Advances in the Shannon Capacity of Graphs

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The Shannon capacity of a graph

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- An independent set in G is composed of input symbols that cannot be confused with each other at the channel output.
- Thus, the largest number of inputs a channel can communicate without error in a single use is $\alpha(\mathrm{G})$.

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- Thus, the Shannon capacity of a graph is defined as

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- The computation of the Shannon capacity is notoriously difficult, let alone the computation of the independence number, which is an NP-hard problem.

Properties of the Shannon Capacity

- For a simple graph G and for any $m, k \in \mathbb{N}$,

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$$\Theta(\mathsf{G} \boxtimes \mathsf{H}) \geq \Theta(\mathsf{G}) \cdot \Theta(\mathsf{H}),$$
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- N. Alon (1998) provided a specific example where

$$\Theta(\mathsf{G} + \mathsf{H}) > \Theta(\mathsf{G}) + \Theta(\mathsf{H}),$$

thus disproving Shannon's conjecture that it holds with equality.

New results on the Shannon capacity of structured graphs

- Define the set $\mathbb{N}[x_1, \ldots, x_\ell]$ as the set of polynomials of variables $x_1, \ldots, x_\ell$ with natural coefficients.
- A polynomial of graphs, $p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)$, is defined such that $+$ is the disjoint union of graphs and $\cdot$ is the strong product of graphs.

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Question

What are the conditions on a sequence of graphs $\mathsf{G}_1, \dots, \mathsf{G}_\ell$ such that for every polynomial $p \in \mathbb{N}[x_1, \dots, x_\ell]$,

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A result by Schrijver (2023) says that this holds for every polynomial if and only if

$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \Theta(\mathsf{G}_1) + \ldots + \Theta(\mathsf{G}_\ell).$$

Thus, it is enough to focus on the disjoint union only.

Theorem

Let $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ be simple graphs. If $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ for every $i \in [\ell]$, then

$$\Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)) = p(\Theta(\mathsf{G}_1), \ldots, \Theta(\mathsf{G}_\ell))$$

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Theorem

Let $\mathsf{G}_1, \dots, \mathsf{G}_\ell$ be simple graphs. If $\Theta(\mathsf{G}_i) = \alpha_{\mathrm{f}}(\mathsf{G}_i)$ for (at least) $\ell - 1$ graphs, then

$$\Theta(p(\mathsf{G}_1, \dots, \mathsf{G}_\ell)) = p(\Theta(\mathsf{G}_1), \dots, \Theta(\mathsf{G}_\ell))$$

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- Thus, if $\Theta(\mathsf{G}) = \vartheta(\mathsf{G})$ and $\Theta(\mathsf{H}) = \vartheta(\mathsf{H})$, or if $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$, then

$$\Theta(\mathsf{G})\,\Theta(\mathsf{H}) = \Theta(\mathsf{G} \boxtimes \mathsf{H}).$$

Example

- If $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ are all Kneser graphs or self-complementary vertex-transitive graphs, then

$$\Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)) = p(\Theta(\mathsf{G}_1), \ldots, \Theta(\mathsf{G}_\ell))$$

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  for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$.
- In particular, if $\mathsf{G}_i = \mathsf{K}(n_i, r_i)$ (with $n_i \geq 2r_i$) for every $i \in [\ell]$, then

$$\Theta(\mathsf{K}(n_1, r_1) + \ldots + \mathsf{K}(n_\ell, r_\ell)) = \binom{n_1 - 1}{r_1 - 1} + \ldots + \binom{n_\ell - 1}{r_\ell - 1}.$$

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- Similarly, if G_i is a self-complementary vertex-transitive graph for every $i \in [\ell]$, then
$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \sqrt{n_1} + \ldots + \sqrt{n_\ell},$$
where n_i is the order of G_i.

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- On the other hand, all graphs have

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \alpha_{\mathrm{f}}(\mathsf{G}).$$

So, the first conditions require a weaker equality than the second conditions.

Example

If $\mathsf{G} = \mathsf{K}(7,2)$ and $\mathsf{H} = \mathsf{K}(5,2)$, then

$$\begin{aligned} \Theta(\mathsf{G}) &= \vartheta(\mathsf{G}) < \alpha_{\mathrm{f}}(\mathsf{G}), \\ \Theta(\mathsf{H}) &= \vartheta(\mathsf{H}) < \alpha_{\mathrm{f}}(\mathsf{H}). \end{aligned}$$

So we get $\Theta(\mathsf{G} + \mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H})$ by the conditions of the first theorem, but not by the second one.

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So we get $\Theta(\mathsf{G}+\mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H})$ by the conditions of the first theorem, but not by the second one.
On the other hand, if $\mathsf{G} = \mathsf{C}_4$ is the 4-cycle graph and H is the complement of the Schlafli graph, then

$$\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G}),$$
$$\Theta(\mathsf{H}) < \vartheta(\mathsf{H}).$$

So we get $\Theta(\mathsf{G}+\mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H})$ by the conditions of the second theorem, but not by the first one.

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Definition: Tadpole graphs

Let k and ℓ be natural numbers such that $k \geq 3$ and $\ell \geq 0$. The Tadpole graph $\mathrm{T}(k, \ell)$ is defined as a cycle of order k, and a path of order $\ell + 1$, such that one endpoint of the path is a vertex of the cycle.

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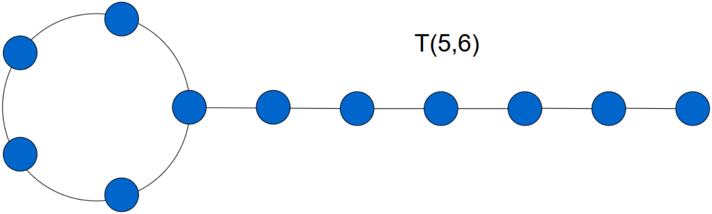

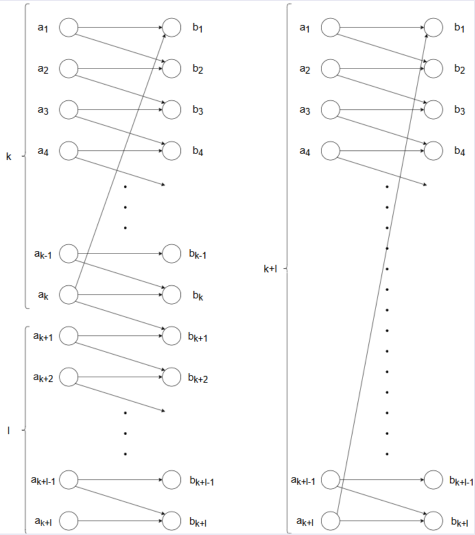

Figure 1: The DMCs of $\mathrm{T}(k,\ell)$ (left plot) and $\mathsf{C}_{k+\ell}$ (right plot).

Theorem

Let $k \geq 3$ and $\ell \geq 1$ be integers.

1. If one of the following two conditions holds
   - $k = 3$ or $k \geq 4$ is even,
   - $k \geq 5$ is odd and $\ell \geq 1$ is odd,

   then
   $$\Theta(\mathrm{T}(k, \ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil.$$

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2. If $k \geq 5$ is odd and $\ell \geq 2$ is even, then
$$\Theta(\mathrm{T}(k, \ell)) = \Theta(\mathsf{C}_k) + \frac{\ell}{2},$$

   which yields
$$\Theta(\mathrm{T}(5, \ell)) = \sqrt{5} + \frac{\ell}{2},$$
$$\frac{k + \ell - 1}{2} \leq \Theta(\mathrm{T}(k, \ell)) \leq \frac{k}{1 + \sec \frac{\pi}{k}} + \frac{\ell}{2}, \quad k \geq 7.$$

   The gap between these upper and lower bounds is less than $\frac{1}{2}$.

Proof of the second case

- If $k \geq 5$ is odd and $\ell \geq 2$ is even, then since $\mathsf{C}_k + \mathsf{P}_\ell$ is a spanning subgraph of $\mathrm{T}(k, \ell)$, we get

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- By our earlier proposition, it then follows that
$$\begin{aligned}\Theta(\mathsf{C}_k + \mathsf{P}_\ell) &= \Theta(\mathsf{C}_k) + \Theta(\mathsf{P}_\ell) \\ &= \Theta(\mathsf{C}_k) + \frac{\ell}{2}.\end{aligned}$$

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- This overall implies that
$$\Theta(\mathrm{T}(k,\ell)) \leq \Theta(\mathsf{C}_k) + \frac{\ell}{2}.$$

Proof of the second case (cont.)

- On the other hand, if $k \geq 5$ is odd and $\ell \geq 2$ is even, since $\mathsf{C}_k + \mathsf{P}_{\ell-1}$ is an induced subgraph of $\mathrm{T}(k, \ell)$, it follows that

$$\begin{aligned}\Theta(\mathrm{T}(k,\ell)) &\geq \Theta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) \\ &\geq \Theta(\mathsf{C}_k) + \Theta(\mathsf{P}_{\ell-1}) \\ &= \Theta(\mathsf{C}_k) + \left\lceil \frac{\ell-1}{2} \right\rceil \\ &= \Theta(\mathsf{C}_k) + \frac{\ell}{2},\end{aligned}$$

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  so the upper & lower bounds on $\Theta(\mathrm{T}(k,\ell))$ coincide, proving equality.
- In this case, if $k \geq 7$ is odd and $\ell \geq 2$ is even,

$$\frac{k-1}{2} = \alpha(\mathsf{C}_k) \leq \Theta(\mathsf{C}_k) \leq \vartheta(\mathsf{C}_k) = \frac{k}{1+\sec\frac{\pi}{k}},$$

$$\Rightarrow \frac{k+\ell-1}{2} \leq \Theta(\mathrm{T}(k,\ell)) \leq \vartheta(\mathsf{C}_k) + \frac{\ell}{2} \leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}.$$

Example

Let $\mathsf{G}_1 = \mathrm{T}(5,6)$, then by our result,

$$\Theta(\mathrm{T}(5,6)) = 3 + \sqrt{5} = 5.23607\ldots$$

For comparison, using SageMath software gives the values

$$\sqrt{\alpha(\mathsf{G} \boxtimes \mathsf{G})} = \sqrt{26} = 5.09902\ldots,$$
$$\sqrt[3]{\alpha(\mathsf{G} \boxtimes \mathsf{G} \boxtimes \mathsf{G})} = \sqrt[3]{136} = 5.14256\ldots$$

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Let $\mathsf{G}_2 = \mathrm{T}(7,6)$, then by our result and the lower bound $\Theta(\mathsf{C}_7) \geq \sqrt[5]{367}$ (Polak & Schrijver, 2019),

$$\begin{aligned} 6.2578659\ldots = \sqrt[5]{367} + 3 &\leq \Theta(\mathsf{C}_7) + 3 \\ &= \Theta(\mathrm{T}(7,6)) \\ &\leq \frac{7}{1 + \sec\frac{\pi}{7}} + 3 = 6.3176672\ldots \end{aligned}$$

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$$\sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2} \leq \Theta(\mathrm{T}(k,\ell)) \leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2},$$

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- By using the independence number of the squares of odd-cycles, which were calculated by Hales (1973), we get

$$\Theta(\mathrm{T}(k,\ell)) = \Theta(\mathsf{C}_k) + \frac{\ell}{2} \geq \sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2}$$

- It is possible to improve the lower bound for certain odd-cycles by using better known bounds.

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- Kneser graphs achieve the Shannon capacity at $k = 1$,

$$\Theta(\mathsf{K}(n,k)) = \alpha(\mathsf{K}(n,k)) = \binom{n-1}{k-1}.$$

- The graph $\mathsf{C}_5 + \mathsf{K}_1$ doesn't achieve the Shannon capacity at any of its powers, that is, for every $k \in \mathbb{N}$,

$$\Theta(\mathsf{C}_5 + \mathsf{K}_1) > \sqrt[k]{\alpha((\mathsf{C}_5 + \mathsf{K}_1)^k)}.$$

Theorem: Guo and Watanabe 1990

Let G be a universal graph ($\alpha(\mathrm{G}) = \alpha_{\mathrm{f}}(\mathrm{G})$), and let H satisfy $\Theta(\mathrm{H}) > \alpha(\mathrm{H})$. The Shannon capacity of $\mathrm{G} + \mathrm{H}$ is not attained at any finite power.

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Example

Let G be a universal graph, and let C_k with an odd $k \geq 5$. Then, the Shannon capacity of $\mathsf{G} + \mathsf{C}_k$ is unattainable at any finite power.

Theorem

Let H be a graph with $\alpha(\mathsf{H}) < \Theta(\mathsf{H})$, $\ell \geq 2$ be an even number, and $v \in \mathsf{V}(\mathsf{H})$. Define G as the connected graph formed from $\mathsf{H} + \mathsf{P}_\ell$ by adding an edge between v and an endpoint of P_ℓ. Then, the capacity of G is unattainable by the independence number of any strong power of G.

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Proof outline

- Since $\mathsf{H} + \mathsf{P}_{\ell-1}$ is an induced subgraph of G and $\mathsf{H} + \mathsf{P}_\ell$ is a spanning subgraph of G, then

$$\Theta(\mathsf{H} + \mathsf{P}_{\ell-1}) \leq \Theta(\mathsf{G}) \leq \Theta(\mathsf{H} + \mathsf{P}_\ell).$$

  These two bounds are shown to be equal to $\Theta(\mathsf{H}) + \frac{\ell}{2}$, hence determining $\Theta(\mathsf{G})$.
- By the same argument, for every $k \in \mathbb{N}$,

$$\alpha((\mathsf{H} + \mathsf{P}_{\ell-1})^k) \leq \alpha(\mathsf{G}^k) \leq \alpha((\mathsf{H} + \mathsf{P}_\ell)^k).$$

Proof outline (cont)

- Since path graphs are universal, and $\alpha(\mathsf{P}_\ell) = \frac{\ell}{2} = \alpha(\mathsf{P}_{\ell-1})$ for every even $\ell \geq 2$, we get

$$\alpha((\mathsf{H} + \mathsf{P}_{\ell-1})^k) = \alpha((\mathsf{H} + \mathsf{P}_\ell)^k)$$
$$\Rightarrow \alpha(\mathsf{G}^k) = \alpha((\mathsf{H} + \mathsf{P}_\ell)^k).$$

- By the Theorem of Guo and Watanabe (1990), for every $k \in \mathbb{N}$,

$$\Theta(\mathsf{G}) = \Theta(\mathsf{H} + \mathsf{P}_\ell) > \sqrt[k]{\alpha((\mathsf{H} + \mathsf{P}_\ell)^k)} = \sqrt[k]{\alpha(\mathsf{G}^k)},$$

so the graph G is unattainable by any of its strong powers.

Corollary

Let $k \geq 5$ be an odd number, and let $\ell \geq 2$ be an even number. Then, the capacity of the Tadpole graph $\mathrm{T}(k, \ell)$ is unattainable by any of its strong powers.

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Proof.

Since for every C_k with $k \geq 5$ odd, $\alpha(\mathsf{C}_k) < \Theta(\mathsf{C}_k)$, then by the theorem above the claim holds.

Corollary

Let $k \geq 5$ be an odd number, and let $\ell \geq 2$ be an even number. Then, the capacity of the Tadpole graph $\mathrm{T}(k, \ell)$ is unattainable by any of its strong powers.

Proof.

Since for every C_k with $k \geq 5$ odd, $\alpha(\mathsf{C}_k) < \Theta(\mathsf{C}_k)$, then by the theorem above the claim holds.

The last corollary provides a countably infinite set of **connected** graphs whose Shannon capacities are unattainable by any of their strong powers. This is the first infinite family of connected graphs with that property. All previous constructions with that property were disconnected graphs.

The next result provides a more general family of graphs whose capacity is computed using results from extremal combinatorics.

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The Gaussian Binomial Coefficient

The number of rank k-subspaces in a rank n-vector space over $\mathbb{F}_q$ (q is a prime power) is given by the Gaussian coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.$$

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Passing to a continuous variable q and letting q tend to 1 gives

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n(n-1) \cdots (n-k+1)}{k!} = \binom{n}{k}.$$

Definition: q-Kneser Graphs

Let $V(n,q)$ be the n-dimensional vector space above the finite field $\mathbb{F}_q$, where q is a prime power. The q-Kneser graph $\mathsf{K}_q(n,k)$ is defined as follows:

- The vertices of $\mathsf{K}_q(n,k)$ are defined as the rank-k subspaces of $V(n,q)$.
- Two vertices are adjacent if their intersection contains only the zero vector.

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- Two vertices are adjacent if their intersection contains only the zero vector.

Theorem

The Shannon capacity of the q-Kneser graph, $\mathsf{K}_q(n,k)$, with $n \geq 2k$ and q a prime power, is

$$\Theta(\mathsf{K}_q(n,k)) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Proof.

- A similar construction of a maximal independent set gives:

$$\alpha(\mathsf{K}_q(n,k)) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

- In a paper from 2012, the eigenvalues of $\mathsf{K}_q(n,k)$ were calculated,

$$\lambda_j = (-1)^j q^{\binom{k}{2}+\binom{k-j+1}{2}} \begin{bmatrix} n-k-j \\ n-2k \end{bmatrix}_q, \quad j = 0, 1, \ldots, k.$$

- Substituting the eigenvalues of $\mathsf{K}_q(n,k)$ in a formula by Lovász gives:

$$\vartheta(\mathsf{K}_q(n,k)) = -\frac{n\lambda_k}{d-\lambda_k} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q,$$

  where the first equality holds since these regular graphs are edge-transitive.

□

Unique Prime Factorization for Connected Graphs

Every connected graph has a unique prime factor decomposition with respect to the strong product.

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In general, many bounds for structured graphs have been proved, for example, N. Alon proved the following,

For every simple graph G of order n,

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{n}$$

N. Alon proved this lower bound using a special construction of independent sets, the following result offers an alternative way.

Theorem

Let $G_1, G_2, \ldots, G_\ell$ be simple graphs, then

$$\Theta(G_1 \boxtimes \ldots \boxtimes G_\ell) \leq \left(\frac{\Theta(G_1 + \ldots + G_\ell)}{\ell}\right)^\ell.$$

Furthermore, if $\Theta(G_i) = \vartheta(G_i)$ for every $i = 1, \ldots, \ell$, then the inequality holds with equality if and only if

$$\Theta(G_1) = \Theta(G_2) = \ldots = \Theta(G_\ell).$$

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- Combining these inequalities gives

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{\Theta(\mathsf{G} \boxtimes \overline{\mathsf{G}})} \geq 2\sqrt{n}$$

Corollary 1

Let $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ be simple graphs, and let $m_1, \ldots, m_\ell \in \mathbb{N}$, and let $m = \sqrt[\ell]{m_1 \cdots m_\ell}$, then

$$\Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell) \leq (m\ell)^{-\ell} \Theta(m_1 \mathsf{G}_1 + \ldots + m_\ell \mathsf{G}_\ell)^\ell.$$

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Corollary 2

Let $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ be simple graphs, with some $\ell \in \mathbb{N}$, and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$ be a probability vector with $\alpha_j \in \mathbb{Q}$ for all $j \in [\ell]$. Let

$$K(\underline{\alpha}) = \{k \in \mathbb{N} : k\alpha_j \in \mathbb{N}, \forall j \in [\ell]\}, \; H(\underline{\alpha}) = -\sum_{j=1}^{\ell} \alpha_j \log \alpha_j.$$

Then, for all $k \in K(\underline{\alpha})$,

$$\Theta(\mathsf{G}_1^{\alpha_1 k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{\alpha_\ell k}) \leq \exp(-kH(\underline{\alpha}))\, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k.$$

Lossless Compression with Side Information

- Alice and Bob teach at a school with n students.
- Each time at recess, exactly $2r$ of these students form two teams of equal size to play a game.
- Bob is on duty for the first half of the recess, so he knows the students in each team. However, he goes for lunch before recess is over.
- Alice is the gym teacher, and later that day she learns which team won.
- Alice only knows the composition of the winning team. Bob knows, on the other hand, the students in both teams but he does not know which team won the game.

Question 1

What is the shortest message (minimum number of bits) that Alice should send to Bob to tell him the winning team?

Graph-Theoretic Formulation of the Problem

- Alice and Bob share a graph G that is constructed as follows:
  1. The n students at their school are labeled with the numbers $1, \ldots, n$.
  2. The vertices of the graph represent the $\binom{n}{r}$ possible teams of all the r-element subsets of $[n]$.
  3. Two vertices are adjacent if they represent disjoint teams, so the two endpoints of an edge in G represent teams that may play together.

  Thus, $\mathsf{G} = \mathsf{K}(n, r)$ is a Kneser graph with $n, r \in \mathbb{N}$ and $n \geq 2r$.
- Alice knows the vertex $v \in \mathsf{V}(\mathsf{G})$, but she does know the edge $e \in \mathsf{E}(\mathsf{G})$ whose endpoints refer to the two teams that played the game. Bob, on the other hand, knows the edge e but he does not know which of its endpoints v represents the winning team.
- What is the minimum number of bits that should send to Bob to tell him the vertex v?

Graph-Theoretic Solution of the Problem

- Alice may take the simplest approach of communicating to Bob the label (number) of her vertex v. It requires $\lceil \log_2 |\mathsf{V}(\mathsf{G})| \rceil$ bits. This does not take, however, the side information that is available to Bob where he knows the edge e in G.
- Claim: The minimum required number of bits to let Bob determine the vertex v is $\lceil \log_2 \chi(\mathsf{G}) \rceil$.

Example

Let $n = 250$ and $r = 100$.

- The simplest approach requires $\lceil \log_2 \binom{n}{r} \rceil = 239$ bits.
- The minimum number of required bits is $\lceil \log_2 (n - 2r + 2) \rceil = 6$ bits.

Proof.

- Preprocessing: Alice and Bob can precolor the vertices of the Kneser graph $\mathsf{G} = \mathsf{K}(n, r)$ using $\chi(\mathsf{G}) = n - 2r + 2$ colors.
- When Alice wants to inform Bob about the vertex v that she knows, she only needs to tell him the color of that vertex. This is sufficient for Bob to recover the vertex v since the edge e is known to Bob, and only one of its two endpoints can be assigned that given color.
- If fewer than $\lceil \log_2 \chi(\mathsf{G}) \rceil$ bits are communicated from Alice to Bob, then there is a pair of adjacent vertices having the same color.

$\square$

On Graph Products and Lossless Compression

Suppose that Alice knows a list of vertices $v_1, \ldots, v_k$ for some $k \in \mathbb{N}$, and Bob knows a list of edges $e_1, \ldots, e_k$ with $v_i \in e_i$ for all $i \in [k]$.

- If Alice wants to communicate her entire list of vertices to Bob, she can do so in k separate messages, for a total of $k\lceil \log_2 \chi(\mathsf{G}) \rceil$ bits.
- Question: Is it possible for Alice to communicate her entire list of vertices to Bob with a smaller number of bits by a use of a single message (instead of k separate messages for each vertex)?

On Graph Products and Lossless Compression

- The endpoints of the list $(e_1, \ldots, e_k)$ of edges that is held by Bob correspond to 2^k possible vertices that Alice might hold.
- If $(v_1, \ldots, v_k)$ and $(v'_1, \ldots, v'_k)$ are two such different vertices, then $\{v_i, v'_i\} = e_i \in \mathsf{E}(\mathsf{G})$ or $v_i = v'_i$ for every $i \in [k]$.

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- These 2^k vertices form a clique in the k-fold strong power G^k.
- Hence, the required minimal number of bits is equal to $\lceil \log_2 \chi(\mathsf{G}^k) \rceil$.

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- These 2^k vertices form a clique in the k-fold strong power G^k.
- Hence, the required minimal number of bits is equal to $\lceil \log_2 \chi(\mathsf{G}^k) \rceil$.
- It is upper bounded by the chromatic number of disjunctive power G^{*k} since the strong power of G is a spanning subgraph of the disjunctive power of G.
- The transition to the chromatic number of the disjunctive product is due to its tensorization properties w.r.t the fractional chromatic number and independence number, in contrast to strong powers.

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- It is upper bounded by the chromatic number of disjunctive power G^{*k} since the strong power of G is a spanning subgraph of the disjunctive power of G.
- The transition to the chromatic number of the disjunctive product is due to its tensorization properties w.r.t the fractional chromatic number and independence number, in contrast to strong powers.
- By precoloring the k-fold disjunctive power of G and sharing it by Alice and Bob, it suffices for Alice to send the color of her vertex (list) to Bob. This will enable him to determine the vertex that Alice holds.

Theorem

Let G and H be simple graphs, then

$$\begin{aligned}
\alpha(\mathsf{G} * \mathsf{H}) &= \alpha(\mathsf{G})\, \alpha(\mathsf{H}), \\
\chi_{\mathrm{f}}(\mathsf{G} * \mathsf{H}) &= \chi_{\mathrm{f}}(\mathsf{G})\ \chi_{\mathrm{f}}(\mathsf{H}), \\
\chi_{\mathrm{f}}(\mathsf{G})\ \chi(\mathsf{H}) &\leq \chi(\mathsf{G} * \mathsf{H}) \leq \chi(\mathsf{G})\ \chi(\mathsf{H}).
\end{aligned}$$

Furthermore,

$$\inf_{k \in \mathbb{N}} \sqrt[k]{\chi(\mathsf{G}^{*k})} = \lim_{k \to \infty} \sqrt[k]{\chi(\mathsf{G}^{*k})} = \chi_{\mathrm{f}}(\mathsf{G}).$$

Theorem

For every graph G,

$$\chi_{\mathrm{f}}(\mathsf{G})^{k-1}\, \chi(\mathsf{G}) \leq \chi(\mathsf{G}^{*k}) \leq \chi_{\mathrm{f}}(\mathsf{G})^{k} \big(1 + k\, \ln \alpha(\mathsf{G})\big),\ \forall k \in \mathbb{N}.$$

On Disjunctive Graph Products and Lossless Compression

- This reduces the number of required bits in a single message to Bob once every k days from $k\lceil \log_2 \chi(\mathsf{G}) \rceil$ to $\lceil \log_2 \chi(\mathsf{G}^{*k}) \rceil$.
- By letting k tend to infinity, the number of required bits per game is reduced from $\lceil \log_2 \chi(\mathsf{G}) \rceil$ to

$$\lim_{k \to \infty} \frac{\lceil \log_2 \chi(\mathsf{G}^{*k}) \rceil}{k} = \log_2 \chi_{\mathrm{f}}(\mathsf{G}).$$

- For $\mathsf{G} = \mathsf{K}(n, r)$, the amount of transmitted information per game is reduced from $\lceil \log_2 (n - 2r + 2) \rceil$ to $\log_2 \frac{n}{r}$ bits.
- For Kneser graphs and $k \to \infty$ the two asymptotic results for $\frac{1}{k} \lceil \log_2 \chi(\mathsf{G}^k) \rceil$ and $\frac{1}{k} \lceil \log_2 \chi(\mathsf{G}^{*k}) \rceil$ coincide, being equal to $\log_2 \frac{n}{r}$.

Example

For $n = 250$ and $r = 100$, the amount of information per game is reduced from 6 to 1.322 bits for $k = 1$ and $k \to \infty$, respectively.

On Disjunctive Graph Products and Lossless Compression

- Practical Limitation: The computational complexity involved in the preprocessing stage where the k-fold disjunctive power G^{*k} needs to be colored, and the information on that particular graph coloring needs to be stored jointly by Alice and Bob, makes it impractical for large k (let alone for $k \to \infty$).
- We next provide a refined analysis, which is valid for all $k \in \mathbb{N}$, enabling to study the reduction in the required number of bits as a function of k, especially for (small) values of k for which the computational complexity at the preprocessing stage of precoloring G^{*k} is still feasible.
- The next analysis suggests a tradeoff between the increase in the computational complexity at the preprocessing stage, and the gain in the reduction of the minimum amount of information per game that Alice needs to deliver in her message to Bob.

Theorem

Let

$$R_k(\mathsf{G}) \triangleq \frac{\lceil \log_2 \chi(\mathsf{G}^{*k}) \rceil}{k}, \quad k \in \mathbb{N}$$

be the number of information bits per game for a list of k games. Then,

$$\begin{aligned} & \log_2 \chi_{\mathrm{f}}(\mathsf{G}) + \frac{1}{k} \log_2 \left(\frac{\chi(\mathsf{G})}{\chi_{\mathrm{f}}(\mathsf{G})} \right) \\ & \leq R_k(\mathsf{G}) \\ & \leq \log_2 \chi_{\mathrm{f}}(\mathsf{G}) + \frac{\log_2(1 + k \ln \alpha(\mathsf{G})) + 1}{k}, \quad \forall k \in \mathbb{N}. \end{aligned}$$

If $\chi(\mathsf{G}) > \chi_{\mathrm{f}}(\mathsf{G})$, then

$$\log_2 \chi_{\mathrm{f}}(\mathsf{G}) + O\left(\frac{1}{k}\right) \leq R_k(\mathsf{G}) \leq \log_2 \chi_{\mathrm{f}}(\mathsf{G}) + O\left(\frac{\log k}{k}\right).$$

Finite-Length Analysis Specialized to Kneser Graphs

The following is a specialized analysis for Kneser graphs. Let $\mathsf{G} = \mathsf{K}(n, r)$ be a Kneser graph with $n \geq 2r$. Then

$$\begin{aligned}
&\log_2 \binom{n}{r} + \frac{\log_2 \left(r\left(1 - \frac{2(r-1)}{n}\right)\right)}{k} \\
&\leq R_k(\mathsf{G}) \\
&\leq \log_2 \binom{n}{r} + \frac{1}{k} \log_2 \left(knH_b\left(\frac{r}{n}\right) + \frac{k}{2} \ln\left(\frac{r}{2\pi n(n-r)}\right) + 1\right) + \frac{1}{k},
\end{aligned}$$

for all $k \in \mathbb{N}$, where $H_b \colon [0,1] \to [0, \ln 2]$ is the binary entropy function on base e, i.e.,

$$H_b(x) = \begin{cases} -x \ln x - (1-x)\ln(1-x), & 0 < x < 1, \\ 0, & x \in \{0, 1\}. \end{cases}$$

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- We provided two families of graphs whose Shannon capacities are not attained by a finite strong power, which includes a countably infinite family of connected such graphs.
- We proved a general inequality between the Shannon capacity of graphs and exemplified some of its consequences.
- We provide finite-length analysis of a problem on lossless compression with side information. This problem was formulated as a problem in graph theory.

Outlook

- Does there exist a graph whose Shannon capacity is noninteger rational number? If there are such graphs, it will provide additional families of graphs whose capacities are not attainable by the independence number of their finite strong powers.

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- It is known that if the capacity of graph is attained at a finite power, then the capacity of its Mycielskian is strictly larger than that of the original graph. In view of our earlier constructions of graphs whose capacity is not attained by the independence number of any finite power, it would be interesting to determine whether this property also holds for such graphs.
- It is known that a graph G is universal if and only if $\alpha(\mathrm{G}) = \alpha_{\mathrm{f}}(\mathrm{G})$. Additionally, we showed that if $\Theta(\mathrm{G}) = \alpha_{\mathrm{f}}(\mathrm{G})$ then for every graph H, $\Theta(\mathrm{G} \boxtimes \mathrm{H}) = \Theta(\mathrm{G})\Theta(\mathrm{H})$. Is the converse true as well?