Optimal Correlators and Waveforms for Mismatched Detection

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The Problem

Consider the following signal detection problem:

\[ \mathcal{H}_0 : Y_t = N_t \]
\[ \mathcal{H}_1 : Y_t = X_t + N_t \equiv s_t + Z_t + N_t \]

where \( N_t = \mathcal{N}(0, \sigma_N^2) \), \( X_t = s_t + Z_t \), with \( s_t = \mathbb{E}\{X_t\} \) and \( Z_t \) is non-Gaussian i.i.d. (SIN, e.g., echo, mul. noise, jamming, cross-talk).

The likelihood ratio test (LRT) is difficult to implement, in general.

We consider the class of correlation detectors,

\[ \sum_{t=1}^{n} w_t Y_t \leq T \equiv n\theta, \quad \theta \geq 0 \]

where the best \( \{w_t\} \) are sought for optimum FA/MD tradeoff.
For Gaussian $Z_t$, $w^*_t \propto s_t$. What if $Z_t$ is non-Gaussian?

What if both $\{w_t\}$ and $\{s_t\}$ are subjected to optimization?

Extending the scope to detectors of the class

$$\sum_{t=1}^{n} w_t Y_t + \alpha \cdot \sum_{t=1}^{n} Y_t^2 \leq n\theta$$

and

$$\sum_{t=1}^{n} w_t Y_t + \alpha \cdot \sum_{t=1}^{n} |Y_t| \leq n\theta$$
Related Work

♣ Similar problem for APD: Merhav ('21).

♣ Mismatched detection due to uncertainty:
  Gini et al. ('98), Bandiera et al. ('09), Liu et al. ('15, '19, '20), ...

♣ Robust detection: Capon ('61), El-Sawy & Vandelinde ('77,'79),
  Geraniotis ('85), Kassam et al. ('76,'81,'82,'85),...

♣ Parametric uncertainty and GLRT: Van Trees ('68),
  Conte & Ricci ('98), Erez & Feder ('00), Zeitouni et al. ('92) ....
Probability of False Alarm (FA)

$\mathcal{H}_0: Y_t = N_t$

FA Probability:

$$P_{FA}(\theta) = \Pr \left\{ \sum_{t=1}^{n} w_t N_t \geq \theta n \right\} = Q \left( \frac{\theta n}{\sigma_N \|w\|} \right) = \exp \left\{ -\frac{\theta^2 n^2}{2\sigma_N^2 \|w\|^2} \right\}$$

$$E_{FA}(\theta) = \frac{\theta^2}{2\sigma_N^2 \cdot (1/n) \|w\|^2}$$

$$E_{FA}(\theta) \geq E_0 \longrightarrow \frac{1}{n} \|w\|^2 \leq \frac{\theta^2}{2\sigma_N^2 E_0} \triangleq P_w.$$
Probability of Missed Detection (MD)

\[ \mathcal{H}_1 : Y_t = s_t + Z_t + N_t \]

MD Probability:

\[ P_{MD}(\theta) = \inf_{\lambda \geq 0} \exp \left\{ \lambda \left[ \theta n - \sum_{t=1}^{n} w_t s_t \right] + \frac{\lambda^2 \sigma_N^2 \| w \|^2}{2} + \sum_{t=1}^{n} C(\lambda w_t) \right\} , \]

where \( C(\lambda) \triangleq \ln(\mathbb{E}\{e^{\lambda Z}\}) \).

Assume that \( \{(w_t, s_t)\} \) have an asymptotic empirical density, \( f_{WS} \):

\[ E_{MD}(\theta) = \sup_{\lambda \geq 0} \left\{ \lambda(\mathbb{E}\{W \cdot S\} - \theta) - \mathbb{E}\{C(\lambda W)\} - \frac{\lambda^2 \sigma_N^2 \mathbb{E}\{W^2\}}{2} \right\} \]

with \( (W, S) \sim f_{WS} \).
Optimization Problems

♣ Optimal $w$ for a given $s$: Given $f_S$, $P_w$, find

$$\max \sup_{fW \mid s} \lambda \geq 0 \left\{ \lambda (\mathbb{E}\{W \cdot S\} - \theta) - \mathbb{E}\{C(\lambda W)\} - \frac{\lambda^2 \sigma^2 \mathbb{E}\{W^2\}}{2} \right\}$$

s.t. $\mathbb{E}\{W^2\} \leq P_w$.

♣ Joint optimization of $(w, s)$: Given $P_s$, $P_w$, find

$$\max \sup_{fWs} \lambda \geq 0 \left\{ \lambda (\mathbb{E}\{W \cdot S\} - \theta) - \mathbb{E}\{C(\lambda W)\} - \frac{\lambda^2 \sigma^2 \mathbb{E}\{W^2\}}{2} \right\}$$

s.t. $\mathbb{E}\{W^2\} \leq P_w$ and $\mathbb{E}\{S^2\} \leq P_s$. 
Optimal $w$ for a Given $s$

For a given Chernoff parameter value, $\lambda$:

$$f^*_W|_S(w|s) = \delta(w - g^{-1}(s|\rho, \lambda))$$

where $\rho$ is Lagrange multiplier chosen to meet the $P_w$-constraint, and

$$g(w|\rho, \lambda) = \dot{C}(\lambda w) + \left(\frac{\rho}{\lambda} + \sigma^2_N \lambda\right) \cdot w,$$

$\dot{C}(\cdot)$ being the derivative of $C(\cdot)$. $C$ is convex $\rightarrow \dot{C}$ is increasing.

Equivalently,

$$w_t = g^{-1}(s_t|\rho, \lambda), \quad t = 1, 2, \ldots, n.$$  

Note that $g$ (and hence also $g^{-1}$) is nonlinear unless $Z_t$ is Gaussian.
4-ASK Signal + Binary Interference, $Z_t = \pm Z_0$
Same + Uniform Interference, $Z_t \sim [-Z_0, +Z_0]$
Same + Laplacian Interference
Joint Optimization of $w$ and $s$

Consider again the problem:

$$\max_{f W s} \sup_{\lambda \geq 0} \left\{ \lambda (E\{W \cdot S\} - \theta) - E\{C(\lambda W)\} - \frac{\lambda^2 \sigma^2_N E\{W^2\}}{2} \right\}.$$ 

♠ The optimal $w$ for a given $s$ is a non-linear function of $s$.
♠ The optimal $s$ for a given $w$ is clearly a linear function of $w$.
♠ $w_t$ and $s_t$ must taken only values according to the solutions of:

$$g(w|\rho, \lambda) = \dot{C}(\lambda w) + \left( \frac{\rho}{\lambda} + \sigma^2_N \lambda \right) w = \zeta \cdot w$$
Joint Optimization of \((w, s)\) (Cont’d)

\(C(\cdot)\) is always convex, but \(C(\sqrt{\cdot})\) is not necessarily.

**Theorem:**

\(\heartsuit\) If \(C(\sqrt{\cdot})\) is convex, both \(w^*\) and \(s^*\) are either DC or bipolar, and

\[
E_{\text{MD}}(\theta) = \sup_{\lambda \geq 0} \sup_{P \leq P_w} \left\{ \lambda(\sqrt{P_sP} - \theta) - C(\lambda \sqrt{P}) - \frac{\lambda^2 \sigma^2 N P}{2} \right\}.
\]

\(\heartsuit\) If \(C(\sqrt{\cdot})\) is concave, \(w^*\) and \(s^*\) are all zero, except one component with the entire energy.

\[
E_{\text{MD}}(\theta) = \sup_{\lambda \geq 0} \sup_{P \leq P_w} \left\{ \lambda(\sqrt{P_sP} - \theta) - \lim_{n \to \infty} \frac{C(\lambda \sqrt{Pn})}{n} - \frac{\lambda^2 \sigma^2 N P}{2} \right\}.
\]

Note that in some cases (like the binary/uniform interference),

\[
\lim_{n \to \infty} \frac{C(\lambda \sqrt{Pn})}{n} = 0
\]

which means that the interference has no effect at all.
Joint Optimization of \((w, s)\) (Cont’d)

\(C(\sqrt{\cdot})\) may be neither convex nor concave, for example,

\[
f_Z(z) = \epsilon \cdot \left[ \frac{1}{2} \cdot \delta(z - z_0) + \frac{1}{2} \cdot \delta(z + z_0) \right] + (1 - \epsilon) \cdot \frac{q}{2} e^{-q|z|}.
\]

Here, the equation

\[
g(w|\rho, \lambda) = \dot{C}(\lambda w) + \left( \frac{\rho}{\lambda} + \sigma^2_N \lambda \right) w = \zeta \cdot w
\]

has more than two (positive) non-zero solutions, which should be time-shared to achieve LCE\(\{C(\lambda \sqrt{\cdot})\}\).
Joint Optimization of \((w, s)\) (Cont’d)
The MD probability

\[ P_{MD}(\theta) = \Pr \left\{ \sum_{t=1}^{n} w_t (s_t + Z_t + N_t) + \alpha \sum_{t=1}^{n} (s_t + Z_t + N_t)^2 < \theta n \right\} \]

\[ = \inf_{\lambda \geq 0} \mathbb{E} \left\{ \prod_{t=1}^{n} \exp \left[ -\alpha \lambda (s_t + Z_t + N_t)^2 + \ldots \right] \right\} \]

The trick is to use the identity

\[ \exp \{-a(s_t + Z_t + N_t)^2\} = \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} \exp \left\{ -j\omega (s_t + Z_t + N_t) - \frac{\omega^2}{4a} \right\} d\omega \]

and commute the integrations. Likewise,

\[ \exp \{-a|s_t + Z_t + N_t|\} = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\exp\{-j\omega (s_t + Z_t + N_t)\}}{\omega^2 + a^2} d\omega. \]
Summary

- We studied optimal correlation-detection for non-Gaussian noise.
- The best $w$ for a given $s$ is non-linear in $s$.
- If $(w, s)$ are optimized jointly, the relation is linear and they are both discrete-valued.
- The form of the solution depends on the convexity/concavity of $C(\sqrt{\cdot})$.
- There are extensions to correlation + energy detectors.