Optimal Correlators and Waveforms for Mismatched Detection

Neri Merhav

The Viterbi Faculty of Electrical and Computer Engineering

Technion – Israel Institute of Technology

Haifa 3200003, Israel

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The Problem

Consider the following signal detection problem:

$$\mathcal{H}_0 : \; Y_t = N_t$$

$$\mathcal{H}_1 : \; Y_t = X_t + N_t \equiv s_t + Z_t + N_t$$

where $N_t = \mathcal{N}(0, \sigma_N^2)$, $X_t = s_t + Z_t$, with $s_t = \mathbf{E}\{X_t\}$ and Z_t is non-Gaussian i.i.d. (SIN, e.g., echo, mul. noise, jamming, cross-talk).

The likelihood ratio test (LRT) is difficult to implement, in general.

We consider the class of correlation detectors,

$$\sum_{t=1}^{n} w_t Y_t \lessgtr T \equiv n\theta, \qquad \theta \geq 0$$

where the best $\{w_t\}$ are sought for optimum FA/MD tradeoff.

The Problem (Cont'd)

- ♠ For Gaussian Z_t, $w_t^* \propto s_t$. What if Z_t is non–Gaussian?
- ♠ What if both $\{w_t\}$ and $\{s_t\}$ are subjected to optimization?
- ♠ Extending the scope to detectors of the class

$$\sum_{t=1}^{n} w_t Y_t + \alpha \cdot \sum_{t=1}^{n} Y_t^2 \lessgtr n\theta$$

and

$$\sum_{t=1}^{n} w_t Y_t + \alpha \cdot \sum_{t=1}^{n} |Y_t| \lessgtr n\theta$$

Related Work

- Similar problem for APD: Merhav ('21).
- Mismatched detection due to uncertainty:

  Gini *et al.* ('98), Bandiera *et al.* ('09), Liu *et al.* ('15, '19, '20), ...
- Robust detection: Capon ('61), El-Sawy & Vandelinde ('77,'79),

  Geraniotis ('85), Kassam *et al.* ('76,'81,'82,'85),...
- Parametric uncertainty and GLRT: Van Trees ('68),

  Conte & Ricci ('98), Erez & Feder ('00), Zeitouni *et al.* ('92) ....

Probability of False Alarm (FA)

$$\mathcal{H}_0 : \; Y_t = N_t$$

FA Probability:

$$P_{\textsf{FA}}(\theta) = \textsf{Pr}\left\{\sum_{t=1}^{n} w_t N_t \geq \theta n\right\} = Q\left(\frac{\theta n}{\sigma_N \|\boldsymbol{w}\|}\right) \doteq \exp\left\{-\frac{\theta^2 n^2}{2\sigma_N^2 \|\boldsymbol{w}\|^2}\right\}$$

$$E_{\textsf{FA}}(\theta) = \frac{\theta^2}{2\sigma_N^2 \cdot (1/n)\|\boldsymbol{w}\|^2}$$

$$E_{\textsf{FA}}(\theta) \geq E_0 \quad \longrightarrow \quad \frac{1}{n}\|\boldsymbol{w}\|^2 \leq \frac{\theta^2}{2\sigma_N^2 E_0} \triangleq P_w.$$

Probability of Missed Detection (MD)

$$\mathcal{H}_1 : \; Y_t = s_t + Z_t + N_t$$

MD Probability:

$$P_{\mathsf{MD}}(\theta) \doteq \inf_{\lambda \geq 0} \exp \left\{ \lambda \left[\theta n - \sum_{t=1}^{n} w_t s_t \right] + \frac{\lambda^2 \sigma_N^2 \|\boldsymbol{w}\|^2}{2} + \sum_{t=1}^{n} C(\lambda w_t) \right\},$$

where $C(\lambda) \stackrel{\triangle}{=} \ln(\mathbf{E}\{e^{\lambda Z}\})$.

Assume that $\{(w_t, s_t)\}$ have an asymptotic empirical density, f_{WS}:

$$E_{\mathsf{MD}}(\theta) = \sup_{\lambda \geq 0} \left\{ \lambda(\mathbf{E}\{W \cdot S\} - \theta) - \mathbf{E}\{C(\lambda W)\} - \frac{\lambda^2 \sigma_N^2 \mathbf{E}\{W^2\}}{2} \right\}$$

with $(W, S) \sim f_{WS}$.

Optimization Problems

- Optimal $\boldsymbol{w}$ for a given $\boldsymbol{s}$: Given $f_S,\ P_w$, find

$$\max_{f_{W\,|\,S}} \sup_{\lambda \geq 0} \left\{ \lambda(\mathbf{E}\{W \cdot S\} - \theta) - \mathbf{E}\{C(\lambda W)\} - \frac{\lambda^2 \sigma_N^2 \mathbf{E}\{W^2\}}{2} \right\}$$

s.t. $\mathbf{E}\{W^2\} \leq P_w$.

- Joint optimization of $(\boldsymbol{w}, \boldsymbol{s})$: Given $P_s,\ P_w$, find

$$\max_{f_{WS}} \sup_{\lambda \geq 0} \left\{ \lambda(\mathbf{E}\{W \cdot S\} - \theta) - \mathbf{E}\{C(\lambda W)\} - \frac{\lambda^2 \sigma_N^2 \mathbf{E}\{W^2\}}{2} \right\}$$

s.t. $\mathbf{E}\{W^2\} \leq P_w$ and $\mathbf{E}\{S^2\} \leq P_s$.

Optimal w for a Given s

For a given Chernoff parameter value, λ:

$$f^*_{W|S}(w|s) = \delta(w - g^{-1}(s|\rho, \lambda))$$

where ρ is Lagrange multiplier chosen to meet the P_w-constraint, and

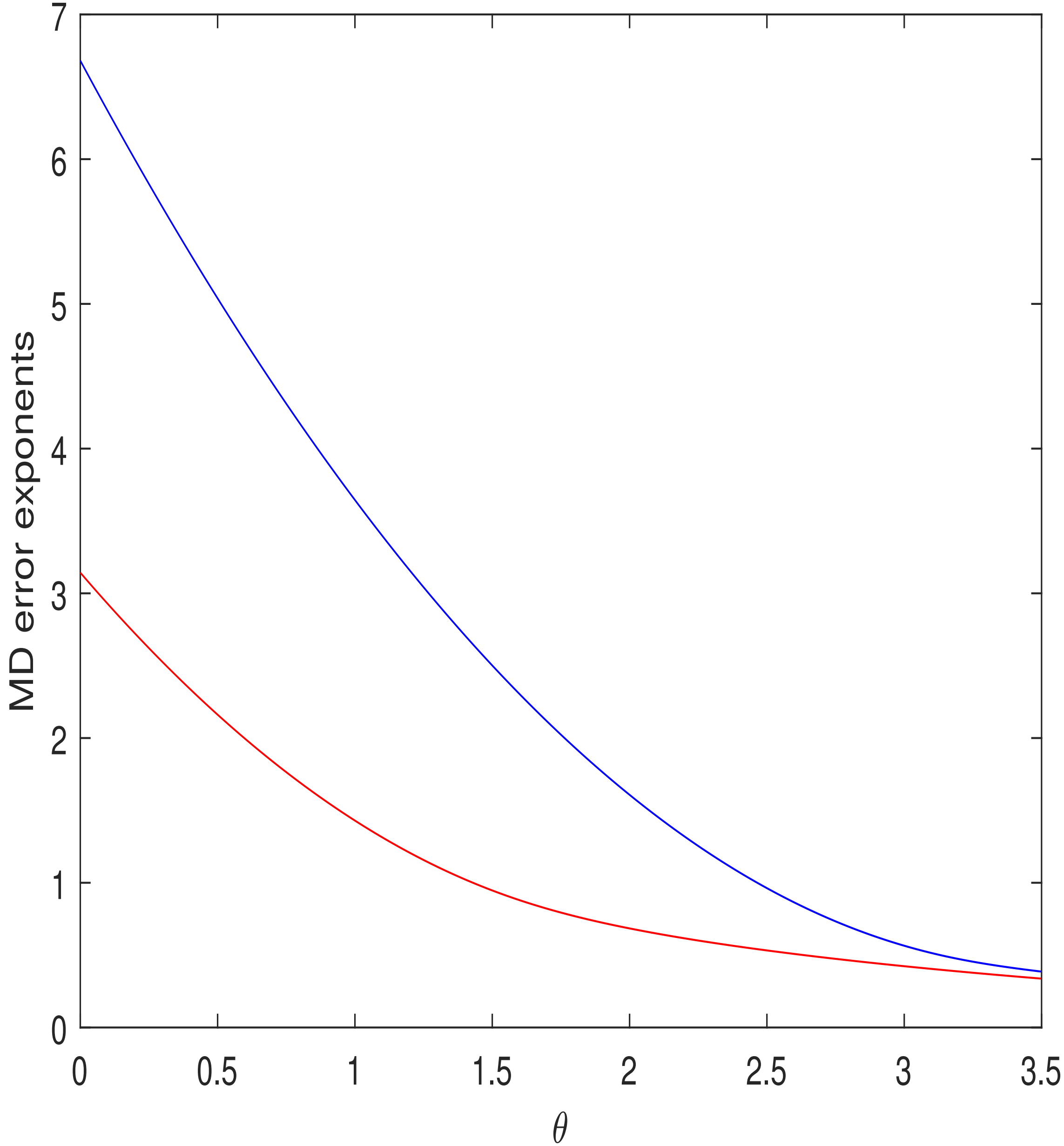
$$g(w|\rho, \lambda) = \dot{C}(\lambda w) + \left(\frac{\rho}{\lambda} + \sigma_N^2 \lambda\right) \cdot w,$$

$\dot{C}(\cdot)$ being the derivative of $C(\cdot)$. C is convex $\rightarrow \dot{C}$ is increasing.
Equivalently,

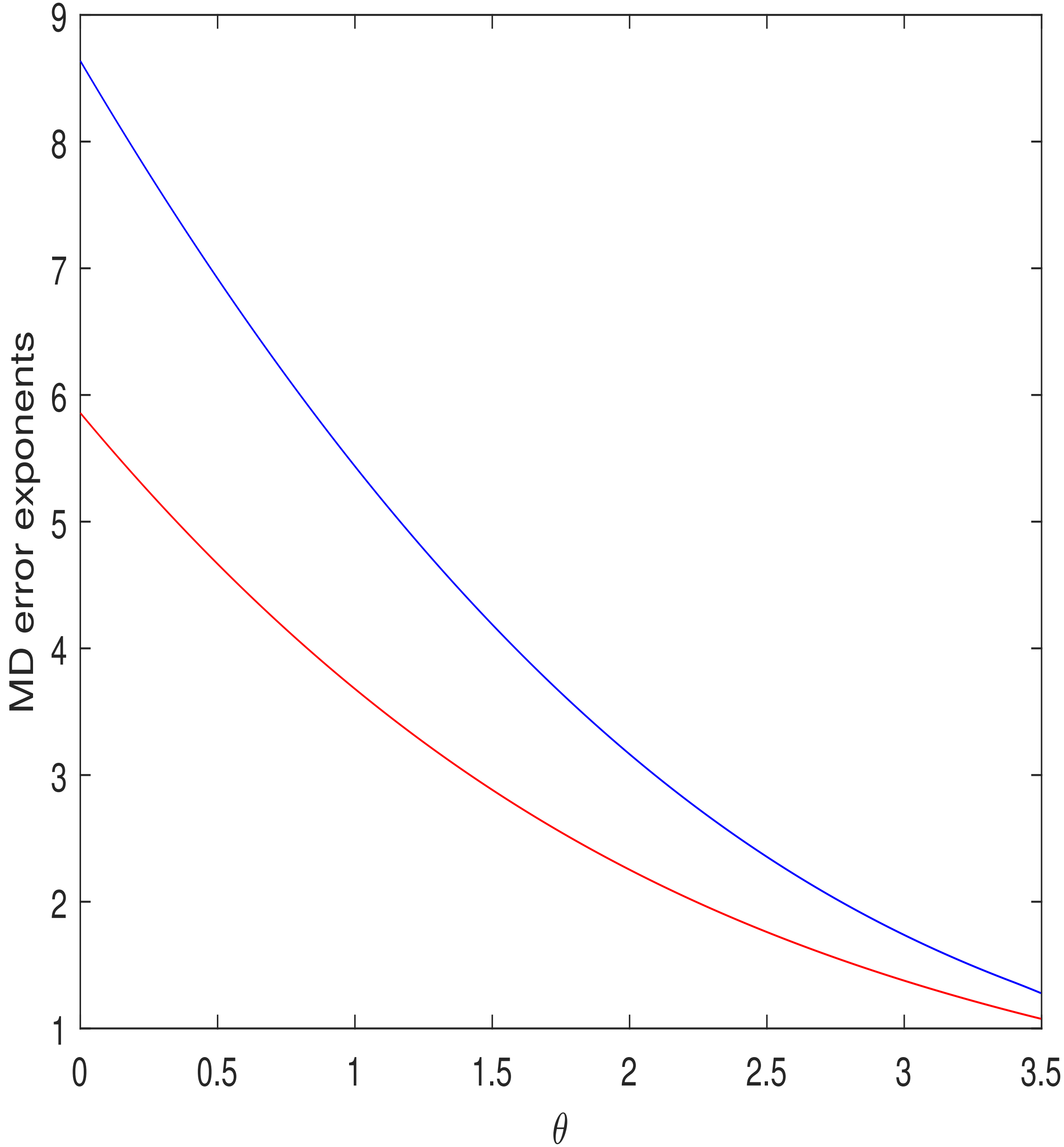
$$w_t = g^{-1}(s_t|\rho, \lambda), \quad t = 1, 2, \ldots, n.$$

Note that g (and hence also g^{-1}) is nonlinear unless Z_t is Gaussian.

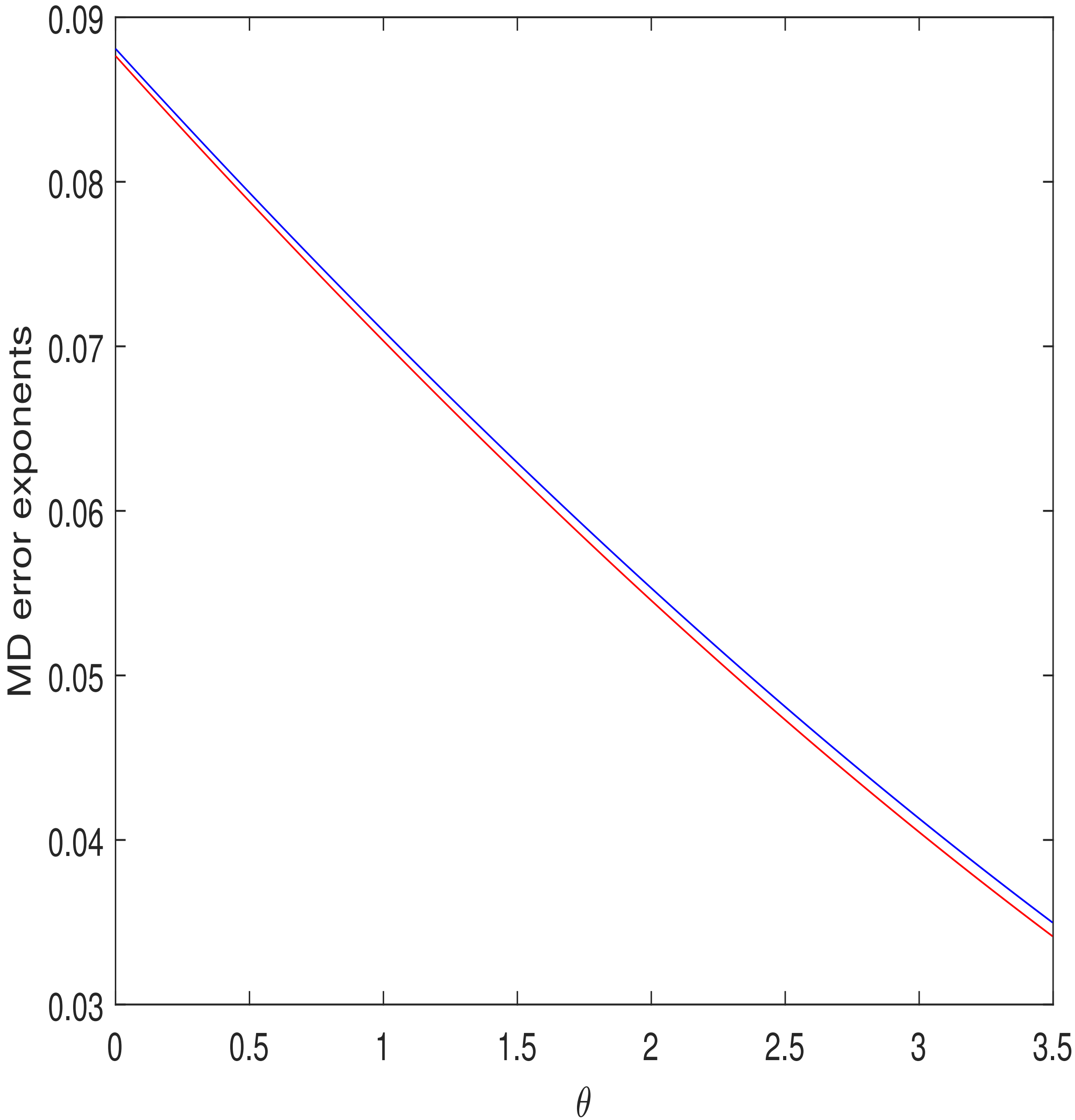
4-ASK Signal + Binary Inteference, $Z_t = \pm Z_0$



Same + Uniform Inteference, $Z_t \sim [-Z_0, +Z_0]$



Same + Laplacian Interference



Joint Optimization of w and s

Consider again the problem:

$$\max_{f w s} \sup_{\lambda \geq 0} \left\{ \lambda(\mathbf{E}\{W \cdot S\} - \theta) - \mathbf{E}\{C(\lambda W)\} - \frac{\lambda^2 \sigma_N^2 \mathbf{E}\{W^2\}}{2} \right\}.$$

- The optimal w for a given s is a non-linear function of s.
- The optimal s for a given w is clearly a linear function of w.
- w_t and s_t must taken only values according to the solutions of:

$$g(w|\rho, \lambda) = \dot{C}(\lambda w) + \left(\frac{\rho}{\lambda} + \sigma_N^2 \lambda \right) w = \zeta \cdot w$$

Joint Optimization of $(\boldsymbol{w}, \boldsymbol{s})$ (Cont'd)

$C(\cdot)$ is always convex, but $C(\sqrt{\cdot})$ - not necessarily.

Theorem:

♡ If $C(\sqrt{\cdot})$ is convex, both $\boldsymbol{w}^*$ and $\boldsymbol{s}^*$ are either DC or bipolar, and

$$E_{\mathsf{MD}}(\theta) = \sup_{\lambda \geq 0} \sup_{P \leq P_w} \left\{ \lambda(\sqrt{P_s P} - \theta) - C\left(\lambda\sqrt{P}\right) - \frac{\lambda^2 \sigma_N^2 P}{2} \right\}.$$

♡ If $C(\sqrt{\cdot})$ is concave, $\boldsymbol{w}^*$ and $\boldsymbol{s}^*$ are all zero, except one component with the entire energy.

$$E_{\mathsf{MD}}(\theta) = \sup_{\lambda \geq 0} \sup_{P \leq P_w} \left\{ \lambda(\sqrt{P_s P} - \theta) - \lim_{n \to \infty} \frac{C\left(\lambda\sqrt{Pn}\right)}{n} - \frac{\lambda^2 \sigma_N^2 P}{2} \right\}.$$

Note that in some cases (like the binary/uniform interference),

$$\lim_{n \to \infty} \frac{C\left(\lambda\sqrt{Pn}\right)}{n} = 0$$

which means that the interference has no effect at all.

Joint Optimization of $(\boldsymbol{w}, \boldsymbol{s})$ (Cont'd)

$C(\sqrt{\cdot})$ may be neither convex nor concave, for example,

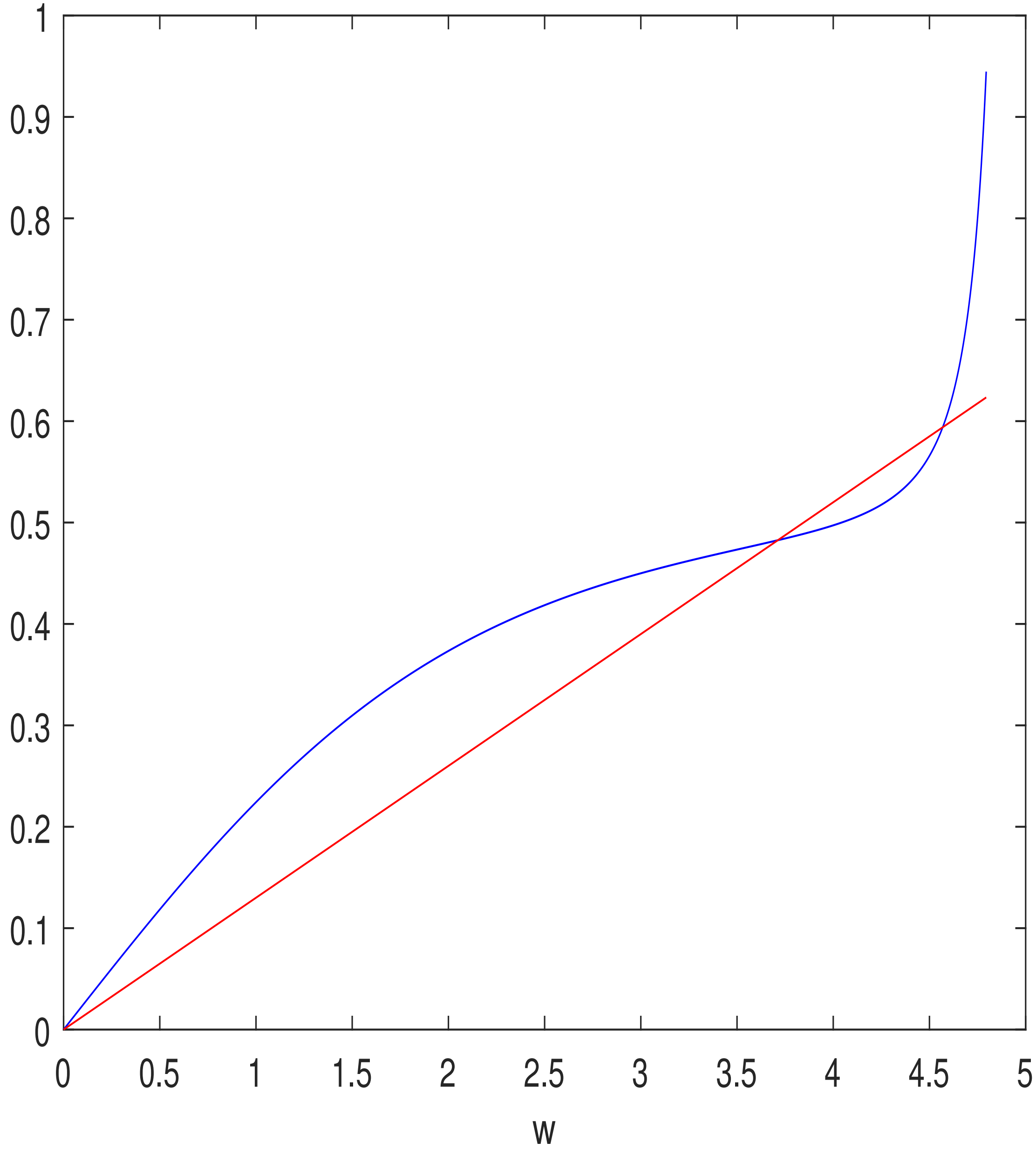
$$f_Z(z) = \epsilon \cdot \left[\frac{1}{2} \cdot \delta(z - z_0) + \frac{1}{2} \cdot \delta(z + z_0) \right] + (1 - \epsilon) \cdot \frac{q}{2} e^{-q|z|}.$$

Here, the equation

$$g(w|\rho, \lambda) = \dot{C}(\lambda w) + \left(\frac{\rho}{\lambda} + \sigma_N^2 \lambda \right) w = \zeta \cdot w$$

has more than two (positive) non-zero solutions, which should be time-shared to achieve LCE$\{C(\lambda\sqrt{\cdot})\}$.

Joint Optimization of (w, s) (Cont'd)



A Word About Correlation+Energy Detectors

The MD probability

$$P_{\mathrm{MD}}(\theta) = \Pr\left\{\sum_{t=1}^{n} w_t(s_t + Z_t + N_t) + \alpha \sum_{t=1}^{n} (s_t + Z_t + N_t)^2 < \theta n\right\}$$

$$\doteq \inf_{\lambda \geq 0} \mathbf{E}\left\{\prod_{t=1}^{n} \exp\left[-\alpha\lambda(s_t + Z_t + N_t)^2 + \ldots\right]\right\}$$

The trick is to use the identity

$$\exp\{-a(s_t + Z_t + N_t)^2\} = \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} \exp\left\{-j\omega(s_t + Z_t + N_t) - \frac{\omega^2}{4a}\right\} \mathrm{d}\omega$$

and commute the integrations. Likewise,

$$\exp\{-a|s_t + Z_t + N_t|\} = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\exp\{-j\omega(s_t + Z_t + N_t)\}}{\omega^2 + a^2} \mathrm{d}\omega.$$

Summary

- We studied optimal correlation-detection for non-Gaussian noise.
- The best $\boldsymbol{w}$ for a given $\boldsymbol{s}$ is non-linear in $\boldsymbol{s}$.
- If $(\boldsymbol{w}, \boldsymbol{s})$ are optimized jointly, the relation is linear and they are both discrete-valued.
- The form of the solution depends on the convexity/concavity of $C(\sqrt{\cdot})$.
- There are extensions to correlation + energy detectors.