

On the Corner Points of the Capacity Region of a Two-User Gaussian Interference Channel

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Abstract

This work considers the corner points of the capacity region of a two-user Gaussian interference channel (GIC). In a two-user GIC, the rate pairs where one user transmits its data at the single-user capacity (without interference), and the other at the largest rate for which reliable communication is still possible are called corner points. This paper relies on existing outer bounds on the capacity region of a two-user GIC that are used to derive informative bounds on the corner points of the capacity region. The new bounds refer to a weak two-user GIC (i.e., when both cross-link gains in standard form are positive and below 1), and a refinement of these bounds is obtained for the case where the transmission rate of one user is within $\varepsilon > 0$ of the single-user capacity. The bounds on the corner points are asymptotically tight as the transmitted powers tend to infinity, and they are also useful for the case of moderate SNR and INR. Upper and lower bounds on the gap (denoted by Δ) between the sum-rate and the maximal achievable total rate at the two corner points are derived. This is followed by an asymptotic analysis analogous to the study of the generalized degrees of freedom (where the SNR and INR scalings are coupled such that $\frac{\log(\text{INR})}{\log(\text{SNR})} = \alpha \geq 0$), leading to an asymptotic characterization of this gap which is exact for the whole range of α . The upper and lower bounds on Δ are asymptotically tight in the sense that they achieve the exact asymptotic characterization. Improved bounds on Δ are derived for finite SNR and INR, and their improved tightness is exemplified numerically.

Keywords: Corner point, Gaussian interference channel, inner/ outer bound, interference to noise ratio, sum-rate.

I. INTRODUCTION

The two-user Gaussian interference channel (GIC) has been extensively studied in the literature during the last four decades (see, e.g., [9, Chapter 6], [22] and references therein). For completeness and to set notation, the model of a two-user GIC in *standard form* is introduced shortly: this discrete-time, memoryless interference channel is characterized by the following equations that relate the paired inputs (X_1, X_2) and outputs (Y_1, Y_2) :

$$Y_1 = X_1 + \sqrt{a_{12}} X_2 + Z_1 \quad (1)$$

$$Y_2 = \sqrt{a_{21}} X_1 + X_2 + Z_2 \quad (2)$$

where the cross-link gains a_{12} and a_{21} are time-invariant, the inputs and outputs are real valued, and Z_1 and Z_2 denote additive real-valued Gaussian noise samples. Let $X_1^n \triangleq (X_{1,1}, \dots, X_{1,n})$ and $X_2^n \triangleq (X_{2,1}, \dots, X_{2,n})$ be two transmitted codewords across the channel where $X_{i,j}$ denotes the symbol that is transmitted by user i at time instant j (here, $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$). The model assumes that there is no cooperation between the transmitters, not between the receivers. It is assumed, however, that the receivers have full knowledge of the codebooks used

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by both users. The power constraints on the inputs are given by $\frac{1}{n} \sum_{j=1}^n \mathbb{E}[X_{1,j}^2] \leq P_1$ and $\frac{1}{n} \sum_{j=1}^n \mathbb{E}[X_{2,j}^2] \leq P_2$ where $P_1, P_2 > 0$. The random vectors Z_1^n and Z_2^n have i.i.d. Gaussian entries with zero mean and unit variance, and they are independent of the inputs X_1^n and X_2^n . Furthermore, Z_1^n and Z_2^n can be assumed to be independent since the capacity region of a two-user, discrete-time, memoryless interference channel depends only on the marginal *pdfs* $p(y_i|x_1, x_2)$ for $i \in \{1, 2\}$ (as the receivers do not cooperate). Finally, perfect synchronization between the pairs of transmitters and receivers allows time-sharing between the users, which implies that the capacity region is convex.

Depending on the values of a_{12} and a_{21} , the two-user GIC is classified into weak, strong, mixed, one-sided and degraded GIC. If $0 < a_{12}, a_{21} < 1$, the channel is called a *weak* GIC. If $a_{12} \geq 1$ and $a_{21} \geq 1$, the channel is a *strong* GIC; furthermore, if $a_{12} \geq 1 + P_1$ and $a_{21} \geq 1 + P_2$ then the channel is a *very strong* GIC, and its capacity region is not harmed (i.e., reduced) as a result of the interference [3]. If either $a_{12} \geq 1$ and $0 < a_{21} < 1$ or $a_{21} \geq 1$ and $0 < a_{12} < 1$, the channel is called a *mixed* GIC; the special case where $a_{12}a_{21} = 1$ is called a *degraded* GIC. It is a *one-sided* GIC if either $a_{12} = 0$ or $a_{21} = 0$; a one-sided GIC is either weak or strong if its non-zero cross-link gain is below or above 1, respectively. Finally, a *symmetric* GIC refers to the case where $a_{12} = a_{21}$ and $P_1 = P_2$.

In spite of the simplicity of the model of a two-user GIC, the exact characterization of its capacity region is yet unknown, except for strong ([12], [20]) or very strong interference [3]. For other GICs, not only the capacity region is yet unknown but even its corner points are not fully determined. For mixed or one-sided GICs, a single corner point of the capacity region is known and it attains the sum-rate of this channel (see [15, Section 6.A], [17, Theorem 2], and [21, Section 2.C]). For weak GICs, both corner points of the capacity region are yet unknown.

The operational meaning of the study of the corner points of the capacity region for a two-user GIC is to explore the situation where one transmitter sends its information at the maximal achievable rate for a single user (in the absence of interference), and the second transmitter maintains a data rate that enables reliable communication to the two non-cooperating receivers [5]. Two questions occur in this scenario:

Question 1: What is the maximal achievable rate of the second transmitter ?

Question 2: Does it enable the first receiver to reliably decode the messages of both transmitters ?

In his paper [5], Costa presented an approach suggesting that when one of the transmitters, say transmitter 1, sends its data over a two-user GIC at the maximal interference-free rate $R_1 = \frac{1}{2} \log(1 + P_1)$ bits per channel use, then the maximal rate R_2 of transmitter 2 is the rate that enables receiver 1 to decode both messages. The corner points of the capacity region are therefore related to a multiple-access channel where one of the receivers decodes correctly both messages. However, [17, pp. 1354–1355] pointed out a gap in the proof of [5, Theorem 1], though it was conjectured that the main result holds. It therefore leads to the following conjecture:

Conjecture 1: For rate pairs (R_1, R_2) in the capacity region of a two-user GIC with arbitrary positive cross-link gains a_{12} and a_{21} , and power constraints P_1 and P_2 , let

$$C_1 \triangleq \frac{1}{2} \log(1 + P_1), \quad C_2 \triangleq \frac{1}{2} \log(1 + P_2) \quad (3)$$

be the capacities of the single-user AWGN channels (in the absence of interference), and let

$$R_1^* \triangleq \frac{1}{2} \log \left(1 + \frac{a_{21}P_1}{1 + P_2} \right) \quad (4)$$

$$R_2^* \triangleq \frac{1}{2} \log \left(1 + \frac{a_{12}P_2}{1 + P_1} \right). \quad (5)$$

Then, the following is conjectured to hold for achieving reliable communication at both receivers:

- 1) If $R_2 \geq C_2 - \varepsilon$, for an arbitrary $\varepsilon > 0$, then $R_1 \leq R_1^* + \delta_1(\varepsilon)$ where $\delta_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.
- 2) If $R_1 \geq C_1 - \varepsilon$, then $R_2 \leq R_2^* + \delta_2(\varepsilon)$ where $\delta_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

The discussion on Conjecture 1 is separated in the continuation to this section into mixed, strong, and weak one-sided GICs. This is done by restating some known results from [5], [12], [15], [17], [19], [20] and [21]. The focus of this paper is on weak GICs. For this class, the corner points of the capacity region are yet unknown, and they are studied in the converse part of this paper by relying on some existing outer bounds on the capacity region. Various outer bounds on the capacity region of GICs that have been introduced in the literature (see, e.g., [1], [4], [9], [10], [13], [15], [16], [19] and [21]–[24]). The analysis in this paper provides informative bounds that are given in closed form, and they are asymptotically tight for sufficiently large SNR and INR. Improvements of these bounds are derived for finite SNR and INR, and these improvements are exemplified numerically.

A. On Conjecture 1 for Mixed GICs

Conjecture 1 is considered in the following for mixed GICs:

Proposition 1: Consider a mixed GIC where $a_{12} \geq 1$ and $a_{21} < 1$, and assume that transmitter 1 sends its message at rate $R_1 \geq C_1 - \varepsilon$ for an arbitrary $\varepsilon > 0$. Then, the following holds:

- 1) If $1 - a_{12} < (a_{12}a_{21} - 1)P_1$, then $R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_2}{1 + a_{21}P_1} \right) + \varepsilon$. This implies that the maximal rate R_2 is *strictly smaller* than the corresponding upper bound in Conjecture 1.
- 2) Otherwise, if $1 - a_{12} \geq (a_{12}a_{21} - 1)P_1$, then $R_2 \leq R_2^* + \varepsilon$. This coincides with the upper bound in Conjecture 1.

The above two items refer to a corner point that achieves the sum-rate. On the other hand, if $R_2 \geq C_2 - \varepsilon$, then

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{P_1}{1 + P_2} \right) + \delta(\varepsilon) \quad (6)$$

where $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof: The first two items of this proposition follow from [15, Theorem 10] or the earlier result in [13, Theorem 1]. Eq. (6) is a consequence of [13, Theorem 2]. ■

B. On Conjecture 1 for Strong GICs

The capacity region of a strong GIC is equal to the intersection of the capacity regions of the two Gaussian multiple-access channels from the two transmitters to each one of the receivers (see [12, Theorem 5.2] and [20]). The two corner points of this capacity region are consistent with Conjecture 1. Question 2 is answered in the affirmative for a strong GIC because each receiver is able to decode the messages of both users.

The capacity region of a very strong GIC, where $a_{12} \geq 1 + P_1$ and $a_{21} \geq 1 + P_2$, is not affected by the interference [3]. This is a trivial case where Conjecture 1 does not provide a tight upper bound on the maximal transmission rate (note that if $a_{12} > 1 + P_1$ and $a_{21} > 1 + P_2$, then $R_1^* > C_1$ and $R_2^* > C_2$).

C. On the Corner Points of Weak One-Sided GICs

In [5], an interesting equivalence has been established between weak one-sided GICs and degraded GICs: a weak one-sided GIC with power constraints P_1 and P_2 , and cross-link gains $a_{12} = 0$ and $a_{21} = a \in (0, 1)$ in standard form, has an identical capacity region to that of a degraded GIC whose standard form is given by

$$Y_1 = X_1 + \sqrt{\frac{1}{a}} X_2 + Z_1, \quad Y_2 = \sqrt{a} X_1 + X_2 + Z_2 \quad (7)$$

with the same power constraints on the inputs, and where $Z_1 Z_2$ are independent Gaussian random variables with zero mean and unit variance. The first part of Proposition 1 implies that one corner point of a weak one-sided GIC is determined exactly, it is achievable by treating the interference as noise, and it is given by

$$\left(\frac{1}{2} \log(1 + P_1), \frac{1}{2} \log \left(1 + \frac{P_2}{1 + aP_1} \right) \right). \quad (8)$$

In [17, Theorem 2], it is shown that this corner point achieves the sum-rate of the weak one-sided GIC. We consider in the following the second corner point of the capacity region: according to Proposition 1, the second corner point of the weak one-sided GIC is given by (R_1, C_2) where (see (6))

$$\frac{1}{2} \log \left(1 + \frac{aP_1}{1 + P_2} \right) \leq R_1 \leq \frac{1}{2} \log \left(1 + \frac{P_1}{1 + P_2} \right). \quad (9)$$

The lower bound on R_1 follows from the achievability of the point (R_1^*, C_2) for the degraded GIC in (7). The following statement summarizes this short discussion on weak one-sided GICs.

Proposition 2: Consider a weak one-sided GIC, which in standard form has power constraints P_1 and P_2 for transmitters 1 and 2, respectively, and whose cross-link gains are $a_{12} = 0$ and $a_{21} = a$ for $0 < a < 1$. One of the two corner points of its capacity region is given in (8), and it achieves the sum-rate. The other corner point is (R_1, C_2) where R_1 satisfies the bounds in (9), and these bounds are tight when $a \rightarrow 1$.

The achievable rate region of Costa [6] for a weak one-sided GIC coincides with the Han-Kobayashi achievable region for i.i.d. Gaussian codebooks (see [27, Section 2]). This region has a corner point at (R_1^*, C_2) where R_1^* is given in (4) with $a_{21} = a$ (note that it is equal to the lower bound in (9)). However, it remains unknown whether the capacity-achieving input distribution is Gaussian.

D. Organization of this paper

The structure of this paper is as follows: Conjecture 1 is considered in Section II for a weak GIC. The excess rate for the sum-rate with respect to the corner points of the capacity region is considered in Section III. A summary is provided in Section IV with some directions for further research. Throughout this paper, two-user GICs are considered.

II. ON THE CORNER POINTS OF THE CAPACITY REGION OF A WEAK GIC

This section considers Conjecture 1 for a weak GIC. It is easy to verify that the points $(R_1, R_2) = (C_1, R_2^*)$ and (R_1^*, C_2) are both included in the capacity region of a weak GIC, and the corresponding receiver of the transmitter that operates at the single-user capacity can be designed to decode the messages of the two users. We proceed in the following to the converse part, which leads to the following statement:

Theorem 1: Consider a weak two-user GIC, and let C_1 , C_2 , R_1^* and R_2^* be as defined in (3)–(5). If $R_1 \geq C_1 - \varepsilon$ for an arbitrary $\varepsilon > 0$, then reliable communication requires that

$$R_2 \leq \min \left\{ R_2^* + \frac{1}{2} \log \left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right) + 2\varepsilon, \right. \\ \left. \frac{1}{2} \log \left(1 + \frac{P_2}{1 + P_1} \right) + \left(1 + \frac{1 + P_1}{a_{21}P_2} \right) \varepsilon \right\}. \quad (10)$$

Similarly, if $R_2 \geq C_2 - \varepsilon$, then

$$R_1 \leq \min \left\{ R_1^* + \frac{1}{2} \log \left(1 + \frac{P_1}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right) + 2\varepsilon, \right. \\ \left. \frac{1}{2} \log \left(1 + \frac{P_1}{1 + P_2} \right) + \left(1 + \frac{1 + P_2}{a_{12}P_1} \right) \varepsilon \right\}. \quad (11)$$

Consequently, the corner points of the capacity region are (R_1, C_2) and (C_1, R_2) where

$$R_1^* \leq R_1 \leq \min \left\{ R_1^* + \frac{1}{2} \log \left(1 + \frac{P_1}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right), \frac{1}{2} \log \left(1 + \frac{P_1}{1 + P_2} \right) \right\} \quad (12)$$

$$R_2^* \leq R_2 \leq \min \left\{ R_2^* + \frac{1}{2} \log \left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right), \frac{1}{2} \log \left(1 + \frac{P_2}{1 + P_1} \right) \right\}. \quad (13)$$

In the limit where P_1 and P_2 tend to infinity, which makes it an interference-limited channel,

- 1) Conjecture 1 holds, and it gives an asymptotically tight bound.
- 2) The rate pairs (C_1, R_2^*) and (R_1^*, C_2) form the corner points of the capacity region.
- 3) The answer to Question 2 is affirmative.

Proof: The proof of this theorem relies on the two outer bounds on the capacity region that are given in [10, Theorem 3] and [13, Theorem 2].

Suppose that $R_1 \geq C_1 - \varepsilon$ bits per channel use. The outer bound by Etkin *et al.* in [10, Theorem 3] (it is also known as the ETW bound) yields that the rates R_1 and R_2 satisfy the inequality constraint

$$2R_1 + R_2 \leq \frac{1}{2} \log(1 + P_1 + a_{12}P_2) + \frac{1}{2} \log \left(\frac{1 + P_1}{1 + a_{21}P_1} \right) + \frac{1}{2} \log \left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2} \right)$$

which therefore yields that (see (3) and (5))

$$R_2 \leq \frac{1}{2} \log(1 + P_1 + a_{12}P_2) + \frac{1}{2} \log \left(\frac{1 + P_1}{1 + a_{21}P_1} \right) \\ + \frac{1}{2} \log \left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2} \right) - (\log(1 + P_1) - 2\varepsilon) \\ = R_2^* + \frac{1}{2} \log \left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right) + 2\varepsilon. \quad (14)$$

The outer bound by Kramer in [13, Theorem 2], formulated here in an equivalent form, states that the capacity region is included in the set $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ where

$$\mathcal{K}_1 = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq \frac{1}{2} \log \left(1 + \frac{(1-\beta)P'}{\beta P' + \frac{1}{a_{21}}} \right) \\ 0 \leq R_2 \leq \frac{1}{2} \log(1 + \beta P') \end{array} \right\} \quad (15)$$

with $P' \triangleq P_2 + \frac{P_1}{a_{21}}$ and $\beta \in \left[\frac{P_2}{(1+P_1)P'}, \frac{P_2}{P'} \right]$ is a free parameter; the set \mathcal{K}_2 is obtained by swapping the indices in \mathcal{K}_1 . From the boundary of the outer bound in (15), the value of β that satisfies the equality

$$\frac{1}{2} \log \left(1 + \frac{(1-\beta)P'}{\beta P' + \frac{1}{a_{21}}} \right) = C_1 - \varepsilon$$

is given by

$$\beta = \frac{2^{2\varepsilon} P_2 + \frac{(2^{2\varepsilon}-1)(1+P_1)}{a_{21}}}{(1+P_1) \left(P_2 + \frac{P_1}{a_{21}} \right)}.$$

The substitution of this value of β into the upper bound on R_2 in (15) implies that if $R_1 \geq C_1 - \varepsilon$ then

$$\begin{aligned} R_2 &\leq \frac{1}{2} \log(1 + \beta P') \\ &= \frac{1}{2} \log \left(1 + \frac{P_2}{1+P_1} \right) + \delta(\varepsilon) \end{aligned} \quad (16)$$

where

$$\delta(\varepsilon) = \frac{1}{2} \log \left(1 + \frac{(2^{2\varepsilon} - 1) \left(P_2 + \frac{1+P_1}{a_{21}} \right)}{1 + P_1 + P_2} \right).$$

The function δ satisfies $\delta(0) = 0$, and straightforward calculus shows that

$$0 < \delta'(c) < 1 + \frac{1+P_1}{a_{21}P_2}, \quad \forall c \geq 0.$$

It therefore follows (from the mean-value theorem of calculus) that

$$0 < \delta(\varepsilon) < \left(1 + \frac{1+P_1}{a_{21}P_2} \right) \varepsilon. \quad (17)$$

A combination of (14), (16), (17) gives the upper bound on the rate R_2 in (10). Similarly, if $R_2 \geq C_2 - \varepsilon$, the upper bound on the rate R_1 in (11) is obtained by swapping the indices in (10).

From the inclusion of the points (C_1, R_2^*) and (R_1^*, C_2) in the capacity region, and the bounds in (10) and (11) in the limit where $\varepsilon \rightarrow 0$, it follows that the corner points of the capacity region are (R_1, C_2) and (C_2, R_1) with the bounds on R_1 and R_2 in (12) and (13), respectively.

Since the point (C_1, R_2^*) is achievable, also is (R_1, R_2^*) for $R_1 < C_1$; hence, if $C_1 - \varepsilon \leq R_1 < C_1$, then the maximal rate R_2 of transmitter 2 satisfies

$$R_2^* \leq R_2 \leq R_2^* + \frac{1}{2} \log \left(1 + \frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)} \right) + 2\varepsilon.$$

The uncertainty in the maximal achievable rate R_2 when $R_1 \geq C_1 - \varepsilon$ and $\varepsilon \rightarrow 0$ is therefore upper bounded by $\Delta R_2 \triangleq \frac{1}{2} \log \left(1 + \frac{P_2}{(1+a_{21}P_1)(1+a_{12}P_2)} \right)$. The asymptotic case when $P_1, P_2 \rightarrow \infty$ and $\frac{P_2}{P_1} \rightarrow k$ for an arbitrary $k > 0$ is examined in the following: In this case, $R_2^* \rightarrow \frac{1}{2} \log(1 + ka_{12})$ and $\Delta R_2 \rightarrow 0$ which proves that Conjecture 1 holds in this asymptotic case where the transmitted powers tend to infinity. Since the points (C_1, R_2^*) and (R_1^*, C_2) are included in the capacity region, it follows from this converse that they asymptotically form the corner points of this region. As is explained above, operating at the points (C_1, R_2^*) or (R_1^*, C_2)

enables receiver 1 or 2, respectively, to decode both messages. This answers Question 2 in the affirmative for the considered asymptotic case. ■

Remark 1: Consider a weak symmetric GIC where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in (0, 1)$. The corner points of the capacity region of this two-user interference channel are given by (C, R_c) and (R_c, C) where $C = \frac{1}{2} \log(1+P)$ is the capacity of a single-user AWGN channel with input power constraint P , and an additive Gaussian noise with zero mean and unit variance. Theorem 1 gives that

$$R_c \leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{aP}{1+P} \right) + \frac{1}{2} \log \left(1 + \frac{P}{(1+aP)^2} \right), \frac{1}{2} \log \left(1 + \frac{P}{1+P} \right) \right\}. \quad (18)$$

In the following, we compare the two terms inside the minimization in (18) where the first term follows from the ETW bound in [10, Theorem 3], and the second term follows from Kramer's bound in [13, Theorem 2]. Straightforward algebra reveals that, for $a \in (0, 1)$, the first term gives a better bound on R_c if and only if

$$P > \frac{2a^2 - a + 1 + \sqrt{5a^2 - 2a + 1}}{2a^2(1-a)}. \quad (19)$$

Hence, for an arbitrary cross-link gain $a \in (0, 1)$ of a symmetric and weak two-user GIC, there exists a threshold for the SNR where above it, the ETW bound provides a better upper bound on the corner points; on the other hand, for values of SNR below this threshold, Kramer's bound provides a better bound on the corner points. The dependence of the threshold for the SNR (P) on the cross-link gain is shown in Figure 1. The threshold for the SNR (P), as is shown in

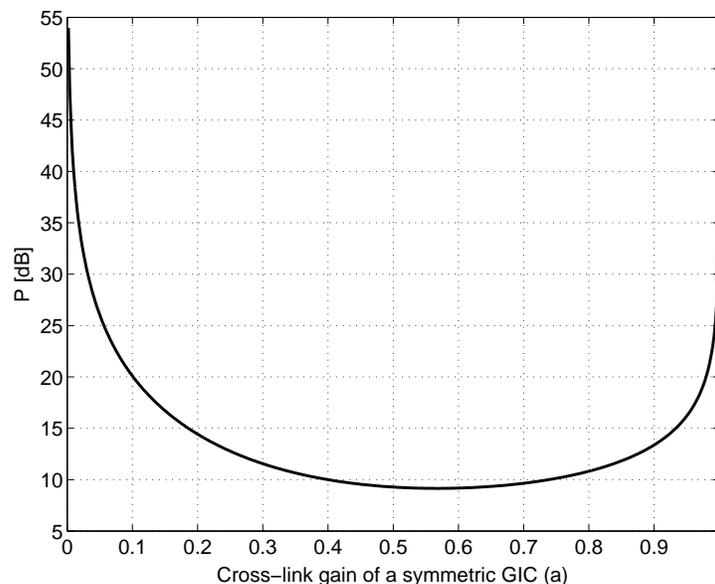


Fig. 1. The curve in this figure shows the threshold for the SNR (P), in decibels, as a function of the cross-link gain (a) for a weak and symmetric GIC. This threshold is given by the right-hand side of (19). For points (a, P) above this curve, the ETW bound is better in providing an upper bound on the corner points of the capacity region, whereas Kramer's bound is better in this respect for points (a, P) below this curve.

Figure 1, tends to infinity if $a \rightarrow 0$ or $a \rightarrow 1$; this implies that in these two cases, Kramer's bound is better for all values of P . This is further discussed in the following:

- 1) If $a \rightarrow 0$ then, for every $P > 0$, the first term on the right-hand side of (18) tends to the capacity C ; this forms a trivial upper bound on the value R_c of the corner point. On the other

hand, the second term on the right-hand side of (18) gives the upper bound of $\frac{1}{2} \log \left(1 + \frac{P}{1+P}\right)$ which is smaller than C for all values of P . Note that the second term in (18) implies that, for a symmetric GIC, $R_c \leq \frac{1}{2}$ bit per channel use for all values of P . In fact, for a given P , the advantage of the second term in the extreme case where $a \rightarrow 0$ served as the initial motivation for incorporating it in Theorem 1.

- 2) If $a \rightarrow 1$ then, for every $P > 0$, the first term tends to $\frac{1}{2} \log \left(1 + \frac{P}{1+P}\right) + \frac{1}{2} \log \left(1 + \frac{P}{(1+P)^2}\right)$ which is larger than the second term. Hence, also in this case, the second term gives a better bound for all values of P .

Example 1: The condition in (19) is consistent with [15, Figs. 10 and 11], as explained in the following:

- 1) According to [15, Fig. 10], for $P = 7$ and $a = 0.2$, Kramer's outer bound gives a better upper bound on the corner point than the ETW bound. For $a = 0.2$, the complementary of the condition in (19) implies that Kramer's bound is indeed better in this respect for $P < 27.725$. This is supported by Figure 1.
- 2) According to [15, Fig. 11], for $P = 100$ and $a = 0.1$, the ETW is nearly as tight as Kramer's bound in providing an upper bound on the corner point. For $a = 0.1$, the complementary of the condition in (19) implies that Kramer's outer bound gives a better upper bound on the corner point than the ETW bound if $P < 102.33$ (as is supported by Figure 1); hence, for $P = 100$, there is only a slight advantage to Kramer's bound over the ETW bound that is not visible in [15, Fig. 11]: Kramer's bound gives an upper bound on R_c that is equal to 0.4964 bits per channel use, and the ETW bound gives an upper bound of 0.5026 bits per channel use.

Remark 2: If $a_{12}a_{21}P_{1,2} \gg 1$, then it follows from (12) and (13) that the two corner points of the capacity region approximately coincide with the points (R_1^*, C_2) and (C_1, R_2^*) in Conjecture 1.

In the following example, we evaluate the bounds in Theorem 1 for finite values of transmitted powers (P_1 and P_2) to illustrate the asymptotic tightness of these bounds.

Example 2: Consider a weak and symmetric GIC where $a = 0.5$ and $P = 100$. Assume that transmitter 1 operates at the single-user capacity $C = \frac{1}{2} \log(1 + P) = 3.33$ bits per channel use. According to (13), the corresponding maximal rate R_2 of transmitter 2 is between 0.292 and 0.317 bits per channel use; the upper bound on R_2 in this case follows from the ETW bound. This gives good accuracy in the assessment of the two corner points of the capacity region (see Remark 2 where, in this case, $a^2P = 25 \gg 1$). If P is increased by 10 dB (to 1000), and transmitter 1 operates at the single-user capacity $C = \frac{1}{2} \log(1 + P) = 5.0$ bits per channel use, then the corresponding maximal rate R_2 is between 0.292 and 0.295 bits per channel use. Hence, the precision of the assessment of the corner points is improved in the latter case. The improved accuracy of the latter assessment when the value of P is increased is consistent with Remark 2, and the asymptotic tightness of the bounds in Theorem 1. Figure 2 refers to a weak and symmetric GIC where $P_1 = P_2 = 100$ and $a_{12} = a_{21} = 0.5$. The solid line in this figure corresponds to the boundary of the ETW outer bound on the capacity region (see [10, Theorem 3]) which is given (in units of bits per channel use) by

$$\mathcal{R}_o = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq 3.3291 \\ 0 \leq R_2 \leq 3.3291 \\ R_1 + R_2 \leq 4.1121 \\ 2R_1 + R_2 \leq 6.9755 \\ R_1 + 2R_2 \leq 6.9755 \end{array} \right\}. \quad (20)$$

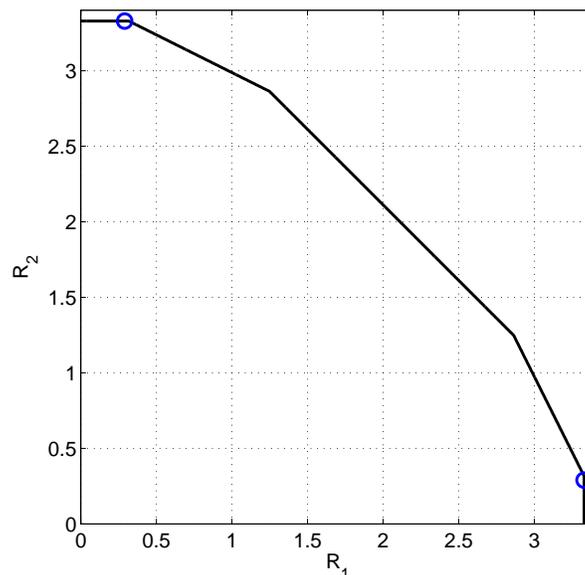


Fig. 2. The figure refers to a GIC with cross-link gains $a_{12} = a_{21} = 0.5$, and a common transmitted power $P_1 = P_2 = 100$ in standard form (see Example 2). The solid curve is the boundary of the outer bound in [10, Theorem 3] (the ETW bound in (20)), and the two circled points refer to Conjecture 1; these points are achievable, and they almost coincide with the boundary of the outer bound.

The two circled points correspond to Conjecture 1; these points are achievable, and (as is verified numerically) they almost coincide with the boundary of the outer bound in [10, Theorem 3].

III. THE EXCESS RATE FOR THE SUM-RATE WITH RESPECT TO THE CORNER POINTS OF THE CAPACITY REGION

The sum-rate of a mixed, strong or one-sided GIC is attained at a corner point of its capacity region. This is in contrast to a (two-sided) weak GIC whose sum-rate is not attained at a corner point of its capacity region. It is therefore of interest to examine the excess rate for the sum-rate with respect to these corner points by measuring the gap between the sum-rate (C_{sum}) and the maximal total rate ($R_1 + R_2$) at the corner points of the capacity region:

$$\Delta \triangleq C_{\text{sum}} - \max\{R_1 + R_2 : (R_1, R_2) \text{ is a corner point}\}. \quad (21)$$

The parameter Δ measures the excess rate for the sum-rate with respect to the case where one transmitter operates at its single-user capacity, and the other reduces its rate to the point where reliable communication is achievable. We have $\Delta = 0$ for mixed, strong and one-sided GICs. This section derives bounds on Δ for weak GICs, and it also provides an asymptotic analysis analogous to the study of the generalized degrees of freedom (where the SNR and INR scalings are coupled such that $\frac{\log(\text{INR})}{\log(\text{SNR})} = \alpha \geq 0$). This leads to an asymptotic characterization of this gap which is demonstrated to be exact for the whole range of α . The upper and lower bounds on Δ are shown in this section to be asymptotically tight in the sense that they achieve the exact asymptotic characterization. Improvements of the bounds on Δ are derived in this section for finite SNR and INR, and these bounds are exemplified numerically.

For the analysis in this section, the bounds in Theorem 1 and bounds on the sum-rate (see, e.g., [7], [10], [11], [21], [25] and [26]) are used to obtain upper and lower bounds on Δ .

A. An Upper Bound on Δ for Weak GICs

The following derivation of an upper bound on Δ relies on an upper bound on the sum-rate, and a lower bound on the maximal value of $R_1 + R_2$ at the two corner points of its capacity region. Since the points (R_1^*, C_2) and (C_1, R_2^*) are achievable for a weak GIC, it follows that

$$\begin{aligned} & \max\{R_1 + R_2 : (R_1, R_2) \text{ is a corner point}\} \\ & \geq \max\{R_1^* + C_2, R_2^* + C_1\} \\ & = \frac{1}{2} \max\left\{\log(1 + P_2 + a_{21}P_1), \log(1 + P_1 + a_{12}P_2)\right\}. \end{aligned} \quad (22)$$

An outer bound on the capacity region of a weak GIC is provided in [10, Theorem 3]. This bound leads to the following upper bound on the sum-rate:

$$\begin{aligned} C_{\text{sum}} \leq \frac{1}{2} \min & \left\{ \log(1 + P_1) + \log\left(1 + \frac{P_2}{1 + a_{21}P_1}\right), \log(1 + P_2) + \log\left(1 + \frac{P_1}{1 + a_{12}P_2}\right), \right. \\ & \left. \log\left(1 + a_{12}P_2 + \frac{P_1}{1 + a_{21}P_1}\right) + \log\left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2}\right) \right\}. \end{aligned} \quad (23)$$

Consequently, combining (21)–(23) gives the following upper bound on Δ :

$$\begin{aligned} \Delta \leq \frac{1}{2} & \left[\min \left\{ \log(1 + P_1) + \log\left(1 + \frac{P_2}{1 + a_{21}P_1}\right), \log(1 + P_2) + \log\left(1 + \frac{P_1}{1 + a_{12}P_2}\right), \right. \right. \\ & \left. \left. \log\left(1 + a_{12}P_2 + \frac{P_1}{1 + a_{21}P_1}\right) + \log\left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2}\right) \right\} \right. \\ & \left. - \max\left\{\log(1 + P_2 + a_{21}P_1), \log(1 + P_1 + a_{12}P_2)\right\} \right]. \end{aligned} \quad (24)$$

For a weak and symmetric GIC, where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a$ ($0 < a < 1$), (24) is simplified to

$$\begin{aligned} \Delta & = \Delta(P, a) \\ & \leq \frac{1}{2} \left[\min \left\{ \log(1 + P) + \log\left(1 + \frac{P}{1 + aP}\right), 2 \log\left(1 + aP + \frac{P}{1 + aP}\right) \right\} \right. \\ & \quad \left. - \log(1 + (1 + a)P) \right] \\ & = \frac{1}{2} \min \left\{ \log\left(\frac{1 + P}{1 + aP}\right), \log\left(1 + \frac{P}{(1 + aP)^2} + \frac{aP[P + (1 + aP)^2]}{(1 + aP)(1 + (a + 1)P)}\right) \right\}. \end{aligned} \quad (25)$$

Hence, in the limit where we let P tend to infinity,

$$\lim_{P \rightarrow \infty} \Delta(P, a) \leq \frac{1}{2} \log\left(\frac{1}{a}\right), \quad \forall a \in (0, 1). \quad (26)$$

Note that, for $a = 1$, the capacity region is the polyhedron that is obtained from the intersection of the capacity regions of the two underlying Gaussian multiple-access channels. This implies that $\Delta(P, 1) = 0$, so the bound in (26) is continuous from the left at $a = 1$.

B. A Lower Bound on Δ for Weak GICs

The following derivation of a lower bound on Δ relies on a lower bound on the sum-rate, and an upper bound on the maximal value of $R_1 + R_2$ at the two corner points of the capacity region. From Theorem 1, the maximal total rate at the corner points of the capacity region of a weak GIC is upper bounded as follows:

$$\begin{aligned}
& \max\{R_1 + R_2 : (R_1, R_2) \text{ is a corner point}\} \\
& \stackrel{(a)}{\leq} \min \left\{ \max \left\{ R_1^* + C_2 + \frac{1}{2} \log \left(1 + \frac{P_1}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right), \right. \right. \\
& \quad \left. \left. R_2^* + C_1 + \frac{1}{2} \log \left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right) \right\}, \frac{1}{2} \log(1 + P_1 + P_2) \right\} \\
& \stackrel{(b)}{=} \frac{1}{2} \min \left\{ \max \left\{ \log(1 + P_2 + a_{21}P_1) + \log \left(1 + \frac{P_1}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right), \right. \right. \\
& \quad \left. \left. \log(1 + P_1 + a_{12}P_2) + \log \left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right) \right\}, \log(1 + P_1 + P_2) \right\}
\end{aligned} \tag{27}$$

where inequality (a) follows from (12), (13), and the equality

$$\max\{\min\{a, c\}, \min\{b, c\}\} = \min\{\max\{a, b\}, c\}, \quad \forall a, b, c \in \mathbb{R}$$

and equality (b) follows from (3)–(5).

In order to get a lower bound on the sum-rate of the capacity region of a weak GIC, we rely on the particularization of the outer bound in [23] for a GIC. This leads to the following outer bound \mathcal{R}_o in [9, Section 6.7.2]:

$$\mathcal{R}_o = \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq \frac{1}{2} \log(1 + P_1) \\ 0 \leq R_2 \leq \frac{1}{2} \log(1 + P_2) \\ R_1 + R_2 \leq \frac{1}{2} \log(1 + P_1 + a_{12}P_2) + \frac{1}{2} \log \left(1 + \frac{P_2}{1 + a_{12}P_2} \right) \\ R_1 + R_2 \leq \frac{1}{2} \log(1 + P_2 + a_{21}P_1) + \frac{1}{2} \log \left(1 + \frac{P_1}{1 + a_{21}P_1} \right) \\ R_1 + R_2 \leq \frac{1}{2} \log \left(1 + a_{12}P_2 + \frac{P_1}{1 + a_{21}P_1} \right) + \frac{1}{2} \log \left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2} \right) \\ 2R_1 + R_2 \leq \frac{1}{2} \log(1 + P_1 + a_{12}P_2) + \frac{1}{2} \log \left(1 + \frac{P_1}{1 + a_{21}P_1} \right) \\ \quad + \frac{1}{2} \log \left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2} \right) \\ R_1 + 2R_2 \leq \frac{1}{2} \log(1 + P_2 + a_{21}P_1) + \frac{1}{2} \log \left(1 + \frac{P_2}{1 + a_{12}P_2} \right) \\ \quad + \frac{1}{2} \log \left(1 + a_{12}P_2 + \frac{P_1}{1 + a_{21}P_1} \right) \end{array} \right\}. \tag{28}$$

The outer bound \mathcal{R}_o has the property that if $(R_1, R_2) \in \mathcal{R}_o$ then $(R_1 - \frac{1}{2}, R_2 - \frac{1}{2}) \in \mathcal{R}_{\text{HK}}$ where \mathcal{R}_{HK} denotes the Han-Kobayashi achievable rate region in [12] (see [23, Remark 2] and [9, Section 6.7.2]). Note that the "within one bit" result in [10] and [23] is per complex dimension, and it is replaced here by half a bit per dimension since all the random variables involved in the calculations of the outer bound on the capacity region of a scalar GIC are real-valued [9,

Theorem 6.6]. Consider the boundary of the outer bound \mathcal{R}_o in (28). If one of the three inequality constraints on $R_1 + R_2$ is active in (28) (this condition is first needed to be verified), then a point on the boundary of the rate region \mathcal{R}_o that is dominated by one of these three inequality constraints satisfies the equality

$$R_1 + R_2 = \frac{1}{2} \min \left\{ \begin{aligned} &\log(1 + P_1 + a_{12}P_2) + \log \left(1 + \frac{P_2}{1 + a_{12}P_2} \right), \\ &\log(1 + P_2 + a_{21}P_1) + \log \left(1 + \frac{P_1}{1 + a_{21}P_1} \right), \\ &\log \left(1 + a_{12}P_2 + \frac{P_1}{1 + a_{21}P_1} \right) + \log \left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2} \right) \end{aligned} \right\}. \quad (29)$$

Since $(R_1 - \frac{1}{2}, R_2 - \frac{1}{2})$ is an achievable rate pair, then the sum-rate is lower bounded by $R_1 + R_2 - 1$. It therefore follows from (29) that

$$C_{\text{sum}} \geq \frac{1}{2} \min \left\{ \begin{aligned} &\log(1 + P_1 + a_{12}P_2) + \log \left(1 + \frac{P_2}{1 + a_{12}P_2} \right), \\ &\log(1 + P_2 + a_{21}P_1) + \log \left(1 + \frac{P_1}{1 + a_{21}P_1} \right), \\ &\log \left(1 + a_{12}P_2 + \frac{P_1}{1 + a_{21}P_1} \right) + \log \left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2} \right) \end{aligned} \right\} - 1. \quad (30)$$

A combination of (21), (27) and (30) leads to the following lower bound on the excess rate for the sum-rate with respect to the corner points:

$$\begin{aligned} \Delta \geq \frac{1}{2} \left[\min \left\{ \begin{aligned} &\log(1 + P_1 + a_{12}P_2) + \log \left(1 + \frac{P_2}{1 + a_{12}P_2} \right), \\ &\log(1 + P_2 + a_{21}P_1) + \log \left(1 + \frac{P_1}{1 + a_{21}P_1} \right), \\ &\log \left(1 + a_{12}P_2 + \frac{P_1}{1 + a_{21}P_1} \right) + \log \left(1 + a_{21}P_1 + \frac{P_2}{1 + a_{12}P_2} \right) \end{aligned} \right\} \right. \\ &- \min \left\{ \max \left\{ \begin{aligned} &\log(1 + P_2 + a_{21}P_1) + \log \left(1 + \frac{P_1}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right), \\ &\log(1 + P_1 + a_{12}P_2) + \log \left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)} \right) \end{aligned} \right\}, \right. \\ &\left. \left. \log(1 + P_1 + P_2) \right\} \right] - 1 \quad (31) \end{aligned}$$

provided that there exists a rate-pair (R_1, R_2) that is dominated by one of the three inequality constraints on $R_1 + R_2$ in (28); as mentioned above, this condition is first needed to be verified for validating both lower bounds in (30) and (31). In the following, the lower bound on Δ is particularized for a weak and symmetric GIC, and a sufficient condition is stated for ensuring that the lower bounds in (30) and (31) hold for this channel. To this end, we state and prove the following lemma:

Lemma 1: For a weak and symmetric two-user GIC with a common power constraint on its inputs that satisfies $P \geq 2.551$, there exists a rate-pair (R_1, R_2) on the boundary of the outer bound \mathcal{R}_o in (28) that is dominated by one of the inequality constraints on $R_1 + R_2$ in \mathcal{R}_o .

Proof: Consider the straight lines that correspond to the inequality constraints on $2R_1 + R_2$ and $R_1 + 2R_2$ in (28). For a weak and symmetric two-user GIC (where $a_{12} = a_{21} = a$ with $0 < a < 1$, and $P_1 = P_2 \triangleq P$), this corresponds to

$$\begin{aligned} 2R_1 + R_2 &= \frac{1}{2} \left[\log(1 + P + aP) + \log \left(1 + \frac{P}{1 + aP} \right) + \log \left(1 + aP + \frac{P}{1 + aP} \right) \right], \\ R_1 + 2R_2 &= \frac{1}{2} \left[\log(1 + P + aP) + \log \left(1 + \frac{P}{1 + aP} \right) + \log \left(1 + aP + \frac{P}{1 + aP} \right) \right]. \end{aligned}$$

These two straight lines intersect at a point (R_1, R_2) where

$$R_1 = R_2 \triangleq R = \frac{1}{6} \left[\log(1 + P + aP) + \log \left(1 + \frac{P}{1 + aP} \right) + \log \left(1 + aP + \frac{P}{1 + aP} \right) \right] \quad (32)$$

and the corresponding value of $R_1 + R_2$ at this point is given by

$$R_1 + R_2 = \frac{1}{3} \left[\log(1 + P + aP) + \log \left(1 + \frac{P}{1 + aP} \right) + \log \left(1 + aP + \frac{P}{1 + aP} \right) \right]. \quad (33)$$

For the considered GIC, the inequality constraints on $R_1 + R_2$ in the outer bound (28) are given by

$$R_1 + R_2 \leq \frac{1}{2} \left[\log(1 + P + aP) + \log \left(1 + \frac{P}{1 + aP} \right) \right], \quad (34)$$

$$R_1 + R_2 \leq \log \left(1 + aP + \frac{P}{1 + aP} \right). \quad (35)$$

The right-hand side of (33) is equal to the weighted average of the right-hand sides of (34) and (35) with weights $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Hence, it follows that one of the two inequality constraints on $R_1 + R_2$ in (34) and (35) should be active in the determination of the boundary of the outer bound in (28), provided that the point (R, R) satisfies the condition $R < \frac{1}{2} \log(1 + P)$ for every $0 < a < 1$ (see the first and second inequality constraints on R_1 and R_2 , respectively, in (28)). By showing this, it implies that the point (R, R) is outside the rate region \mathcal{R}_o in (28). Consequently, it ensures the existence of a point (R_1, R_2) , located at the boundary of the rate region in (28), that is dominated by one of the inequality constraints on $R_1 + R_2$ in (34) and (35). In order to verify that indeed the condition $R < \frac{1}{2} \log(1 + P)$ holds for every $0 < a < 1$, where R is given in (32), let

$$\begin{aligned} f_P(a) \triangleq \frac{1}{2} \log(1 + P) - \frac{1}{6} \left[\log(1 + P + aP) \right. \\ \left. + \log \left(1 + \frac{P}{1 + aP} \right) + \log \left(1 + aP + \frac{P}{1 + aP} \right) \right], \quad \forall a \in [0, 1] \end{aligned} \quad (36)$$

where $P > 0$ is arbitrary; the satisfiability of this condition requires that f_P is positive over the interval $[0, 1]$. The function f_P satisfies $f_P(0) = 0$, and it is concave over the interval $[0, 1]$ if and only if $P \geq 0.680$ (one can show that this is a necessary and sufficient condition such that $f_P''(1) \leq 0$; furthermore, under the latter condition, the third derivative of f_P is also positive on

$[0,1]$, which implies that $f_P'' \leq 0$ over this interval, so the function f_P is concave). This implies that $f_P(a) > 0$ for all $a \in (0, 1)$ if $f_P(1) \geq 0$ and $P \geq 0.680$. Straightforward algebra shows that $f_P(1) \geq 0$ if and only if $P^4 + P^3 - 6P^2 - 7P - 2 \geq 0$, which is satisfied if and only if $P \geq 2.55003$ (the other solutions of this inequality are infeasible for P since it is real and positive). This completes the proof of the lemma. \blacksquare

Lemma 1 yields that the lower bound on the sum-rate in (30) is satisfied for a weak and symmetric GIC if $P \geq 2.551$. Consequently, also the lower bound on Δ in (31) holds for a weak and symmetric GIC under the same condition on P . In this case, the lower bound in (31) is simplified to

$$\begin{aligned} \Delta &= \Delta(P, a) \\ &\geq \frac{1}{2} \left[\min \left\{ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{1+aP} \right), 2 \log \left(1 + aP + \frac{P}{1+aP} \right) \right\} \right. \\ &\quad \left. - \min \left\{ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{(1+aP)^2} \right), \log(1 + 2P) \right\} \right] - 1. \end{aligned} \quad (37)$$

In the following, we consider the limit of the lower bound on Δ in the asymptotic case where we let P tend to infinity, while $a \in (0, 1)$ is kept fixed. In this case, we have from the lower bound in (37)

$$\begin{aligned} &\lim_{P \rightarrow \infty} \Delta(P, a) \\ &\geq \frac{1}{2} \lim_{P \rightarrow \infty} \left[\min \left\{ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{1+aP} \right), \log \left(\left(1 + aP + \frac{P}{1+aP} \right)^2 \right) \right\} \right. \\ &\quad \left. - \min \left\{ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{(1+aP)^2} \right), \log(1 + 2P) \right\} \right] - 1 \\ &\stackrel{(a)}{=} \frac{1}{2} \lim_{P \rightarrow \infty} \log \left(1 + \frac{P}{1+aP} \right) - \log \left(1 + \frac{P}{(1+aP)^2} \right) - 1 \\ &= \frac{1}{2} \log \left(1 + \frac{1}{a} \right) - 1. \end{aligned} \quad (38)$$

Equality (a) holds since, for large enough P ,

$$\begin{aligned} \log(1 + (a+1)P) + \log \left(1 + \frac{P}{1+aP} \right) &\approx \log((a+1)P) + \log \left(1 + \frac{1}{a} \right) = \log \left(\frac{(a+1)^2 P}{a} \right), \\ 2 \log \left(1 + aP + \frac{P}{1+aP} \right) &\approx \log(a^2 P^2), \\ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{(1+aP)^2} \right) &\approx \log((a+1)P) \end{aligned}$$

so, if $a \in (0, 1)$ and P is large enough, each minimization of the pair of terms in the two lines before equality (a) is equal to its first term.

For a weak and symmetric GIC, a comparison of the asymptotic upper and lower bounds on Δ in (26) and (38) yields that these two asymptotic bounds differ by at most 1 bit per channel use; this holds irrespectively of the cross-link gain $a \in (0, 1)$. Note that the upper bound is tight for a close to 1, and also both asymptotic bounds scale like $\frac{1}{2} \log \left(\frac{1}{a} \right)$ for small values of a (so, they tend to infinity as $a \rightarrow 0$).

C. An Analogous Measure to the Generalized Degrees of Freedom and its Implications

This section is focused on the model of a two-user symmetric GIC, and it provides an asymptotic analysis of the excess rate for the sum-rate with respect to the corner points of its capacity region. The asymptotic analysis of this excess rate (Δ) is analogous to the study of the generalized degrees of freedom where the SNR and INR scalings are coupled such that

$$\frac{\log(\text{INR})}{\log(\text{SNR})} = \alpha \geq 0. \quad (39)$$

The main results of this section is a derivation of an exact asymptotic characterization of Δ for the whole range of α (see Theorem 2), and a demonstration that the closed-form expressions for the upper and lower bounds on Δ in Sections III-A and III-B are asymptotically tight in the sense of achieving the exact asymptotic characterization of Δ (see Theorem 3). Implications of the asymptotic analysis and the main results of this section are further discussed in the following.

Consider a two-user symmetric GIC whose cross-link gain a scales like $P^{\alpha-1}$ for some fixed value of $\alpha \geq 0$. For this GIC, the generalized degrees of freedom (GDOF) is defined as the asymptotic limit of the normalized sum-rate $\frac{C_{\text{sum}}(P, P^{\alpha-1})}{\log P}$ when $P \rightarrow \infty$. This GDOF refers to the case where the SNR (P) tends to infinity, and the interference to noise ratio ($\text{INR} = aP$) scales according to (39) while $\alpha \geq 0$ is kept fixed. The GDOF of a two-user symmetric GIC (without feedback) is defined as follows:

$$d(\alpha) \triangleq \lim_{P \rightarrow \infty} \frac{C_{\text{sum}}(P, P^{\alpha-1})}{\log P} \quad (40)$$

and this limit exists for every $\alpha \geq 0$ (see [10, Section 3.G]).

For large P , let us consider in an analogous way the asymptotic scaling of the normalized excess rate for the sum-rate with respect to the corner points of the capacity region. To this end, we study the asymptotic limit of the ratio $\frac{\Delta(P, P^{\alpha-1})}{\log P}$ for a fixed $\alpha \geq 0$ when P tends to infinity. Similarly to (40), the denominator of this ratio is equal to the asymptotic sum-rate of two parallel AWGN channels with no interference. However, in the latter expression, the excess rate for the sum-rate with respect to the corner points is replacing the sum-rate that appears in the numerator on the right-hand side of (40). Correspondingly, for an arbitrary $\alpha \geq 0$, let us define

$$\delta(\alpha) \triangleq \lim_{P \rightarrow \infty} \frac{\Delta(P, P^{\alpha-1})}{\log P}. \quad (41)$$

provided that this limit exists. In the following, we demonstrate the existence of this limit and provide a closed-form expression for δ .

Theorem 2: The limit in (41) exists for every $\alpha \geq 0$, and the function δ admits the following closed-form expression:

$$\delta(\alpha) = \begin{cases} |\frac{1}{2} - \alpha|, & \text{if } 0 \leq \alpha < \frac{2}{3} \\ \frac{1-\alpha}{2}, & \text{if } \frac{2}{3} \leq \alpha < 1 \\ 0, & \text{if } \alpha \geq 1 \end{cases}. \quad (42)$$

Proof: If $\alpha \geq 1$ and $P \geq 1$, the cross-link gain is $a = P^{\alpha-1} \geq 1$, and the channel is a strong and symmetric two-user GIC. The capacity region of a strong two-user GIC is equal to the intersection of the capacity regions of the two Gaussian multiple-access channels from the two transmitters to each one of the receivers (see [12, Theorem 5.2] and [20]). The sum-rate of

this GIC is therefore equal to the total rate ($R_1 + R_2$) at each of the corner points of its capacity region. Hence, if $\alpha \geq 1$ and $P > 1$ then $\Delta(P, P^{\alpha-1}) = 0$, and (41) implies that

$$\delta(\alpha) = 0, \quad \forall \alpha \geq 1. \quad (43)$$

For a symmetric two-user GIC with an input power constraint $P > 1$ and an interference level $\alpha \in [0, 1)$, the cross-link gain is $a = P^{\alpha-1} < 1$. This refers to a weak and symmetric two-user GIC. From Theorem 1 (see (12) and (13)), the bounds on the corner points of the capacity region of a weak and symmetric two-user GIC imply that the maximal total rate at these corner points satisfies the inequality

$$\frac{1}{2} \log(1 + P) \leq \max\{R_1 + R_2 : (R_1, R_2) \text{ is a corner point}\} \leq \frac{1}{2} \log(1 + 2P). \quad (44)$$

From (21) and (44), it follows that for $P > 1$ and $\alpha \in [0, 1)$

$$C_{\text{sum}}(P, P^{\alpha-1}) - \frac{1}{2} \log(1 + 2P) \leq \Delta(P, P^{\alpha-1}) \leq C_{\text{sum}}(P, P^{\alpha-1}) - \frac{1}{2} \log(1 + P). \quad (45)$$

Consequently, for $\alpha \in (0, 1)$, a division by $\log P$ of the three sides of the inequality in (45) and a calculation of the limit as $P \rightarrow \infty$ gives that (see (40) and (41))

$$\delta(\alpha) = d(\alpha) - \frac{1}{2}, \quad \forall \alpha \in (0, 1). \quad (46)$$

The limit in (40) for the GDOF of a two-user symmetric GIC (without feedback) exists, and it admits the following closed-form expression (see [10, Theorem 2]):

$$\begin{aligned} d(\alpha) &= \min\left\{1, \max\left\{\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right\}, \max\{\alpha, 1 - \alpha\}\right\} \\ &= \begin{cases} 1 - \alpha, & \text{if } 0 \leq \alpha < \frac{1}{2} \\ \alpha, & \text{if } \frac{1}{2} \leq \alpha < \frac{2}{3} \\ 1 - \frac{\alpha}{2}, & \text{if } \frac{2}{3} \leq \alpha < 1 \\ \frac{\alpha}{2}, & \text{if } 1 \leq \alpha < 2 \\ 1, & \text{if } \alpha \geq 2 \end{cases} . \end{aligned} \quad (47)$$

A combination of (43), (46) and (47) proves the closed-form expression for δ in (42). \blacksquare

Figure 3 provides a comparison of the GDOF and the function δ for an interference level α (i.e., the cross-link gain is $a = P^{\alpha-1}$). Equation (46) shows that, for an interference level $\alpha \in [0, 1]$, the difference between the GDOF (denoted here by $d(\alpha)$) and $\delta(\alpha)$ is half a bit per channel use (see Figure 3). In light of the closed-form expression of δ in (42), the asymptotic tightness of the bounds in (25) and (37) is demonstrated in the following:

Theorem 3: Consider a weak and symmetric two-user GIC where the SNR and INR scalings are coupled according to (39). Then, the upper and lower bounds on the excess rate Δ in (25) and (37) are asymptotically tight in the sense that, in the limit where $P \rightarrow \infty$, the normalization of these bounds by $\log P$ tend to δ in (41) and (42).

Proof: Substituting $a = P^{\alpha-1}$ into the upper bound on $\Delta(P, a)$ in (25) gives that, for

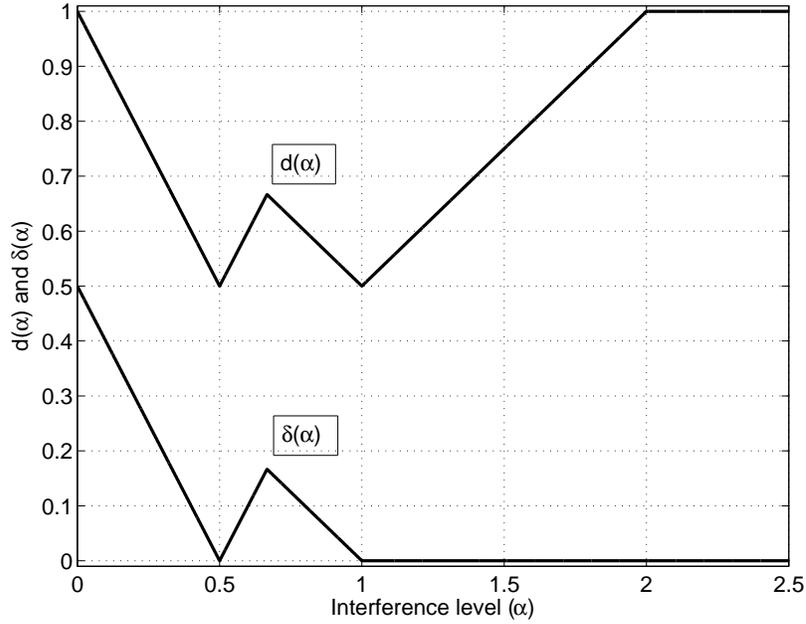


Fig. 3. A comparison of the generalized degrees of freedom (GDOF) in (47), and the exact asymptotic characterization of δ in (41) (see (42)) as a function of the non-negative interference level α .

$P > 1$ and $\alpha \in [0, 1)$,

$$\begin{aligned}
 & \Delta(P, P^{\alpha-1}) \\
 & \leq \frac{1}{2} \left[\min \left\{ \log(1+P) + \log \left(1 + \frac{P}{1+P^\alpha} \right), 2 \log \left(1 + P^\alpha + \frac{P}{1+P^\alpha} \right) \right\} \right. \\
 & \quad \left. - \log(1+P+P^\alpha) \right] \\
 & \triangleq \bar{\Delta}(P, P^{\alpha-1}).
 \end{aligned} \tag{48}$$

Consequently, for $\alpha \in [0, 1)$, we get from (48) that

$$\begin{aligned}
 & \lim_{P \rightarrow \infty} \frac{\bar{\Delta}(P, P^{\alpha-1})}{\log P} \\
 & = \frac{1}{2} \left[\min \left\{ 2 - \alpha, 2 \max \{ \alpha, 1 - \alpha \} \right\} - 1 \right] \\
 & = \min \left\{ \frac{1 - \alpha}{2}, \left| \frac{1}{2} - \alpha \right| \right\} \\
 & = \delta(\alpha)
 \end{aligned} \tag{49}$$

where the last equality follows from (42).

The substitution of the cross-link gain $a = P^{\alpha-1}$ into the lower bound on $\Delta(P, a)$ in (37)

gives that, for $P \geq 2.551$ and $\alpha \in [0, 1)$,

$$\begin{aligned} & \Delta(P, P^{\alpha-1}) \\ & \geq \frac{1}{2} \left[\min \left\{ \log(1 + P + P^\alpha) + \log \left(1 + \frac{P}{1 + P^\alpha} \right), 2 \log \left(1 + P^\alpha + \frac{P}{1 + P^\alpha} \right) \right\} \right. \\ & \quad \left. - \min \left\{ \log(1 + P + P^\alpha) + \log \left(1 + \frac{P}{(1 + P^\alpha)^2} \right), \log(1 + 2P) \right\} \right] - 1 \\ & \triangleq \underline{\Delta}(P, P^{\alpha-1}). \end{aligned} \tag{50}$$

Consequently, for $\alpha \in [0, 1)$, it follows from (50) that

$$\begin{aligned} & \lim_{P \rightarrow \infty} \frac{\underline{\Delta}(P, P^{\alpha-1})}{\log P} \\ & \stackrel{(a)}{=} \frac{1}{2} \left[\min \left\{ 2 - \alpha, 2 \max\{\alpha, 1 - \alpha\} \right\} - 1 \right] \\ & \stackrel{(b)}{=} \delta(\alpha) \end{aligned} \tag{51}$$

where the subtraction by 1 in equality (a) follows from the satisfiability of the inequality

$$\log(1 + P) \leq \min \left\{ \log(1 + P + P^\alpha) + \log \left(1 + \frac{P}{(1 + P^\alpha)^2} \right), \log(1 + 2P) \right\} \leq \log(1 + 2P)$$

which therefore implies that

$$\lim_{P \rightarrow \infty} \frac{\min \left\{ \log(1 + P + P^\alpha) + \log \left(1 + \frac{P}{(1 + P^\alpha)^2} \right), \log(1 + 2P) \right\}}{\log P} = 1.$$

Equality (b) in (51) follows from the last two equalities in (49).

To conclude, (49) and (51) demonstrate the asymptotic tightness of the upper and lower bound in (25) and (37), respectively, for the considered coupling of the SNR and INR in (39). This completes the proof of the theorem. \blacksquare

Remark 3: The following is a discussion on Theorem 3. Consider the case where the SNR and INR scalings are coupled such that (39) holds. Under this assumption, the reason for the asymptotic tightness of the upper and lower bounds in (48) and (50) is twofold. The first reason is related to the ETW bound that provides the exact asymptotic linear growth of the sum-rate with $\log P$ (see [10, Theorem 2]). The second reason is attributed to the fact that, for a weak and symmetric two-user GIC, the total rate at the two corner points of the capacity region is bounded between $\frac{1}{2} \log(1 + P)$ and $\frac{1}{2} \log(1 + 2P)$ (see (12) and (13)) and both scale like $\frac{1}{2} \log P$ for large P . It is noted that an ignorance of the effect of Kramer's bound in the derivation of the upper bounds on the right-hand sides of (12) and (13) would have weakened the lower bound on $\Delta(P, P^{\alpha-1})$ by a removal of the term $\frac{1}{2} \log(1 + 2P)$ from the right-hand side of (50). Consequently, for $\alpha \in [0, \frac{1}{2}]$, this removal would have reduced the asymptotic limit in (51) from $\delta(\alpha) = \frac{1}{2} - \alpha$ (see (42)) to zero.

As a consequence of the asymptotic analysis in this sub-section, some implications are provided in the following:

- 1) Consider a two-user symmetric GIC where the cross-link gain is equal to $a = P^{\alpha-1}$ for $\alpha \geq 0$. From (40), (41), (42) and (47), it follows that

$$\lim_{P \rightarrow \infty} \frac{\Delta(P, P^{\alpha-1})}{C_{\text{sum}}(P, P^{\alpha-1})} = \frac{\delta(\alpha)}{d(\alpha)} = \begin{cases} \frac{1-2\alpha}{2(1-\alpha)}, & \text{if } 0 \leq \alpha < \frac{1}{2} \\ 1 - \frac{1}{2\alpha}, & \text{if } \frac{1}{2} \leq \alpha < \frac{2}{3} \\ \frac{1-\alpha}{2-\alpha}, & \text{if } \frac{2}{3} \leq \alpha < 1 \\ 0, & \text{if } \alpha \geq 1 \end{cases} \quad (52)$$

is the asymptotic fractional loss in the total rate at the corner points of the capacity region.

- 2) Analogously to the GDOF of a two-user symmetric GIC in (40), the function δ is defined in (41) by replacing the sum-rate with the excess rate for the sum-rate with respect to the corner points; in both cases, it is assumed that the cross-link gain is $a = P^{\alpha-1}$ for some interference level $\alpha \geq 0$. The GDOF is known to be a non-monotonic function of α over the interval $[0,1]$ (see [10, pp. 5542–5543] and (47)). From (42), it also follows that δ is a non-monotonic function over this interval. For $P > 1$, the cross-link gain $a = P^{\alpha-1}$ forms a monotonic increasing function of $\alpha \in [0,1]$, and it is a one-to-one mapping from the interval $[0,1]$ to itself. This implies that, for large P , the excess rate for the sum-rate with respect to the corner points (denoted by $\Delta(P, a)$) is a non-monotonic function of a over the interval $[0,1]$. This observation is supported by numerical results in Section III-E. A discussion on this phenomenon is provided later in this section (see Remark 4).
- 3) Consider the closed-form expression in (47) for the GDOF of a symmetric two-user GIC. For large P , the worst interference with respect to the sum-rate is known to occur when the cross-link gain scales like $\frac{1}{\sqrt{P}}$ or it is 1 (this refers to $\alpha = \frac{1}{2}$ or $\alpha = 1$, respectively). If $\alpha = \frac{1}{2}$, we have from (47)

$$d\left(\frac{1}{2}\right) = \lim_{P \rightarrow \infty} \frac{C_{\text{sum}}\left(P, \frac{1}{\sqrt{P}}\right)}{\log P} = \frac{1}{2} \quad (53)$$

and, from (52),

$$\lim_{P \rightarrow \infty} \frac{\Delta\left(P, \frac{1}{\sqrt{P}}\right)}{C_{\text{sum}}\left(P, \frac{1}{\sqrt{P}}\right)} = 0. \quad (54)$$

The same also holds for the case where $\alpha = 1$ (i.e., when the cross-link gain is $a = 1$). It therefore follows that, for the worst interference with respect to the sum-rate, there is asymptotically no loss in the total rate ($R_1 + R_2$) when the users operate at one of the corner points of the capacity region.

- 4) The limit on the left-hand side of (52) is bounded between zero and one-half for $a = P^{\alpha-1}$ with $\alpha \geq 0$, and it gets a local maximal value at $\alpha = \frac{2}{3}$ (which is global maximum for $\alpha \geq \frac{1}{2}$). From Theorem 2, we have

$$\lim_{P \rightarrow \infty} \frac{\Delta\left(P, \frac{1}{\sqrt[3]{P}}\right)}{\log P} = \frac{1}{6}. \quad (55)$$

- 5) From the asymptotic upper and lower bounds on $\Delta(P, a)$ for large P and a fixed $a \in (0, 1)$ (see (26) and (38)), we have

$$\frac{1}{2} \log \left(1 + \frac{1}{a}\right) - 1 \leq \lim_{P \rightarrow \infty} \Delta(P, a) \leq \frac{1}{2} \log \left(\frac{1}{a}\right), \quad \forall a \in (0, 1).$$

Since also the equality $\Delta(P, a) = 0$ holds for every $a \geq 1$, then it follows that

$$\lim_{P \rightarrow \infty} \frac{\Delta(P, a)}{\log P} = 0, \quad \forall a > 0.$$

This is consistent with the equality $\delta(1) = 0$ in (42).

- 6) Consider the capacity region of a weak and symmetric two-user GIC, and the bounds on the excess rate for the sum-rate with respect to the corner points of its capacity region (see Sections III-A and III-B). In this case, the transmission rate of one of the users is assumed to be equal to the single-user capacity of the respective AWGN channel. Consider now the case where the transmission rate of this user is reduced by no more than $\varepsilon > 0$, so it is within ε of the single-user capacity. Then, from Theorem 1, it follows that the upper bound on the transmission rate of the other user cannot increase by more than

$$f(\varepsilon) \triangleq \max \left\{ 2\varepsilon, \left(1 + \frac{1+P}{aP} \right) \varepsilon \right\}.$$

Consequently, the lower bound on the excess rate for the sum-rate in (37) is reduced by no more than $f(\varepsilon)$. Furthermore, the upper bound on this excess rate cannot increase by more than ε (note that if the first user reduces its transmission rate by no more than ε , then the other user can stay at the same transmission rate; overall, the total transmission rate is decreased by no more than ε , and consequently the excess rate for the sum-rate cannot increase by more than ε). Revisiting the analysis in this sub-section by introducing a positive $\varepsilon \triangleq \varepsilon(P)$ to the calculations, before taking the limit of P to infinity, leads to the conclusion that the corresponding characterization of δ in (42) stays un-affected as long as

$$\lim_{P \rightarrow \infty} \frac{\varepsilon(P)}{\log P} = 0$$

which then implies that

$$\lim_{P \rightarrow \infty} \frac{f(\varepsilon(P))}{\log P} = 0$$

when the value of the cross-link gain a is fixed. For example, this happens to be the case if ε scales like $(\log P)^\beta$ for an arbitrary $\beta \in (0, 1)$ (so, in the limit where $P \rightarrow \infty$, we have $\varepsilon(P) \rightarrow \infty$ but $\frac{\varepsilon(P)}{\log P} \rightarrow 0$).

Consider a weak and symmetric GIC where, in standard form, $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in (0, 1)$. Let Δ denote the excess rate for the sum-rate with respect to the corner points of the capacity region, as it is defined in (21). The following summarizes the results that are introduced in this section so far for this channel model:

- The excess rate Δ satisfies the upper bound in (25).
- If $P \geq 2.551$, it also satisfies the lower bound in (37).
- For large enough P , $\Delta = \Delta(P, a)$ is a non-monotonic function of a over the interval $(0, 1]$.
- The upper and lower bounds on $\Delta(P, P^{\alpha-1})$ in (48) and (50), respectively, imply the exact asymptotic scaling of $\Delta(P, P^{\alpha-1})$ with $\log P$ for an arbitrary $\alpha \geq 0$ (note that these bounds apply to $\alpha \in [0, 1)$, but $\Delta(P, P^{\alpha-1}) = 0$ when $\alpha \geq 1$ and $P \geq 1$).
- The asymptotic linear growth of $\Delta(P, P^{\alpha-1})$ with $\log P$, for $\alpha \geq 0$, is given by $\delta(\alpha)$ in (42). Furthermore, a connection between the function δ and the symmetric GDOF is given in (46) (see Fig. 3).
- When the value of the cross-link gain is kept fixed between 0 and 1, the excess rate Δ satisfies the upper and lower bounds in (26) and (38), respectively. These asymptotic bounds on Δ scale like $\frac{1}{2} \log \left(\frac{1}{a} \right)$, and they differ by at most 1 bit per channel use, irrespectively of the fixed value of $a \in (0, 1]$.
- Let $a = P^{\alpha-1}$ for some $\alpha \geq 0$ and $P > 1$. Consider the loss in the total rate, expressed as a fraction of the sum-rate, when the users operate at one of the corner points of the

capacity region. This asymptotic normalized loss is provided in (52), and it is bounded between 0 and $\frac{1}{2}$. For large values of P , it roughly varies from 0 to $\frac{1}{4}$ by letting a grow (only slightly) from $\frac{1}{\sqrt{P}}$ to $\frac{1}{\sqrt[3]{P}}$.

The following remark refers to the third item above:

Remark 4: For a weak and symmetric two-user GIC, the excess rate for the sum-rate with respect to the corner points is the difference between the sum-rate of the capacity region and the total rate at any of the two corner points of the capacity region. According to Theorem 1, for large P , the total rate at a corner point is an increasing function of $a \in (0, 1]$. Although it is known that, for large P , the sum-rate of the capacity region is not monotonic decreasing in a , a priori, there was a possibility that by subtracting from it a monotonic increasing function in a , the difference (that is equal to the excess rate Δ) would be monotonic decreasing in a . However, it is shown not to be the case. The fact that, for large P , the excess rate $\Delta(P, a)$ is not a monotonic decreasing function of a is a stronger property than the non-monotonicity of the sum-rate.

D. A Tightening of the Bounds on the Excess Rate (Δ) for Weak and Symmetric GICs

In Sections III-A and III-B, closed-form expressions for upper and lower bounds on Δ are derived for weak GICs. These expressions are used in Section III-C for an asymptotic analysis where we let P tend to infinity. In the following, the bounds on the excess rate Δ are improved for finite P at the cost of introducing bounds that are subject to numerical optimizations. For simplicity, we focus on the model of a weak and symmetric GIC. In light of Theorem 3, a use of improved bounds does not imply any asymptotic improvement as compared to the bounds in Section III-C that are expressed in closed form. Nevertheless, the new bounds are improved for finite SNR and INR, as is illustrated in Section III-E.

1) *An improved lower bound on Δ :* An improvement of the lower bound on the excess rate for the sum-rate with respect to the corner points (Δ) is obtained by relying on an improved lower bound on the sum-rate in comparison to (37). For tightening the lower bound on the sum-rate, it is suggested to combine (37) with the lower bound in [17, Eq. (32)] (the latter bound follows from the Han-Kobayashi achievable region, see [17, Table 1]):

$$C_{\text{sum}} \geq \max_{\alpha, \beta, \delta} \rho(P, a, \alpha, \beta, \delta) \quad (56)$$

where

$$\begin{aligned} & \rho(P, a, \alpha, \beta, \delta) \\ & \triangleq \delta \log \left(1 + \frac{2\alpha\delta P}{1 + 2a\beta\delta P} \right) + \delta \log \left(1 + \frac{2\beta\delta P}{1 + 2a\alpha\delta P} \right) + \left(\frac{1 - 2\delta}{2} \right) \log(1 + 2(1 + 2\delta)P) \\ & + \min \left\{ \frac{\delta}{2} \cdot \log \left(1 + \frac{2\bar{\alpha}\delta P + 2a\bar{\beta}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P} \right) + \frac{\delta}{2} \cdot \log \left(1 + \frac{2\bar{\beta}\delta P + 2a\bar{\alpha}\delta P}{1 + 2\beta\delta P + 2a\alpha\delta P} \right), \right. \\ & \quad \delta \log \left(1 + \frac{2\bar{\alpha}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P} \right) + \delta \log \left(1 + \frac{2\bar{\beta}\delta P}{1 + 2\beta\delta P + 2a\alpha\delta P} \right), \\ & \quad \left. \delta \log \left(1 + \frac{2a\bar{\alpha}\delta P}{1 + 2a\alpha\delta P + 2\beta\delta P} \right) + \delta \log \left(1 + \frac{2a\bar{\beta}\delta P}{1 + 2\alpha\delta P + 2a\beta\delta P} \right) \right\} \\ & \forall (\alpha, \beta, \delta) \quad \text{s.t.} \quad 0 \leq \alpha \leq 1, \quad 0 \leq \beta \leq 1, \quad 0 \leq \delta \leq \frac{1}{2}. \end{aligned}$$

A combination of (21), (27) and (56) gives the following lower bound on Δ for a weak and symmetric GIC:

$$\Delta = \Delta(P, a) \geq \max_{\alpha, \beta, \delta} \left\{ \rho(P, a, \alpha, \beta, \delta) \right\} - \frac{1}{2} \min \left\{ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{(1+aP)^2} \right), \log(1+2P) \right\}.$$

Furthermore, it follows from Lemma 1 that if $P \geq 2.551$, a combination of (21) and (27) with the two lower bounds on the sum-rate in (37) and (56) gives the following tightened lower bound on Δ (as compared to (37)):

$$\Delta \geq \max \left\{ \max_{\alpha, \beta, \delta} \left\{ \rho(P, a, \alpha, \beta, \delta) \right\}, \frac{1}{2} \min \left\{ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{1+aP} \right), 2 \log \left(1 + aP + \frac{P}{1+aP} \right) \right\} - 1 \right\} - \frac{1}{2} \min \left\{ \log(1 + (a+1)P) + \log \left(1 + \frac{P}{(1+aP)^2} \right), \log(1+2P) \right\}. \quad (57)$$

2) *An improved upper bound on Δ* : An improvement of the upper bound on the excess rate for the sum-rate with respect to the corner points (Δ) is obtained by relying on an improved upper bound on the sum-rate (as compared to (23)). This is obtained by calculating the minimum of Etkin's bound in [11] and Kramer's bound in [13, Theorem 2]. Following the discussion in [11], Etkin's bound outperforms the upper bounds on the sum-rate in [1], [10], [15], [21]; nevertheless, for values of a that are close to 1, Kramer's bound in [13, Theorem 2] outperforms the other known bounds on the sum-rate (see [11, Fig. 1]). Consequently, the minimum of Etkin's and Kramer's bounds in [11] and [13, Theorem 2] is calculated as an upper bound on the sum-rate. Combining [11, Eqs. (14)-(16)] (while adapting notation, and dividing the bound by 2 for a real-valued GIC), the simplified version of Etkin's upper bound on the sum-rate for real-valued, weak and symmetric GICs gets the form

$$C_{\text{sum}} \leq \min_{\alpha, \sigma, \rho} \left\{ \min \left\{ \frac{1}{2} \log \left(1 + \frac{P(1+\alpha^2)\gamma}{(1-\rho^2)\sigma^2} \right), \log \left(1 + \frac{\alpha^2 P \gamma}{(1-\rho^2)\sigma^2} \right) \right\} + \log \left(\frac{(1+P(1+a))(P(1+\alpha^2)+\sigma^2) - (P(1+\alpha\sqrt{a})+\rho\sigma)^2}{P(1+\alpha^2)\gamma + (1-\rho^2)\sigma^2} \right) \right\} \quad (58)$$

where

$$\gamma = \alpha^2 - 2\alpha\rho\sigma\sqrt{a} + \sigma^2 a, \quad \rho = \alpha\sigma\sqrt{a} \pm \sqrt{(1-\alpha^2)(1-\sigma^2 a)}, \quad \sigma \in [0, \frac{1}{\sqrt{a}}], \quad \alpha \in [-1, 1]. \quad (59)$$

The two possible values of ρ in (59) need to be checked in the optimization of the parameters. For a weak and symmetric GIC, Kramer's upper bound on the sum-rate (see [13, Eqs. (44) and (45)]) is simplified to

$$C_{\text{sum}} \leq \frac{1}{2} \log \left(1 + 2P + \frac{B}{2} - \frac{1}{2} \sqrt{B^2 - 4P^2 \left(\frac{1}{a} - 1 \right)^2} \right) \quad (60)$$

where $B = \frac{1}{a^2} + 2P \left(\frac{1}{a} - 1 \right) - 1$. An improvement of the upper bound on the sum-rate in (23) follows by taking the minimal value of the bounds in (58) and (60); consequently, a combination

of (21) and (22) with this improved upper bound on the sum-rate provides an improved upper bound on Δ (as compared to the bound in (25)).

3) *A simplification of the improved upper bound on Δ for a sub-class of weak and symmetric GICs:* The following simplifies the improved upper bound on the excess rate (Δ) for a sub-class of weak and symmetric GICs. It has been independently demonstrated in [1], [15] and [21] that if

$$0 < a < \frac{1}{4}, \quad 0 < P \leq \frac{\sqrt{a} - 2a}{2a^2} \quad (61)$$

then the sum-rate of the GIC is equal to

$$C_{\text{sum}} = \log \left(1 + \frac{P}{1 + aP} \right). \quad (62)$$

This sum-rate is achievable by using single-user Gaussian codebooks, and treating the interference as noise. Under the conditions in (61), the exact sum-rate coincides with the upper bound given in (58). Hence, a replacement of the upper bound on the sum-rate in (23) with the exact sum-rate in (62), followed by a combination of (21) and (22) gives that

$$\Delta \leq \frac{1}{2} \log \left(\frac{1}{1 + aP} + \frac{P}{(1 + aP)^2} \right). \quad (63)$$

One can verify that, under the conditions in (61), the upper bound on Δ in (63) is indeed positive.

E. Numerical Results

The following section presents numerical results for the bounds on the excess rate for the sum-rate with respect to the corner points (denoted by Δ) while focusing on weak and symmetric two-user GICs.

Figure 4 compares upper and lower bounds on Δ as a function of the cross-link gain for a weak and symmetric GIC. The upper and lower plots of this figure correspond to $P = 50$ and $P = 500$, respectively. The upper and lower bounds on Δ rely on (25) and (37), respectively, and the improved upper and lower bounds on Δ are based on Section III-D. For $P = 50$ (see the upper plot of Figure 4), the advantage of the improved bounds on Δ is exemplified; the lower bound on Δ for the case where $P = 50$ is almost useless (it is zero unless the interference is very weak). The improved upper and lower bounds on Δ for $P = 50$ do not enable to conclude whether Δ is a monotonic decreasing function of a (for weak interference where $a \in [0, 1]$). For $P = 500$ (see the lower plot of Figure 4), the improved bounds on Δ indicate that it is not a monotonic decreasing function of a ; this follows by noticing that the improved upper bound on Δ at $a = 0.045$ is equal to 0.578 bits per channel use, and its improved lower bound at $a = 0.110$ is equal to 0.620 bits per channel use. The observation that, for large P , the function of Δ is not monotonic decreasing in $a \in (0, 1)$ is supported by the asymptotic analysis in Section III-C. This conclusion is stronger than the observation that, for large enough P , the sum-rate is not a monotonic decreasing function of $a \in [0, 1]$ (see [10, pp. 5542–5543]), as it is discussed in Remark 4 (see Section III-C). Figures 4 and 5 show that the phenomenon of the non-monotonicity of Δ as a function of a is more dominant when the value of P is increased. These figures also illustrate the advantage of the improved upper and lower bounds on Δ in Section III-D in comparison to the simple bounds on Δ in (25) and (37). Note, however, that the simple bounds on Δ that are given in closed-form expressions are asymptotically tight as is demonstrated in Theorem 3.

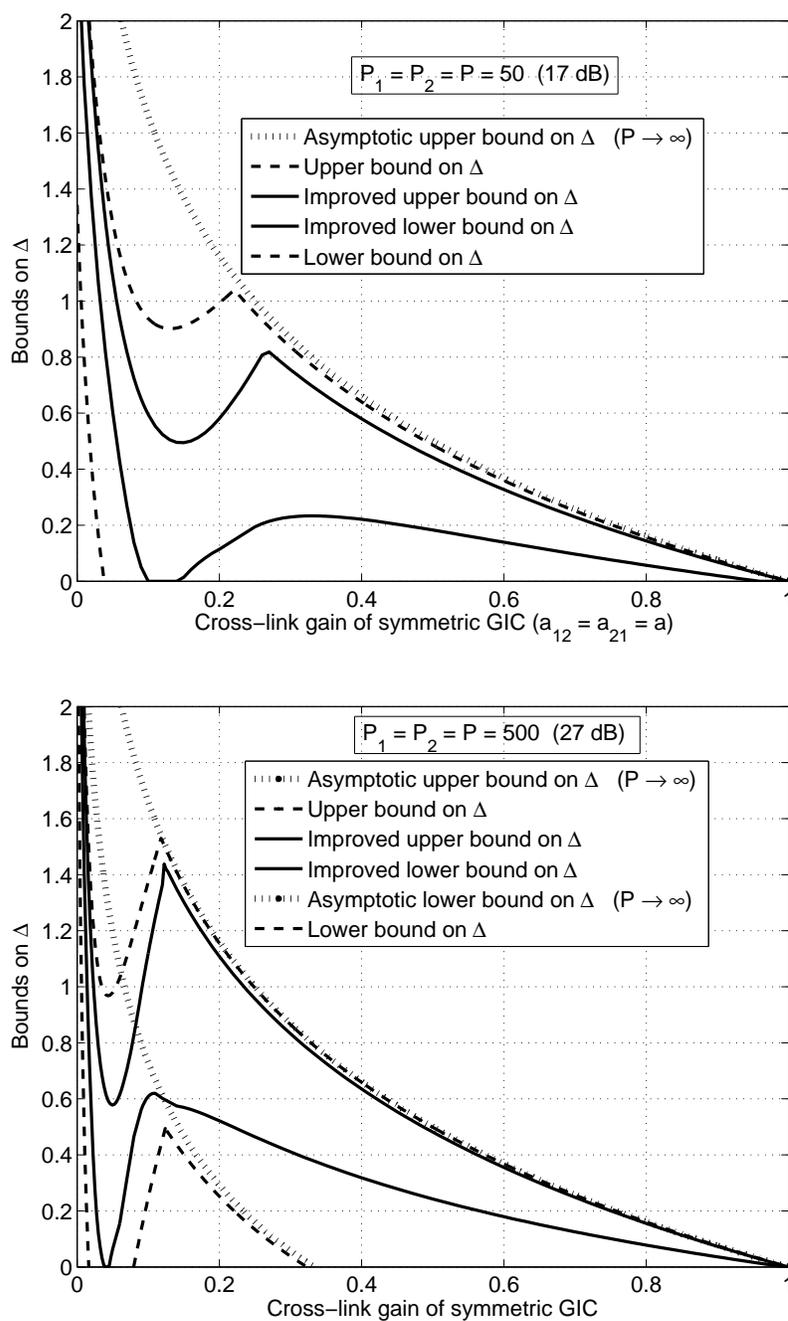


Fig. 4. Upper and lower bounds on the excess rate for the sum-rate with respect to the corner points (Δ) as a function of the cross-link gain (a). The plots refer to a weak and symmetric GIC where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in [0, 1]$ in standard form. The upper and lower plots refer to $P = 50$ and $P = 500$, respectively. The upper and lower bounds on Δ rely on (25) and (37), respectively, and the improved bounds on Δ rely on Section III-D. The dashed lines refer to the asymptotic upper and lower bounds on Δ in (26) and (38), respectively.

Table I compares the asymptotic approximation of Δ with its improved upper bound in Section III-D2. It verifies that, for large P , the minimal value of Δ is obtained at $a \approx \frac{1}{\sqrt{P}}$; it also

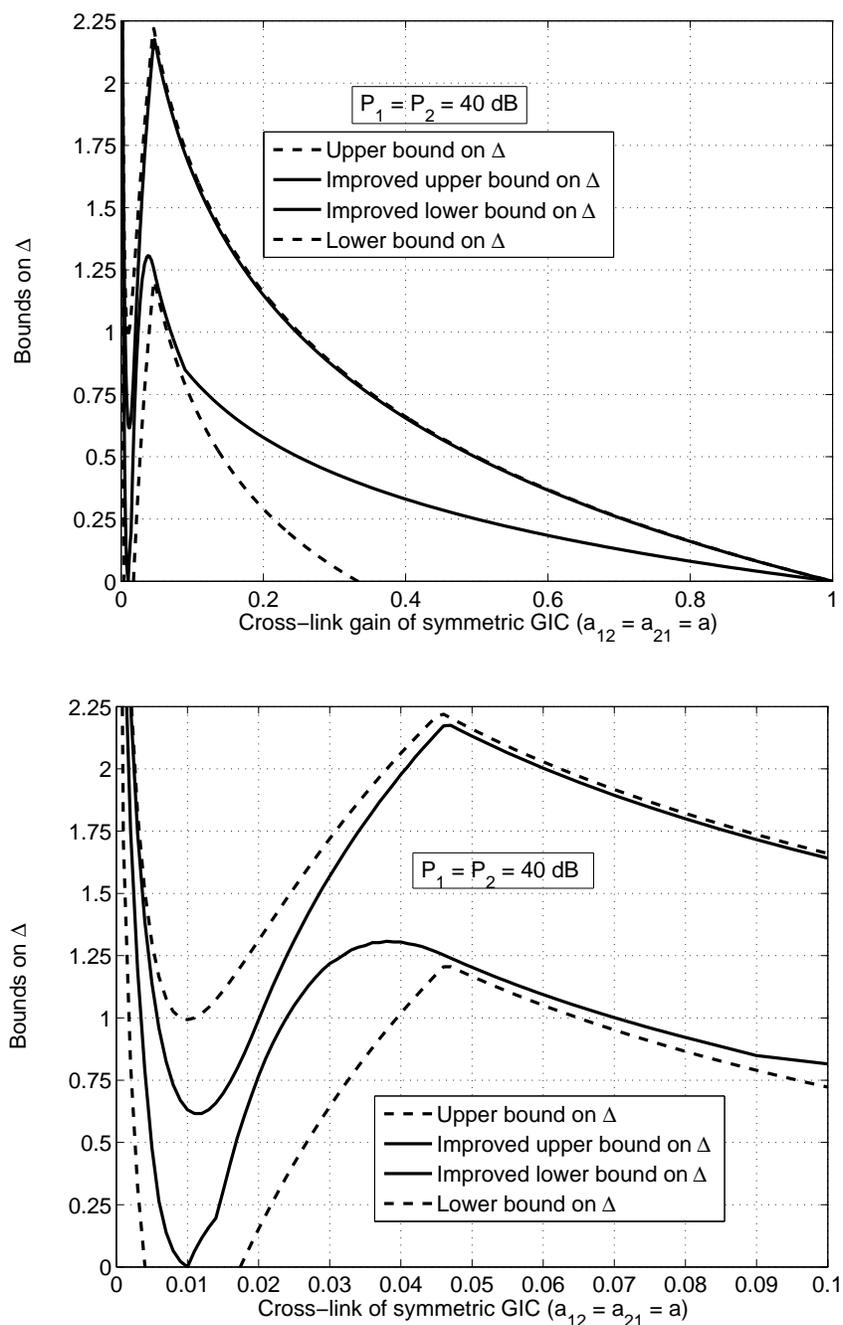


Fig. 5. Upper and lower bounds on the excess rate for the sum-rate with respect to the corner points (Δ) as a function of the cross-link gain (a). This figure refers to a weak and symmetric GIC where $P_1 = P_2 = P = 40$ dB and $a_{12} = a_{21} = a \in [0, 1]$ in standard form. The upper and lower bounds on Δ are given in (25) and (37), respectively, and the improved bounds on Δ rely on Section III-D. The upper plot shows upper and lower bounds on Δ over the range of weak interference ($0 \leq a \leq 1$), and the lower plot zooms in the upper plot for $a \in [0, 0.1]$; it shows that Δ is a non-monotonic function of a in the weak interference regime.

TABLE I. COMPARISON OF THE ASYMPTOTIC APPROXIMATION OF THE EXCESS RATE FOR THE SUM-RATE WITH RESPECT TO THE CORNER POINTS (Δ) WITH ITS IMPROVED UPPER BOUND ON Δ IN SECTION III-D2.

Power constraint in standard form	Value of a achieving minimum of Δ		Normalized Δ by $\log P$		Value of a maximizing Δ for $a \geq \frac{1}{\sqrt{P}}$		Normalized Δ by $\log P$	
	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value
(P)	$(a = \frac{1}{\sqrt{P}})$		Eq. (54)		$(a = \frac{1}{\sqrt[3]{P}})$		Eq. (55)	
27 dB	0.045	0.050	0	0.065	0.126	0.140	0.167	0.154
40 dB	0.010	0.011	0	0.046	0.046	0.042	0.167	0.164
60 dB	0.001	0.001	0	0.032	0.010	0.010	0.167	0.166

verifies that, for large P , the maximal value of Δ for $a \geq \frac{1}{\sqrt{P}}$ is obtained at $a \approx \frac{1}{\sqrt[3]{P}}$. Table I also supports the asymptotic limits in (54) and (55), showing how close are the numerical results for large P to their corresponding asymptotic limits: specifically, for large P , at $a = \frac{1}{\sqrt{P}}$ and $\frac{1}{\sqrt[3]{P}}$, the ratio $\frac{\Delta}{\log P}$ tends to zero or $\frac{1}{6}$, respectively; this is supported by the numerical results in the 5th and 9th columns of Table I. The asymptotic approximations in Table I are consistent with the overshoots observed in the plots of Δ when the cross-link gain a varies between $\frac{1}{\sqrt{P}}$ and $\frac{1}{\sqrt[3]{P}}$; this interval is narrowed as the value of P is increased (see Figures 4 and 5). Finally, it is also shown in Figures 4 and 5 that the curves of the upper and lower bounds on Δ , as a function of the cross-link gain a , do not converge uniformly to their asymptotic upper and lower bounds in (26) and (38), respectively. This non-uniform convergence is noticed by the large deviation of the bounds for finite P from the asymptotic bounds where this deviation takes place over an interval of small values of a ; however, this interval of a shrinks when the value of P is increased, and its length is approximately $\frac{1}{\sqrt[3]{P}}$ for large P . This conclusion is consistent with the asymptotic analysis in Section III-C (see the items that correspond to Eqs. (52) and (55)), and it is also supported by the numerical results in Table I.

IV. SUMMARY AND OUTLOOK

This paper considers the corner points of the capacity region of a two-user Gaussian interference channel (GIC). The operational meaning of the corner points is a study of the situation where one user sends its information at the single-user capacity (in the absence of interference), and the other user transmits its data at the largest rate for which reliable communication is possible at the two non-cooperating receivers. The approach used in this work for the study of the corner points relies on some existing outer bounds on the capacity region of a two-user GIC.

In contrast to strong, mixed or one-sided GICs, the two corner points of the capacity region of a weak GIC have not been determined yet. This paper is focused on the latter model that refers to a two-user GIC in standard form whose cross-link gains are positive and below 1. Theorem 1 provides rigorous bounds on the corner points of the capacity region, whose tightness is especially remarkable at high SNR and INR.

The sum-rate of a GIC with either strong, mixed or one-sided interference is attained at one of the corner points of the capacity region, and this corner point is known exactly (see [12], [15], [17], [20] and [21]). This is in contrast to a weak GIC whose sum-rate is not attained at any of the corner points of its capacity region. This motivates the study in Section III which introduces and analyzes the *excess rate for the sum-rate with respect to the corner points*. This measure, denoted by Δ , is defined to be the gap between the sum-rate and the maximal total rate obtained by the two corner points of the capacity region. Simple upper and lower bounds on Δ are derived in Section III, which are expressed in closed form, and the asymptotic characterization of these bounds is analyzed. In the asymptotic case where the channel is interference limited

(i.e., $P \rightarrow \infty$) and symmetric, the corresponding upper and lower bounds on Δ differ by at most 1 bit per channel use (irrespective of the value of the cross-link gain a); in this case, both asymptotic bounds on Δ scale like $\frac{1}{2} \log\left(\frac{1}{a}\right)$ for small a .

Analogously to the study of the generalized degrees of freedom (GDOF), an asymptotic characterization of Δ is provided in this paper. More explicitly, under the setting where the SNR and INR scalings are coupled such that $\frac{\log(\text{SNR})}{\log(\text{INR})} = \alpha$ for an arbitrary non-negative α , the exact asymptotic characterization of Δ is provided in Theorem 2. Interestingly, the upper and lower bounds on Δ are demonstrated to be *asymptotically tight for the whole range of this scaling* (see Theorem 3).

For high SNR, the non-monotonicity of Δ as a function of the cross-link gain follows from the asymptotic analysis, and it is shown to be a stronger result than the non-monotonicity of the sum-rate in [10, Section 3].

Improved upper and lower bounds on Δ are introduced for finite SNR and INR, and numerical results of these bounds are exemplified. The numerical results in Section III-E verify the effectiveness of the approximations for high SNR that follow from the asymptotic analysis of Δ .

This paper supports in general Conjecture 1 whose interpretation is that if one user transmits at its single-user capacity, then the other user should decrease its rate such that both decoders can reliably decode its message.

A recent work by Bustin *et al.* studied the corner points via the connection between the minimum mean square error and mutual information [2], providing another support (endorsement) to Costa's conjecture from a different perspective.

We list in the following some directions for further research:

- 1) A possible tightening of the bound in (6) for a mixed GIC is of interest. It is motivated by the fact that the upper bound for the corresponding corner point is above the one in Conjecture 1.
- 2) The unknown corner point of a weak one-sided GIC satisfies the bounds in Proposition 2; it is given by (R_1, C_2) where the gap between the upper and lower bounds on R_1 in (9) is large for small values of a . An improvement of these bounds is of interest (see the last paragraph in Section I-C).
- 3) A possible extension of this work to the class of semi-deterministic interference channels in [23], which includes the two-user GICs and the deterministic interference channels in [8].

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