, w^\ell, y^\ell, z) \left[ \rho(u^\ell - d, m(w^\ell, y^\ell, z)) + \rho_{\text{max}} \cdot d \right] \\
= E \left\{ \frac{1}{n} \sum_{i=0}^{n/\ell-1} \left[ \sum_{\tau=1}^{\ell-d} \rho(u_{i\ell+\tau}, f_{\tau+d}(W_{i\ell+\tau+d}, y_{i\ell+\tau+d}, z_{i\ell+\tau+d})) + \rho_{\text{max}} \cdot d \right] \right\} \\
\leq E_W \left\{ \frac{1}{n} \sum_{i=1}^{n} \rho(u_i, V_i) \right\} + \frac{\rho_{\text{max}} \cdot d}{\ell},
\] where $E_W$ denotes expectation w.r.t. $W^n$ only. Thus,

$$I(\hat{U}^\ell; \hat{Y}^\ell|W^\ell) + \log s \geq \ell \cdot R_{U^\ell|W^\ell}^{WZ} \left( E_W \left\{ \frac{1}{n} \sum_{i=1}^{n} \rho(u_i, V_i) \right\} + \frac{\rho_{\text{max}} \cdot d}{\ell} \right).$$

Finally, combining this with (39) and (40), and taking the expectation w.r.t. the randomness of the channels, we have

$$\ell \cdot C(\Gamma) + \log s + \ell \cdot c(\ell, n) \geq E \left\{ R_{U^\ell|W^\ell}^{WZ} \left( E_W \left\{ \frac{1}{n} \sum_{i=1}^{n} \rho(u_i, V_i) \right\} + \frac{\rho_{\text{max}} \cdot d}{\ell} \right) \right\} \\
\geq R_{U^\ell|W^\ell}^{WZ} \left( \frac{1}{n} \sum_{i=1}^{n} E \left\{ \rho(u_i, V_i) \right\} + \frac{\rho_{\text{max}} \cdot d}{\ell} \right),$$

where in the last inequality we have used Jensen’s inequality and the convexity of the W-Z rate–distortion function [2, Lemma 15.9.1]. The assertion of Theorem 2 now follows immediately.
5 Variations, Modifications and Extensions

In this section, we outline a few variants of our main results to address several possible changes in the model considered. We discuss the following modifications: (i) the additional constraint of common reconstruction, (ii) the case where side information is available to the encoder too, and (iii) the G-P channel. In all these cases, we discuss the changes needed in proof of Theorem 2, and one can also obtain a lower bound to the excess distortion probability by applying the appropriate change of measures, following the same ideas as in the proof of Theorem 1, and as discussed after Theorem 2.

5.1 Common Reconstruction

In [19], Steinberg studied a version of the W-Z problem [20], where there is an additional constraint that the encoder would be capable of generating an exact copy of the reconstruction sequence to be generated by the decoder, with motivation in medical imaging, etc. In the ordinary W-Z setting, this is not the case since the reconstruction depends on the side information, which is not available to the encoder. Steinberg’s solution to the W-Z problem with common reconstruction is very similar to the solution of the regular W-Z problem: the only difference is that the estimator at the decoder is allowed to be a function of the compressed representation only, rather than being a function of both the compressed representation and the side information vector. In other words, in Steinberg’s scheme, the side information serves the decoder only for the purpose of binning, and not for both binning and estimation, as in the classical W-Z achievability scheme. For a pair of memoryless correlated sources, \( \{(U_i, W_i)\} \), Steinberg’s coding theorem [19, Theorem 1] for coding under the common reconstruction constraint, asserts that the corresponding rate–distortion function is given by

\[
R_{U|W}^{WZ,cr}(D) = \min \{I(U; V|W) \equiv \min \{I(U; V) - I(W; V)\},
\]

where the minimum is over all conditional distributions, \( \{P_{V|U}\} \), such that \( V \rightarrow U \rightarrow W \) is a Markov chain and \( E\{\rho(U, V)\} \leq D \).

Equipped with this background, we can impose the common reconstruction constraint in our setting too, provided that the model of the finite–state decoder is somewhat altered: Instead of feeding the finite–state decoder sequentially by \( \{(w_i, y_i)\} \), as in (2), we now feed it by a single
sequence, \( \{r_i\} \), where \( r^n = (r_1, \ldots, r_n) \) is a deterministic function of \( u^n \), which with very high probability (for large \( n \)), can be reconstructed faithfully at the decoder as a function of \( (u^n, y^n) \).

The modifications needed in the proof of Theorem 2 are in two places only: The first modification is that in the last line of eq. (40),

\[
R_{\hat{U}^\ell | W^\ell}^{\text{WZ}}[\Delta(\hat{U}^\ell | \hat{Y}^\ell, W^\ell) / \ell]
\]

should be replaced by

\[
R_{\hat{U}^\ell | W^\ell}^{\text{WZ,cr}}[E\{\rho(\hat{U}^\ell, \hat{V}^\ell)\} / \ell],
\]

where following [19], \( R_{\hat{U}^\ell | W^\ell}^{\text{WZ,cr}}(D) \) is defined according to

\[
R_{\hat{U}^\ell | W^\ell}^{\text{WZ,cr}}(D) = \frac{1}{\ell} \min I(\hat{U}^\ell; V^\ell | W^\ell),
\]

where the minimum is over all \( \{P_{\hat{V}^\ell | \hat{U}^\ell}\} \) such that \( \hat{V}^\ell \rightarrow \hat{U}^\ell \rightarrow W^\ell \) is a Markov chain and

\[
E\{\rho(\hat{U}^\ell, \hat{V}^\ell)\} \leq \ell \cdot D.
\]

The second modification is in eq. (48), where

\[
G(w^\ell, y^\ell, z), m(w^\ell, y^\ell, z)
\]

and

\[
f_{\tau+d}(w_{i\ell+\tau+d}, y_{i\ell+\tau+d}, z_{i\ell+\tau+d})
\]

should be replaced by

\[
G(r^\ell, z), m(r^\ell, z),
\]

and

\[
f_{\tau+d}(r_{i\ell+\tau+d}, z_{i\ell+\tau+d}),
\]

respectively.

The achievability is based on source coding using Steinberg’s coding scheme [19], followed by a capacity–achieving channel code.

### 5.2 Side Information at Both Ends

So far, we have considered the case where the side information is available at the decoder only. On the face of it, one might argue that there is no much point to address the case where side information is available to both encoder and decoder, because it is much easier and it can even be viewed as a special case of side information at the decoder only (simply by redefining \( \{u_i, w_i\} \) as the “source”). Nevertheless, we mention the case of two–sided side information for two reasons:

1. We can allow \( w \) to be an individual sequence too, in addition to \( u \), as opposed to our assumption so far that it is generated by a DMC fed by \( u \).

2. We can derive more explicit lower bounds to the distortion.

In the purely probabilistic setting, the rate–distortion function in the presence of side information at both ends is given by the so called conditional rate–distortion function. As discussed in [21], the only difference between the W-Z rate–distortion function and the conditional rate–distortion function is that in the former, there is the constraint of the Markov structure, whereas the conditional rate–distortion function this constraint is dropped. The proof of Theorem 2 can
easily be altered to incorporate two-sided availability of side information, with both \( u \) and \( w \) being deterministic sequences. The only modification needed is in eq. (40), which will now read as follows:

\[
I(\hat{U}^\ell;\hat{Y}^\ell) \geq \ell \cdot R_{U^\ell|\hat{Y}^\ell}[\Delta(\hat{U}^\ell|\hat{Y}^\ell)/\ell] \geq \ell \cdot R_{\hat{U}^\ell|\hat{W}^\ell}[\Delta(\hat{U}^\ell|\hat{Y}^\ell,\hat{W}^\ell)/\ell],
\]

(53)

where the second inequality is due to the fact that ignoring side information at both ends cannot be better than using it optimally. Note that we have also replaced \( W^\ell \) by \( \hat{W}^\ell \), to account for the fact that we allow it to be a deterministic sequence too, as mentioned before. Obviously, achievability is by conditional rate-distortion coding followed by capacity-achieving channel coding.

For the purpose of the lower bound to the distortion, we can further lower bound the empirical conditional rate-distortion function as follows in the spirit of the conditional Shannon lower bound [8]. First, we can represent it as

\[
R_{U^\ell|\hat{W}^\ell}(D) = H(\hat{U}^\ell|\hat{W}^\ell) - \max_{\{P_{\hat{V}^\ell|\hat{U}^\ell,\hat{W}^\ell}: E[\rho(\hat{U}^\ell,\hat{V}^\ell) \leq \ell D]} H(\hat{U}^\ell|\hat{W}^\ell,\hat{V}^\ell).
\]

(54)

Now, the first term, \( H(\hat{U}^\ell|\hat{W}^\ell) \), can be further lower bounded (within an asymptotically negligible term) in terms of the conditional Lempel-Ziv code-length function of \( u^n \) given \( w^n \) (as side information at both ends), as defined in [25]. Specifically, the following inequality is derived in [13, eq. (17)]:

\[
H(\hat{U}^\ell|\hat{W}^\ell) \geq \frac{\ell}{n} \sum_{j=1}^{c(w^n)} \left[c_j(u^n|w^n) + q^2 \right] \log \frac{c_j(u^n|w^n)}{4q^2},
\]

(55)

where \( q \) is a constant that depends only on \( \ell \) and on the sizes of \( \mathcal{U} \) and \( \mathcal{W} \), \( c(w^n) \) denotes the number of distinct phrases of \( w^n \) that appear in joint incremental parsing [26] of \( (u^n, w^n) \), and \( c_j(u^n|w^n) \) is the number of distinct phrases of \( u^n \) that appear jointly with the \( j \)-th distinct phrase of \( w^n \). Similarly as in [14, Theorem 2], for the case where \( \mathcal{U} = \mathcal{V} = \{0,1,\ldots,\alpha-1\} \) and a difference distortion measure, \( \rho(u,v) = \varrho(u-v) \) (where the subtraction is defined modulo \( \alpha \)), the second, subtracted term of (54), can be easily upper bounded by the constrained maximum entropy function, i.e.,

\[
\max_{\{P_{\hat{V}^\ell|\hat{U}^\ell,\hat{W}^\ell}: E[\rho(\hat{U}^\ell,\hat{V}^\ell) \leq \ell D]} H(\hat{U}^\ell|\hat{W}^\ell,\hat{V}^\ell) \leq \ell \cdot \Phi(D),
\]

(56)

where

\[
\Phi(D) = \sup_{\theta \geq 0} \left[ \theta D + \log \left( \sum_{u \in \mathcal{U}} 2^{-\theta \varrho(u)} \right) \right],
\]

(57)
as can easily be shown by the standard solution to the problem of maximum entropy under a moment constraint. It then follows that the expected distortion is lower bounded in terms of the inverse function, \( \Psi = \Phi^{-1} \), computed at the difference,

\[
\frac{1}{n} \sum_{j=1}^{c(w^n)} c_j(u^n|w^n) \log c_j(u^n|w^n) - C(\Gamma) - \eta(s, d, \ell, n)
\]

where \( \eta(s, d, \ell, n) \) accounts for all resulting redundancy terms (similarly as in [14, proof of Theorem 2]). Specifically, the inverse function, \( \Psi \), is given by (see Appendix)

\[
\Psi(R) = \inf_{\vartheta \geq 0} \left[ R - \log \left( \sum_{u \in U} 2^{-g(u)/\vartheta} \right) \right], \quad (58)
\]

and so,

\[
\frac{1}{n} \sum_{i=1}^{n} E\{\rho(u_i, V_i)\} \geq \sup_{\vartheta \geq 0} \frac{1}{n} \sum_{j=1}^{c(w^n)} c_j(u^n|w^n) \log c_j(u^n|w^n) - C(\Gamma) - \\
\eta(s, d, \ell, n) - \log \left( \sum_{u \in U} 2^{-g(u)/\vartheta} \right) - \frac{\rho_{\text{max}} d}{\ell}. \quad (59)
\]

Of course, instead of maximizing over \( \vartheta \), one may select any arbitrary positive \( \vartheta \) and thereby obtain a valid lower bound, albeit not as tight. We observe that whenever the conditional LZ complexity, \( \frac{1}{n} \sum_{j=1}^{c(w^n)} c_j(u^n|w^n) \log c_j(u^n|w^n) \), exceeds the channel capacity (plus the redundancy terms), the expected distortion has a non-trivial, strictly positive lower bound.

The advantage of the use of the conditional LZ complexity is that it is easier to calculate than the \( \ell \)-th order empirical entropy, especially when \( \ell \) is large, as the super-alphabet size grows exponentially with \( \ell \). In other words, we sacrifice tightness to a certain extent, at the benefit of facilitating the calculation of the bound.

5.3 The Gel’fand-Pinsker Channel

The last extension that we discuss in this work is from the DMC, \( P_{Y|X} \) to the G-P channel model, \( P_{Y^n|X^nS^n} = [P_{Y|XS}]^n \), where the state sequence, \( s^n \), which is governed by a discrete memoryless source, \( P_{S^n} = [P_S]^n \) of alphabet \( S \), is fed non-causally into the encoder as well. The extension of our results to the G-P channel is not trivial if we insist of allowing arbitrary reference block encoders that map \( (u^n, s^n) \) to \( x^n \). However, if we limit the reference encoder to operate on \( \ell \)-blocks, this
extension is possible. In such a case, eq. (39) in the proof of Theorem 2 should be modified as follows. Let \( P_{\hat{Y}t|S^t} = P_{\hat{Y}t} \times P_{S^t} = P_{\hat{Y}t} \times [P_S]^{\ell} \). We also define

\[
\mathcal{P}_t(\Gamma) = \{ P_{BX|S^t} : P_{BS^t|Y^t} = P_{S^t|P_{BX|S^t}P_{Y^t|X^t,S^t}}, E_{P_{X^t}} \phi(X^t) \leq \Gamma \},
\]

where \( B \) is an auxiliary random variable whose alphabet size need not be larger than \( \min\{|X|^\ell, |S|^\ell + 1, |Y|^\ell + |S|^\ell\} \) [5, Theorem 7.3], and where \( \phi(\hat{X}^t) \) is the additive extension of the single-letter transmission cost function, that is, \( \phi(\hat{X}^t) = \sum_{i=1}^{\ell} \phi(\hat{X}_i) \). Then,

\[
E\{I(\hat{U}^t; \hat{Y}^t)\} = E\{I(\hat{U}^t; \hat{Y}^t)\} - I(\hat{U}^t; S^t) \tag{61}
\]

\[
= E\{H(\hat{Y}^t)\} - E\{H(\hat{Y}^t|\hat{U}^t)\} - I(\hat{U}^t; S^t) \tag{62}
\]

\[
\leq H(Y^t) - E\{H(\hat{Y}^t|\hat{U}^t)\} - I(\hat{U}^t; S^t) \tag{63}
\]

\[
\leq H(Y^t) - H(Y^t|\hat{U}^t) + o(n) - I(\hat{U}^t; S^t) \tag{64}
\]

\[
= I(\hat{U}^t; Y^t) - I(\hat{U}^t; S^t) + o(n) \tag{65}
\]

\[
\leq \max_{P_{BX|S^t} \in \mathcal{P}_t(\Gamma)} [I(B; Y^t) - I(B; S^t)] + o(n) \tag{66}
\]

\[
\leq \ell \cdot [C_{GP}(\Gamma) + o(n)], \tag{67}
\]

where \( C_{GP}(\Gamma) \) is capacity of the G-P channel [7] with average transmission cost limited by \( \Gamma \) [9, p. 23, eq. (3.33)]. The four inequalities of this chain are explained as follows. The first inequality is due to the the concavity of the entropy as a functional of the underlying distribution. The second one is obtained by the weak law of large numbers, which guarantees that \( P_{Y^t|X^t,S^t} \rightarrow P_{Y^t|X^t,S^t} \), in probability, as \( n \rightarrow \infty \) (for fixed \( \ell \)), and therefore,

\[
P_{Y^t|\hat{U}^t}(y^t|u^t) = \sum_{s^t} P_{S^t}(s^t)P_{Y^t|X^t,S^t}(y^t|x^t[u^t, s^t], s^t)
\]
tends to \( P_{Y^t|\hat{U}^t}(y^t|u^t) \) in probability for all \( u^t \in \mathcal{U}^t \), \( y^t \in \mathcal{Y}^t \). The third inequality is due to the fact that \( \hat{U}^t \rightarrow (X^t, S^t) \rightarrow Y^t \) is a Markov chain in this order, and so, \( P_{\hat{U}^t} \in \mathcal{P}_t(\Gamma) \), since the reference encoder is assumed to comply with the transmission cost constraint. Finally, the last inequality is due to the fact that the multi-letter extension of the G-P capacity formula cannot improve on the single-letter version, as can easily been seen by comparing the highest rate achievable by multi-letter random coding (over the superalphabet of \( \ell \)-vectors) to the converse bound on the highest achievable rate, which is given by the single-letter formula.

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Appendix

In this appendix, we prove that the function $\Psi(\cdot)$, defined in (58), is the inverse of the function $\Phi(D)$, defined in (57). Let us denote

$$R = \Phi(D) = \sup_{\theta \geq 0} \left[ \theta D + \log \left( \sum_{u \in \mathcal{U}} 2^{-\theta g(u)} \right) \right]. \quad (A.1)$$

This means that:

1. $\forall \theta \geq 0,$
   $$R \geq \theta D + \log \left( \sum_{u \in \mathcal{U}} 2^{-\theta g(u)} \right). \quad (A.2)$$

2. There exists a positive sequence, $\{\theta_n\}$, such that
   $$\lim_{n \to \infty} \left[ \theta_n D + \log \left( \sum_{u \in \mathcal{U}} 2^{-\theta_n g(u)} \right) \right] = R. \quad (A.3)$$

But this is clearly equivalent to the set of statements:

1. $\forall \theta \geq 0,$
   $$D \leq \frac{R - \log \left( \sum_{u \in \mathcal{U}} 2^{-\theta g(u)} \right)}{\theta}. \quad (A.4)$$

2. $\exists \theta \geq 0,$
   $$\lim_{n \to \infty} \frac{R - \log \left( \sum_{u \in \mathcal{U}} 2^{-\theta_n g(u)} \right)}{\theta_n} = D. \quad (A.5)$$

This in turn is equivalent to the statement that

$$D = \inf_{\theta \geq 0} \frac{R - \log \left( \sum_{u \in \mathcal{U}} 2^{-\theta g(u)} \right)}{\theta} = \inf_{\theta \geq 0} \theta \cdot \left[ R - \log \left( \sum_{u \in \mathcal{U}} 2^{-g(u)/\theta} \right) \right] = \Psi(R). \quad (A.6)$$

References


