Advances in the Shannon Capacity of Graphs

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Advances in the Shannon Capacity of Graphs

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The author of this thesis states that the research, including the collection, processing and presentation of data, addressing and comparing to previous research, etc., was done entirely in an honest way, as expected from scientific research that is conducted according to the ethical standards of the academic world. Also, reporting the research and its results in this thesis was done in an honest and complete manner, according to the same standards.

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Abstract

The Shannon capacity of graphs, introduced by Claude Shannon in 1956, equals the maximum transmission rate at which a receiver can accurately recover information without error, with the communication channel represented as a graph. In this graph, the vertices represent the input symbols and any two vertices are adjacent if and only if the corresponding input symbols can be confused by the channel with some positive probability. This concept establishes a significant link among zero-error problems in information theory and graph theory. The calculation of the Shannon capacity of a graph is notoriously difficult, and only the capacity of a some families of graphs are known. The computational complexity of the Shannon capacity has extended the research towards finding computable bounds on the capacity and exploring its properties.

This thesis studies several research directions concerning the Shannon capacity of graphs. Building on Schrijver's recent framework, we establish sufficient conditions under which the Shannon capacity of a polynomial in graphs equals the corresponding polynomial of the individual capacities, thereby simplifying their evaluation. We derive exact values and new bounds for the Shannon capacity of two families of graphs: the q-Kneser graphs and the Tadpole graphs. Furthermore, we construct graphs whose Shannon capacity is never attained by the independence number of any finite power of these graphs, including a countably infinite family of connected graphs with this property. We further prove an inequality relating the Shannon capacities of the strong product of graphs and their disjoint union, leading to streamlined proofs of known bounds. Motivated by problems in lossless data compression, we introduce an approach that incorporates computational complexity through bounds on the chromatic number of small disjunctive powers. These results on the Shannon capacity of graphs underscore fundamental connections between graph theory and information theory.

List of Symbols

DMC	Discrete memoryless channel
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
G	A simple graph
$\overline{\mathsf{G}}$	The complement graph of G
$\mathsf{V}(\mathsf{G})$	The vertex set of G
$\mathsf{E}(\mathsf{G})$	The edge set of G
K_k	The complete graph on k vertices
P_ℓ	The path graph on ℓ vertices
$\mathsf{K}(n,r)$	The Kneser graph
$\mathsf{K}_q(n,r)$	The q-Kneser graph
$\mathrm{T}(k,\ell)$	The Tadpole graph
$\mathsf{srg}(n,d,\lambda,\mu)$	The strongly regular graph
$P(q)$	The Paley graph on q vertices
$\binom{n}{r}$	The binomial coefficient
$\begin{bmatrix} n \\ r \end{bmatrix}_q$	The Gaussian binomial coefficient
$\mathbb{F}_q$	Galois field of order q
$V(n,q)$	The n-dimensional vector space over the finite field $\mathbb{F}_q$
$\mathsf{G} \boxtimes \mathsf{H}$	The strong product graph of G and H
$\mathsf{G} * \mathsf{H}$	The disjunctive product graph of G and H
$\mathsf{G} + \mathsf{H}$	The disjoint union graph of G and H
$\alpha(\mathsf{G})$	The independence number of G
$\omega(\mathsf{G})$	The clique number of G
$\chi(\mathsf{G})$	The chromatic number of G
$\sigma(\mathsf{G})$	The clique-cover number of G
$\alpha_{\mathrm{f}}(\mathsf{G})$	The fractional independence number of G
$\omega_{\mathrm{f}}(\mathsf{G})$	The fractional clique number of G
$\chi_{\mathrm{f}}(\mathsf{G})$	The fractional chromatic number of G
$\sigma_{\mathrm{f}}(\mathsf{G})$	The fractional clique-cover number of G
$\Theta(\mathsf{G})$	The Shannon capacity of G
$\vartheta(\mathsf{G})$	The Lovász ϑ-function of G
$\vartheta'(\mathsf{G})$	The Schrijver ϑ-function of G

Chapter 1

Introduction

The Shannon capacity of graphs, introduced by Claude Shannon in [1], equals the maximum transmission rate at which a receiver can accurately recover information without error, with the communication channel represented as a graph. In this graph, the vertices represent the input symbols and any two vertices are adjacent if and only if the corresponding input symbols can be confused by the channel with some positive probability. This concept establishes a significant link among zero-error problems in information theory and graph theory, prominently featured in multiple surveys [2–6].

The calculation of the Shannon capacity of a graph is notoriously difficult (see, e.g., [7–10]), and only the capacity of a few families of graphs are known, like Kneser graphs and self-complementary vertex-transitive graphs [11]. The computational complexity of the Shannon capacity has extended the research towards finding computable bounds on the capacity and exploring its properties [11–20].

While investigating the properties of the Shannon capacity, it is worth keeping in mind the more general concept of the asymptotic spectrum of graphs, introduced by Zuiddam [19–22], which delineates a space of graph parameters that remain invariant under graph isomorphisms. This space is characterized by the following unique properties: additivity under disjoint union of graphs, multiplicativity under strong product of graphs, normalization for a simple graph with a single vertex, and monotonicity under graph complement homomorphisms. Building upon Strassen's theory of the asymptotic spectra [23], a novel dual characterization of the Shannon capacity of a graph is derived in [21], expressed as the minimum over the elements of its asymptotic spectrum. By confirming that various graph invariants, including the Lovász ϑ-function [11] and the fractional Haemers bound [13], are elements of the asymptotic spectrum of a graph (spectral points), it can be deduced that these elements indeed serve as upper bounds on the Shannon capacity of a graph. For further exploration, the comprehensive paper by Wigderson and Zuiddam [22] provides a survey on Strassen's theory of the asymptotic spectra and its application areas, including the Shannon capacity of graphs.

Several properties of the Shannon capacity have been explored, and many questions are left open. One of the most basic open questions is the determination of the Shannon capacity of cycle graphs of odd length larger than 5.

In addition to zero-order error capacity of a graph, using Graph Theory allows us to investigate many other problems in Information and Communication Theory, for example, in [32, Section 3.4], Graph Theory is used to explore a problem in Lossless data compression, using known results regarding the chromatic number and the disjunctive power of graphs.

This paper studies several research directions regarding the Shannon capacity of graphs, and it is structured as follows.

- Chapter 2 provides preliminaries that are required for the analysis in this paper. Its focus is on graph invariants, and classes of graphs that are used throughout this paper.

- Chapter 3 builds on a recent paper by Schrijver [24], and it explores conditions under which, for a family of graphs, the Shannon capacity of any polynomial in these graphs equals the corresponding polynomial of their individual Shannon capacities. This equivalence can substantially simplify the computation of the Shannon capacity for some of structured graphs. Two sufficient conditions are presented, followed by a comparison of their differences and illustrative examples of their use.

- Chapter 4 explores Tadpole graphs. Exact values and bounds on the capacity of Tadpole graphs are derived, and a direct relation between the capacity of odd-cycles and the capacity of a countably infinite subfamily of the Tadpole graphs is proved, providing an important property of that subfamily that is further discussed in the following section.

- Chapter 5 determines sufficient conditions for the unattainability of the Shannon capacity by the independence number of any finite strong power of a graph. It first presents in an alternative streamlined way an approach by Guo and Watanabe [10]. It then introduces two other original approaches. One of the novelties in this section is the construction of an infinite family of *connected* graphs whose capacity is unattainable by any finite strong power of these graphs.

- Chapter 6 derives the Shannon capacity of the q-Kneser graphs, using a generalized result of Erdos-Ko-Rado [25], and a result on the spectrum of the q-Kneser graphs [26]. This broadens the class of graphs for which the Shannon capacity is explicitly known.

- Chapter 7 introduces a new inequality relating the Shannon capacity of the strong power of graphs to that of their disjoint union, and it identifies several conditions under which equality holds. As an illustration of the applicability of this inequality, this section also presents an alternative shorter proof of a lower bound for the disjoint union of a graph with its complement, originally obtained by N. Alon [7], along with analogous bounds on the Lovász ϑ-function of graphs.

- Chapter 8 explores a problem presented in [32, Section 3.4] regarding lossless data compression using the chromatic number of the disjunctive powers of a graph. This section presents the problem and then offers a new approach to this problem, that takes into account the computation complexity, and uses finite (small) powers of the disjunctive power, to provide a tradeoff between the computational complexity and the Data Compression.

- Chapter 9 provides a summary of this work and also suggests some directions for further research that are related to the findings in this thesis.

Chapter 2

Preliminaries

2.1 Basic definitions and graph families

2.1.1 Terminology

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph.

- $\mathsf{V} = \mathsf{V}(\mathsf{G})$ is the *vertex set* of G, and $\mathsf{E} = \mathsf{E}(\mathsf{G})$ is the *edge set* of G.
- An *undirected graph* is a graph whose edges are undirected.
- A *self-loop* is an edge that connects a vertex to itself.
- A *simple graph* is a graph having no self-loops and no multiple edges between any pair of vertices.
- A *finite graph* is a graph with a finite number of vertices.
- The *order* of a finite graph is the number of its vertices, $|\mathsf{V}(\mathsf{G})| = n$.
- The *size* of a finite graph is the number of its edges, $|\mathsf{E}(\mathsf{G})| = m$.
- Vertices $i, j \in \mathsf{V}(\mathsf{G})$ are *adjacent* if they are the endpoints of an edge in G; it is denoted by $\{i, j\} \in \mathsf{E}(\mathsf{G})$ or $i \sim j$.
- An *empty graph* is a graph without edges, so its size is equal to zero.
- The *degree of a vertex v* in G is the number of adjacent vertices to v in G, denoted by $\mathrm{d}_v = \mathrm{d}_v(\mathsf{G})$.

- A graph is *regular* if all its vertices have an identical degree.
- A *d-regular* graph is a regular graph whose all vertices have a fixed degree d.
- A *walk* in a graph G is a sequence of vertices in G, where every two consecutive vertices in the sequence are adjacent in G.
- A *trail* in a graph is a walk with no repeated edges.
- A *path* in a graph is a walk with no repeated vertices; consequently, a path has no repeated edges, so every path is a trail but a trail is not necessarily a path.
- A *cycle* C in a graph G is obtained by adding an edge to a path P such that it gives a closed walk.
- The *length of a path or a cycle* is equal to its number of edges. A *triangle* is a cycle of length 3.
- A *connected graph* is a graph where every two distinct vertices are connected by a path.
- An *r-partite graph* is a graph whose vertex set is a disjoint union of r subsets such that no two vertices in the same subset are adjacent. If $r = 2$, then G is a *bipartite graph*.
- A *complete graph* on n vertices, denoted by K_n, is a graph whose all n distinct vertices are pairwise adjacent. Hence, K_n is an $(n-1)$-regular graph of order n.
- A *path graph* on n vertices is denoted by P_n, and its size is equal to $n - 1$.
- A *cycle graph* on n vertices is called an n-cycle, and it is denoted by C_n with an integer $n \geq 3$. The order and size of C_n are equal to n, and C_n is a bipartite graph if and only if $n \geq 4$ is even.
- A *complete r-partite graph*, denoted by $\mathsf{K}_{n_1,\ldots,n_r}$ with $n_1, \ldots n_r \in \mathbb{N}$, is an r-partite graph whose vertex set is partitioned into r disjoint subsets of cardinalities $n_1, \ldots, n_r$, such that every two vertices in the same subset are not adjacent, and every two vertices in distinct subsets are adjacent.

Throughout this thesis, the graphs under consideration are finite, simple, and undirected. The standard notation $[n] \triangleq \{1, \dots, n\}$, for every $n \in \mathbb{N}$, is also used.

Definition 2.1 (Subgraphs and graph connectivity)**.** *A graph* F *is a* subgraph *of a graph* G*, and it is denoted by* $\mathsf{F} \subseteq \mathsf{G}$*, if* $\mathsf{V}(\mathsf{F}) \subseteq \mathsf{V}(\mathsf{G})$ *and* $\mathsf{E}(\mathsf{F}) \subseteq \mathsf{E}(\mathsf{G})$*.*

- *A* spanning subgraph *of* G *is obtained by edge deletions from* G*, while its vertex set is left unchanged. A* spanning tree *in* G *is a spanning subgraph of* G *that forms a tree.*

- *An* induced subgraph *is obtained by removing vertices from the original graph, followed by the deletion of their incident edges.*

Definition 2.2 (Isomorphic graphs)**.** *Graphs* G *and* H *are* isomorphic *if there exists a bijection* $f \colon \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{H})$ *(i.e., a one-to-one and onto mapping) such that* $\{i, j\} \in \mathsf{E}(\mathsf{G})$ *if and only if* $\{f(i), f(j)\} \in \mathsf{E}(\mathsf{H})$*. It is denoted by* $\mathsf{G} \cong \mathsf{H}$*, and* f *is said to be an* isomorphism *from* G *to* H*.*

Definition 2.3 (Complement and self-complementary graphs)**.** *The* complement *of a graph* G*, denoted by* $\overline{\mathsf{G}}$*, is a graph whose vertex set is* $\mathsf{V}(\mathsf{G})$*, and its edge set is the complement set* $\overline{\mathsf{E}(\mathsf{G})}$*. Every vertex in* $\mathsf{V}(\mathsf{G})$ *is nonadjacent to itself in* G *and* $\overline{\mathsf{G}}$*, so* $\{i, j\} \in \mathsf{E}(\overline{\mathsf{G}})$ *if and only if* $\{i, j\} \notin \mathsf{E}(\mathsf{G})$ *with* $i \neq j$*. A graph* G *is* self-complementary *if* $\mathsf{G} \cong \overline{\mathsf{G}}$ *(i.e.,* G *is isomorphic to* $\overline{\mathsf{G}}$*).*

Example 2.1. *It can be verified that* P_4 *and* C_5 *are self-complementary graphs.*

2.1.2 Graph operations

This subsection presents the basic graph operations used throughout this thesis.

Definition 2.4 (Strong product of graphs)**.** *Let* G *and* H *be simple graphs. The strong product* $\mathsf{G} \boxtimes \mathsf{H}$ *is a graph whose vertices set is* $\mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$*, and two distinct vertices* $(g_1, h_1), (g_2, h_2)$ *are adjacent if one of the following three conditions is satisfied:*

1. $g_1 = g_2$ *and* $\{h_1, h_2\} \in \mathsf{E}(\mathsf{H})$*,*
2. $\{g_1, g_2\} \in \mathsf{E}(\mathsf{G})$ *and* $h_1 = h_2$*,*

3. $\{g_1, g_2\} \in \mathsf{E}(\mathsf{G})$ *and* $\{h_1, h_2\} \in \mathsf{E}(\mathsf{H})$.

The interested reader is referred to [27] for an extensive textbook on graph products and their properties.

Define the k-fold strong power of G as

$$\mathsf{G}^k \triangleq \underbrace{\mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}}_{k-1 \text{ strong products}}. \tag{2.1}$$

Throughout this thesis, we will use another graph operation.

Definition 2.5 (Disjoint union of graphs)**.** *Let* G *and* H *be two simple graphs. The disjoint union* $\mathsf{G} + \mathsf{H}$ *is a graph whose vertices set is* $\mathsf{V}(\mathsf{G}) \cup \mathsf{V}(\mathsf{H})$*, and the edges set is* $\mathsf{E}(\mathsf{G}) \cup \mathsf{E}(\mathsf{H})$.

Let $m \in \mathbb{N}$, throughout this thesis we use the following notation:

$$m\mathsf{G} \triangleq \underbrace{\mathsf{G} + \ldots + \mathsf{G}}_{m-1 \text{ disjoint unions}}. \tag{2.2}$$

2.1.3 Basic graph invariants under isomorphism

Definition 2.6 (Independent sets)**.** *Let* G *be a simple graph. Define*

- *A set* $\mathcal{I} \subseteq \mathsf{V}(\mathsf{G})$ *is an independent set in* G *if every pair of vertices in* $\mathcal{I}$ *are nonadjacent in* G.
- *The set* $\mathcal{I}(\mathsf{G})$ *is the set of independent sets in* G.

Now we define the independence number of a graph.

Definition 2.7 (Independence number)**.** *The independence number of a graph* G*, denoted* $\alpha(\mathsf{G})$ *is the order of a largest independent set in* G*, i.e.*

$$\alpha(\mathsf{G}) \triangleq \max\{|\,\mathcal{I}\,| : \mathcal{I} \in \mathcal{I}(\mathsf{G})\}. \tag{2.3}$$

Definition 2.8 (Cliques)**.** *Let* G *be a simple graph. Define*

- *A set* $\mathcal{C} \subseteq \mathsf{V}(\mathsf{G})$ *is a clique in* G *if every pair of vertices in* $\mathcal{C}$ *are adjacent in* G.
- *The set* $\mathcal{C}(\mathsf{G})$ *is the set of cliques in* G.

Definition 2.9 (Clique number)**.** *The clique number of a graph* G*, denoted* $\omega(\mathsf{G})$ *is the order of a largest clique in* G*, i.e.*

$$\omega(\mathsf{G}) \triangleq \max\{|\mathcal{C}| : \mathcal{C} \in \mathcal{C}(\mathsf{G})\}. \tag{2.4}$$

Definition 2.10 (Chromatic number)**.** *The chromatic number of* G*, denoted* $\chi(\mathsf{G})$*, is the smallest cardinality of an independent set partition of* G*.*

Definition 2.11 (Clique-cover number)**.** *The clique-cover number of* G*, denoted* $\sigma(\mathsf{G})$*, is the smallest number of cliques needed to cover all the vertices of* G*. Hence,* $\sigma(\mathsf{G}) = \chi(\overline{\mathsf{G}})$*.*

We next provide required properties of the independence number of a graph.

Theorem 2.1. *Let* G *and* H *be simple graphs. Then,*

$$\alpha(\mathsf{G} \boxtimes \mathsf{H}) \geq \alpha(\mathsf{G})\,\alpha(\mathsf{H}), \tag{2.5}$$

$$\alpha(\mathsf{G} + \mathsf{H}) = \alpha(\mathsf{G}) + \alpha(\mathsf{H}). \tag{2.6}$$

Theorem 2.2. *Let* G *be a simple graph, and let* H_1 *and* H_2 *be induced and spanning subgraphs of* G*, respectively. Then*

$$\alpha(\mathsf{H}_1) \leq \alpha(\mathsf{G}) \leq \alpha(\mathsf{H}_2). \tag{2.7}$$

Furthermore, for every $k \in \mathbb{N}$*,*

$$\alpha(\mathsf{H}_1^k) \leq \alpha(\mathsf{G}^k) \leq \alpha(\mathsf{H}_2^k). \tag{2.8}$$

2.1.4 Fractional invariants of graphs

To properly define the four basic fractional invariants of a graph, we need to define the following four sets of functions.

Definition 2.12. *Let* G *be a simple graph. Define the four following sets of functions:*

- *Define* $\mathcal{F}_I(\mathsf{G})$ *as the set of non-negative functions,* $f \colon \mathsf{V}(\mathsf{G}) \to \mathbb{R}$*, such that for every* $\mathcal{I} \in \mathcal{I}(\mathsf{G})$*,*

$$\sum_{v \in \mathcal{I}} f(v) \leq 1. \tag{2.9}$$

- *Define* $\mathcal{F}_C(\mathsf{G})$ *as the set of non-negative functions,* $f \colon \mathsf{V}(\mathsf{G}) \to \mathbb{R}$*, such that for every* $\mathcal{C} \in \mathcal{C}(\mathsf{G})$*,*

$$\sum_{v \in \mathcal{C}} f(v) \leq 1. \tag{2.10}$$

- *Define* $\mathcal{G}_I(\mathsf{G})$ *as the set of non-negative functions,* $g \colon \mathcal{I}(\mathsf{G}) \to \mathbb{R}$*, such that for every* $v \in \mathsf{V}(\mathsf{G})$*,*

$$\sum_{\mathcal{I} \in \mathcal{I}(\mathsf{G}) : v \in \mathcal{I}} g(\mathcal{I}) \geq 1. \tag{2.11}$$

- *Define* $\mathcal{G}_C(\mathsf{G})$ *as the set of non-negative functions,* $g \colon \mathcal{C}(\mathsf{G}) \to \mathbb{R}$*, such that for every* $v \in \mathsf{V}(\mathsf{G})$*,*

$$\sum_{\mathcal{C} \in \mathcal{C}(\mathsf{G}) : v \in \mathcal{C}} g(\mathcal{C}) \geq 1. \tag{2.12}$$

Four basic fractional invariants are next defined by linear programming.

Definition 2.13 (Fractional invariants of graphs)**.** *For a simple graph* G*,*

- *The fractional independence number of* G *is*

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \sup \left\{ \sum_{v \in \mathsf{V}(\mathsf{G})} f(v) : f \in \mathcal{F}_C(\mathsf{G}) \right\}. \tag{2.13}$$

- *The fractional clique-cover number of* G *is*

$$\sigma_{\mathrm{f}}(\mathsf{G}) = \inf \left\{ \sum_{\mathcal{C} \in \mathcal{C}(\mathsf{G})} g(\mathcal{C}) : g \in \mathcal{G}_C(\mathsf{G}) \right\}. \tag{2.14}$$

- *The fractional clique number of* G *is*

$$\omega_{\mathrm{f}}(\mathsf{G}) = \sup \left\{ \sum_{v \in \mathsf{V}(\mathsf{G})} f(v) : f \in \mathcal{F}_I(\mathsf{G}) \right\}. \tag{2.15}$$

- *The fractional chromatic number of* G *is*

$$\chi_{\mathrm{f}}(\mathsf{G}) = \inf \left\{ \sum_{\mathcal{I} \in \mathcal{I}(\mathsf{G})} g(\mathcal{I}) : g \in \mathcal{G}_I(\mathsf{G}) \right\}. \tag{2.16}$$

Using strong duality of linear programming we get the following important theorem.

Theorem 2.3. *Let* G *be a simple graph. Then*

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \sigma_{\mathrm{f}}(\mathsf{G}), \tag{2.17}$$

$$\chi_{\mathrm{f}}(\mathsf{G}) = \omega_{\mathrm{f}}(\mathsf{G}). \tag{2.18}$$

So, from now on we will use mostly the fractional independence number and the fractional chromatic number. And, when we calculate them we will use both of their equivalent linear programming representations.

Another useful property, that is obvious from Definition 2.13, is provided next.

Theorem 2.4. *For a simple graph* G*, the following holds:*

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \sigma_{\mathrm{f}}(\mathsf{G}) = \chi_{\mathrm{f}}(\overline{\mathsf{G}}) = \omega_{\mathrm{f}}(\overline{\mathsf{G}}). \tag{2.19}$$

Finally, some properties of the fractional independence number are presented.

Theorem 2.5. *[28] Let* G *and* H *be simple graphs. Then*

$$\alpha_{\mathrm{f}}(\mathsf{G} \boxtimes \mathsf{H}) = \alpha_{\mathrm{f}}(\mathsf{G})\, \alpha_{\mathrm{f}}(\mathsf{H}), \tag{2.20}$$

$$\alpha(\mathsf{G} \boxtimes \mathsf{H}) \leq \alpha_{\mathrm{f}}(\mathsf{G})\, \alpha(\mathsf{H}). \tag{2.21}$$

2.1.5 Graph spectrum

Definition 2.14 (Adjacency matrix)**.** *Let* G *be a simple undirected graph on* n *vertices. The* adjacency matrix *of* G*, denoted by* $\mathbf{A} = \mathbf{A}(\mathsf{G})$*, is an* $n \times n$ *symmetric matrix* $\mathbf{A} = (\mathrm{A}_{i,j})$ *where* $\mathrm{A}_{i,j} = 1$ *if* $\{i, j\} \in \mathsf{E}(\mathsf{G})$*, and* $\mathrm{A}_{i,j} = 0$ *otherwise (so, the entries in the principal diagonal of* $\mathbf{A}$ *are zeros).*

Definition 2.15 (Graph spectrum)**.** *Let* G *be a simple undirected graph on* n *vertices. The* spectrum *of* G *is defined as the multiset of eigenvalues of the adjacency matrix of* G*.*

2.1.6 Some structured families of graphs

Vertex- and edge-transitivity, defined as follows, play an important role in characterizing graphs.

Definition 2.16 (Automorphism). *An* automorphism *of a graph* G *is an isomorphism from* G *to itself.*

Definition 2.17 (Vertex-transitivity). *A graph* G *is said to be* vertex-transitive *if, for every two vertices* $i, j \in \mathsf{V}(\mathsf{G})$*, there is an automorphism* $f\colon \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{G})$ *such that* $f(i) = j$*.*

Definition 2.18 (Edge-transitivity). *A graph* G *is* edge-transitive *if, for every two edges* $e_1, e_2 \in \mathsf{E}(\mathsf{G})$*, there is an automorphism* $f\colon \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{G})$ *that maps the endpoints of* e_1 *to the endpoints of* e_2*.*

Definition 2.19 (Kneser graphs). *Let* $[n]$ *be the set with natural numbers from* 1 *to* n*, and let* $1 \leq r \leq n$*. The Kneser graph* $\mathsf{K}(n, r)$ *is the graph whose vertex set is composed of the different* r*-subsets of* $[n]$*, and every two vertices* u, v *are adjacent if and only if the respective* r*-subsets are disjoint.*

Kneser graphs are vertex- and edge-transitive.

Definition 2.20 (Perfect graphs). *A graph* G *is perfect if for every induced subgraph* H *of* G*,*

$$\omega(\mathsf{H}) = \chi(\mathsf{H}). \tag{2.22}$$

Definition 2.21 (Universal graphs). *A graph* G *is universal if for every graph* H*,*

$$\alpha(\mathsf{G} \boxtimes \mathsf{H}) = \alpha(\mathsf{G})\, \alpha(\mathsf{H}). \tag{2.23}$$

Lemma 2.1. *If* G *is a universal graph, then* G^k *is universal for all* $k \in \mathbb{N}$*.*

Proof. This follows easily by Definition 2.21 and mathematical induction on k. □

A corollary by Hales from 1973 regarding the connection between graph universality and the fractional independence number is presented next.

Theorem 2.6. *[28] A graph* G *is universal if and only if* $\alpha(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$*.*

Definition 2.22 (Strongly regular graphs)**.** *A graph* G *is strongly regular with parameters* $\mathsf{srg}(n, d, \lambda, \mu)$ *if it suffices:*

- *The order of* G *is* n*.*
- G *is* d*-regular.*
- *Every pair of adjacent vertices has exactly* λ *common neighbors.*
- *Every pair of distinct, nonadjacent vertices has exactly* μ *common neighbors.*

Definition 2.23 (Paley graphs)**.** *Let* q *be a prime power, i.e.* $q = p^n$ *where* p *is prime and* $n \in \mathbb{N}$*, with* $q \equiv 1 \bmod 4$*. The Paley graph of order* q*, denoted* $P(q)$*, is defined as*

- *The vertex set of* $P(q)$ *is* $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$*.*
- *Two vertices* a *and* b *are adjacent if and only if* $a - b \in (\mathbb{F}_q^{\times})^2$*.*

Theorem 2.7. *[29] Let* $\mathsf{G} = P(q)$ *be a Paley graph with* $q \equiv 1 \bmod 4$ *a prime power. Then*

- $P(q)$ *is a self-complementary graph.*
- $P(q)$ *is strongly regular with parameters* $\mathsf{srg}(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$*.*
- $P(q)$ *is vertex-transitive.*
- $P(q)$ *is edge-transitive.*

2.2 The Shannon capacity of graphs

The concept of the Shannon capacity of a graph G was introduced by Claude E. Shannon in [1] to consider the largest information rate that can be achieved with zero-error communication. A discrete channel consists of

- A finite input set $\mathcal{X}$.
- A (possibly infinite) output set $\mathcal{Y}$.
- A non-empty fan-out set $\mathcal{S}_x \subseteq \mathcal{Y}$ for every $x \in \mathcal{X}$.

In each channel use, a sender transmits an input $x \in \mathcal{X}$ and a receiver receives an arbitrary output in $\mathcal{S}_x$. It is possible to represent a DMC (discrete memoryless channel) by a *confusion graph* G, which will be defined as follows

- $\mathsf{V}(\mathsf{G}) = \mathcal{X}$ represents the symbols of the input alphabet to that channel.
- $\mathsf{E}(\mathsf{G})$ is the edge set of G, where two distinct vertices in G are adjacent if the corresponding two input symbols from $\mathcal{X}$ are not distinguishable by the channel, i.e., they can produce an identical output symbol with some positive probability. Formally,

$$\mathsf{E}(\mathsf{G}) = \Big\{ \{x, x'\} : x, x' \in \mathcal{X}, x \neq x', \mathcal{S}_x \cap \mathcal{S}_{x'} \neq \varnothing \Big\}. \tag{2.24}$$

Thus, the largest number of inputs a channel can communicate without error in a single use is the independence number $\alpha(\mathsf{G})$.

In the considered setting, the sender and the receiver agree in advance on an independent set $\mathcal{I}$ of a maximum size $\alpha(\mathsf{G})$, the sender transmits only inputs in $\mathcal{I}$, every received output is in the fan-out set of exactly one input in $\mathcal{I}$, and the receiver can correctly determine the transmitted input. Next, consider a transmission of k-length strings over a channel, where the channel is used $k \geq 1$ times, the sender transmits a sequence $x_1 \ldots x_k$, and the receiver gets a sequence $y_1 \ldots y_k$ of outputs, where $y_i \in \mathcal{S}_{x_i}$ for all $i \in [k]$. In this setup, k uses of the channel are viewed as a single use of a super-channel:

- Its input set is $\mathcal{X}^k$, and its output set is $\mathcal{Y}^k$.
- The fan-out set of $(x_1, \ldots, x_k) \in \mathcal{X}^k$ is the cartesian product

$$\mathcal{S}_{x_1} \times \ldots \times \mathcal{S}_{x_k}. \tag{2.25}$$

Note that two sequences $(x_1, \ldots, x_k), (x'_1, \ldots, x'_k) \in \mathcal{X}^k$ are distinguishable by the channel if and only if there exists $i \in [k]$ such that

$$S_{x_i} \cap S_{x'_i} = \varnothing. \tag{2.26}$$

Thus, it is possible to represent the larger channel by the strong powers of the confusion graph, and the *k-th confusion graph* is defined as thr k-fold strong power of G:

$$\mathsf{G}^k \triangleq \mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}. \tag{2.27}$$

Using the k-th confusion graph, the largest amount of information we can send through the channel with k-uses is the independence number of the k-th confusion graph, $\alpha(\mathsf{G}^k)$. Thus, the maximum information rate per symbol that is achievable by using input strings of length k is equal to

$$\frac{1}{k}\log\alpha(\mathsf{G}^k) = \log\sqrt[k]{\alpha(\mathsf{G}^k)}, \quad k \in \mathbb{N}.$$

And, by omitting the logarithm (as an increasing function) and supremizing over k, the Shannon capacity is defined as follows.

Definition 2.24 (Shannon capacity)**.** *Let* G *be a simple graph. The Shannon capacity of* G *is defined as*

$$\Theta(\mathsf{G}) \triangleq \sup_{k\in\mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^k)} = \lim_{k\to\infty} \sqrt[k]{\alpha(\mathsf{G}^k)}. \tag{2.28}$$

Remark 2.1. *The righthand equality in* (2.28) *is derived from Theorem 2.1 and Fekete's Lemma.*

Now we provide several basic properties of the Shannon capacity which will be used throughout this thesis.

Theorem 2.8. *Let* G *be a simple graph and let* $\ell \in \mathbb{N}$*. Then*

$$\Theta(\mathsf{G}^\ell) = \Theta(\mathsf{G})^\ell. \tag{2.29}$$

Theorem 2.9. *Let* G *be a simple graph and let* $m \in \mathbb{N}$*, then*

$$\Theta(m\mathsf{G}) = m\Theta(\mathsf{G}). \tag{2.30}$$

Theorem 2.10. *Let* G_1 *and* G_2 *be simple graphs. Then*

$$\Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2) \geq \Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2). \tag{2.31}$$

Theorem 2.11 (Shannon's inequality)**.** *[1] Let* G_1 *and* G_2 *be simple graphs. Then*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2) \geq \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2). \tag{2.32}$$

For an elegant proof of Shannon's theorem, see [24].

The following theorem is very important and central in this thesis, thus, its proof is presented as well.

Theorem 2.12 (Duality theorem)**.** *[24] Let* G *and* H *be simple graphs. Then*

$$\Theta(\mathsf{G}+\mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H}) \iff \Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H}). \tag{2.33}$$

Proof. By Theorems 2.10 and 2.11, the claim is equivalent to

$$\Theta(\mathsf{G}+\mathsf{H}) > \Theta(\mathsf{G}) + \Theta(\mathsf{H}) \iff \Theta(\mathsf{G} \boxtimes \mathsf{H}) > \Theta(\mathsf{G})\,\Theta(\mathsf{H}). \tag{2.34}$$

We next prove both directions of the equivalence in (2.34).

1. Assume that $\Theta(\mathsf{G} \boxtimes \mathsf{H}) > \Theta(\mathsf{G})\,\Theta(\mathsf{H})$. By Theorem 2.8,

$$\begin{aligned} \Theta(\mathsf{G}+\mathsf{H})^2 &= \Theta((\mathsf{G}+\mathsf{H})^2) \\ &= \Theta(\mathsf{G}^2 + 2\mathsf{G} \boxtimes \mathsf{H} + \mathsf{H}^2), \end{aligned}$$

   and by Theorems 2.10 and 2.11,

$$\Theta(\mathsf{G}^2 + 2\mathsf{G} \boxtimes \mathsf{H} + \mathsf{H}^2) \geq \Theta(\mathsf{G})^2 + 2\Theta(\mathsf{G} \boxtimes \mathsf{H}) + \Theta(\mathsf{H})^2.$$

   Using the above assumption, we get

$$\begin{aligned} \Theta(\mathsf{G})^2 + 2\Theta(\mathsf{G} \boxtimes \mathsf{H}) + \Theta(\mathsf{H})^2 &> \Theta(\mathsf{G})^2 + 2\Theta(\mathsf{G})\,\Theta(\mathsf{H}) + \Theta(\mathsf{H})^2 \\ &= (\Theta(\mathsf{G}) + \Theta(\mathsf{H}))^2, \end{aligned}$$

   which gives

$$\Theta(\mathsf{G}+\mathsf{H}) > \Theta(\mathsf{G}) + \Theta(\mathsf{H}).$$

2. Next, assume $\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H})$. Then, for all $i, j \in \mathbb{N}$, we get

$$\begin{aligned} \Theta(\mathsf{G}^i \boxtimes \mathsf{H}^j)\,\Theta(\mathsf{G})^j\,\Theta(\mathsf{H})^i &= \Theta(\mathsf{G}^i \boxtimes \mathsf{H}^j)\,\Theta(\mathsf{G}^j)\,\Theta(\mathsf{H}^i) \\ &\leq \Theta(\mathsf{G}^{i+j} \boxtimes \mathsf{H}^{i+j}) \\ &= \Theta(\mathsf{G} \boxtimes \mathsf{H})^{i+j} \\ &= \Theta(\mathsf{G})^{i+j}\,\Theta(\mathsf{H})^{i+j}, \end{aligned}$$

   thus

$$\Theta(\mathsf{G}^i \boxtimes \mathsf{H}^j) \leq \Theta(\mathsf{G})^i\,\Theta(\mathsf{H})^j. \tag{2.35}$$

Using property (2.35), for all $k \in \mathbb{N}$, we get

$$\begin{aligned}
\alpha\Big((\mathsf{G}+\mathsf{H})^k\Big) &= \alpha\left(\sum_{\ell=0}^{k}\binom{k}{\ell}\mathsf{G}^{\ell}\boxtimes\mathsf{H}^{k-\ell}\right) \\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}\alpha(\mathsf{G}^{\ell}\boxtimes\mathsf{H}^{k-\ell}) \\
&\leq \sum_{\ell=0}^{k}\binom{k}{\ell}\Theta\left(\mathsf{G}^{\ell}\boxtimes\mathsf{H}^{k-\ell}\right) \\
&\leq \sum_{\ell=0}^{k}\binom{k}{\ell}\Theta\left(\mathsf{G}^{\ell}\right)\,\Theta\left(\mathsf{H}^{k-\ell}\right) \\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}\Theta(\mathsf{G})^{\ell}\,\Theta(\mathsf{H})^{k-\ell} \\
&= (\Theta(\mathsf{G})+\Theta(\mathsf{H}))^k.
\end{aligned}$$

Finally, letting $k \to \infty$ gives

$$\Theta(\mathsf{G}+\mathsf{H}) \leq \Theta(\mathsf{G})+\Theta(\mathsf{H}),$$

and by Theorem 2.11,

$$\Theta(\mathsf{G}+\mathsf{H}) = \Theta(\mathsf{G})+\Theta(\mathsf{H}).$$

□

Theorem 2.13. *Let* G *be a simple graph, and let* H_1 *and* H_2 *be induced and spanning subgraphs of* G*, respectively. Then*

$$\Theta(\mathsf{H}_1) \leq \Theta(\mathsf{G}) \leq \Theta(\mathsf{H}_2). \tag{2.36}$$

2.3 The Lovász function of graphs

Before we present the Lovász ϑ-function of graphs, we define an orthogonal representation of a graph [11].

Definition 2.25 (Orthogonal representations)**.** *Let* G *be a simple graph. An* orthogonal representation *of* G *in* $\mathbb{R}^d$ *assigns each vertex* $i \in \mathsf{V}(\mathsf{G})$ *to* *a nonzero vector $u_i \in \mathbb{R}^d$ such that, for every distinct nonadjacent vertices $i, j \in \mathsf{V}(\mathsf{G})$, the vectors u_i, u_j are orthogonal. An orthogonal representation is called an* orthonormal representation *if all the representing vectors of* G *have a unit length.*

In an orthogonal representation of a graph G, distinct nonadjacent vertices are mapped to orthogonal vectors, but adjacent vertices may not be necessarily mapped to non-orthogonal vectors. If the latter condition also holds, then it is called a *faithful* orthogonal representation.

Definition 2.26 (Lovász function)**.** *Let* G *be a simple graph of order* n*. The Lovász function of* G *is defined as*

$$\vartheta(\mathsf{G}) \triangleq \min_{u,\mathbf{c}} \max_{1 \leq i \leq n} \frac{1}{(\mathbf{c}^{\mathrm{T}} \mathbf{u}_i)^2}, \tag{2.37}$$

where the minimum is taken over

- *all orthonormal representations* $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ *of* G*.*
- *all unit vectors* $\mathbf{c}$*.*

The unit vector $\mathbf{c}$ *that achieves the minimum is called the handle of the orthonormal representation.*

An orthonormal representation of the pentagon C_5, along with its handle $\mathbf{c}$, is shown in Figure 2.1.

The Lovász ϑ-function can be expressed as a solution of an SDP problem. To that end, let $\mathbf{A} = (A_{i,j})$ be the $n \times n$ adjacency matrix of G with $n \triangleq |\,\mathsf{V}(\mathsf{G})|$. The Lovász ϑ-function $\vartheta(\mathsf{G})$ can be expressed by the following convex optimization problem:

$$\boxed{\begin{array}{l} \text{maximize } \ \mathrm{Tr}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \succeq 0, \\ \mathrm{Tr}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \ \Rightarrow \ B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}} \tag{2.38}$$

The SDP formulation in (2.38) yields the existence of an algorithm that computes $\vartheta(\mathsf{G})$, for every graph G, with a precision of r decimal digits, and a

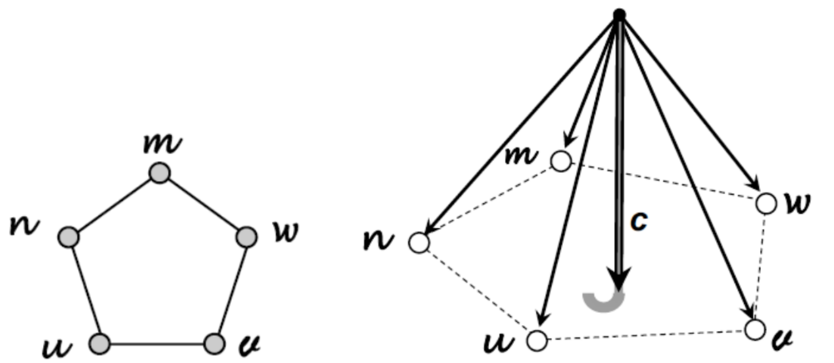

Figure 2.1: A 5-cycle graph and its orthonormal representation (Lovász umbrella).

computational complexity that is polynomial in n and r. Thus, the Lovász ϑ-function can be computed in polynomial time in n, and by [11], it is an upper bound on the Shannon capacity, whose computation requires the computation of an infinite series of independence number, which is a known NP-hard problem.

Adding the inequality constraints $B_{i,j} \geq 0$ for all $i, j \in [n]$ to (2.38) yields the Schrijver ϑ-function of G, denoted $\vartheta'(\mathsf{G})$, which therefore yields

$$\vartheta'(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.39}$$

In light of (2.39), one can ask if the Schrijver ϑ-function can serve as a better upper bound for the Shannon capacity. However, in a recent paper by I. Sason [30], it was shown that the Schrijver ϑ-function need note be an upper bound on the Shannon capacity, by providing a graph whose Shannon capacity is strictly larger than the Schrijver ϑ-function. Lastly, the Schrijver ϑ-function is an upper bound on the independence number, so it can be used for constructing better bounds on the independence number.

We next provide an alternative representation of the Lovász ϑ-function of a graph [11].

Theorem 2.14. *[11] Let $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ range over all orthonormal representations of $\overline{\mathsf{G}}$ and $\mathbf{d}$ over all unit vectors. Then*

$$\vartheta(\mathsf{G}) = \max \sum_{i=1}^{n} (\mathbf{d}^{\mathrm{T}} \mathbf{v}_i)^2. \tag{2.40}$$

Next, we provide several properties of the Lovász number, regarding graph operations and subgraphs.

Theorem 2.15. *[11] Let G_1 and G_2 be simple graphs. Then*

$$\vartheta(\mathsf{G}_1 \boxtimes \mathsf{G}_2) = \vartheta(\mathsf{G}_1)\,\vartheta(\mathsf{G}_2). \tag{2.41}$$

The following result was first stated and proved by Knuth (Section 18 of [16]). We suggest an alternative elementary proof in Appendix A.

Theorem 2.16. *[16] Let G and H be simple graphs. Then*

$$\vartheta(\mathsf{G} + \mathsf{H}) = \vartheta(\mathsf{G}) + \vartheta(\mathsf{H}). \tag{2.42}$$

Theorem 2.17. *Let G be a simple graph, and let H_1 and H_2 be induced and spanning subgraphs of G, respectively. Then*

$$\vartheta(\mathsf{H}_1) \leq \vartheta(\mathsf{G}) \leq \vartheta(\mathsf{H}_2). \tag{2.43}$$

Next, we present a few known formulas and bounds on the Lovász function.

Theorem 2.18. *[11, 31] Let G be a simple graph of order n, then*

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) \geq n, \tag{2.44}$$

with an equality in (2.44) *if G is a vertex-transitive or strongly regular graph.*

Theorem 2.19. *[11] Let G be a d-regular graph of order n, and let λ_n be its smallest eigenvalue. Then*

$$\vartheta(\mathsf{G}) \leq -\frac{n\lambda_n}{d - \lambda_n}, \tag{2.45}$$

with an equality in (2.45) *if G is an edge-transitive graph.*

2.4 Concluding preliminaries

We next provide a useful lower bound for the fractional independence number and the fractional chromatic number (see [32, Proposition 3.1.1]).

Theorem 2.20. *Let* G *be a simple graph of order* n*, then*

$$\alpha_{\mathrm{f}}(\mathsf{G}) \geq \frac{n}{\omega(\mathsf{G})},$$
$$\chi_{\mathrm{f}}(\mathsf{G}) \geq \frac{n}{\alpha(\mathsf{G})}.$$

Both inequalities hold with equality for vertex-transitive graphs.

Theorem 2.21. *Let* G *be a simple graph. Then,*

$$\alpha(\mathsf{G}) \leq \Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \alpha_{\mathrm{f}}(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}) = \sigma(\mathsf{G}). \tag{2.46}$$

In continuation to Definition 2.19, we next provide some known invariants of Kneser graphs.

Theorem 2.22. *Let* $\mathsf{G} = \mathsf{K}(n, r)$ *be a Kneser graph with* $n \geq 2r$*. The invariants of* G *are*

$$\alpha(\mathsf{G}) = \binom{n-1}{r-1},$$
$$\omega(\mathsf{G}) = \left\lfloor \frac{n}{r} \right\rfloor,$$
$$\Theta(\mathsf{G}) = \binom{n-1}{r-1},$$
$$\vartheta(\mathsf{G}) = \binom{n-1}{r-1},$$
$$\vartheta(\overline{\mathsf{G}}) = \frac{n}{r},$$
$$\alpha_{\mathrm{f}}(\mathsf{G}) = \frac{\binom{n}{r}}{\left\lfloor \frac{n}{r} \right\rfloor},$$
$$\chi_{\mathrm{f}}(\mathsf{G}) = \frac{n}{r},$$
$$\chi(\mathsf{G}) = n - 2r + 2,$$
$$\sigma(\mathsf{G}) = \left\lceil \frac{\binom{n}{r}}{\left\lfloor \frac{n}{r} \right\rfloor} \right\rceil.$$

Proof. See [11, Theorem 13], [33] and Theorem 2.20. $\square$

Theorem 2.23. *Let $\ell \geq 1$. Then the path graph P_ℓ is a universal graph with parameters*

$$\alpha(\mathsf{P}_\ell) = \Theta(\mathsf{P}_\ell) = \vartheta(\mathsf{P}_\ell) = \alpha_{\mathrm{f}}(\mathsf{P}_\ell) = \sigma(\mathsf{P}_\ell) = \left\lceil \frac{\ell}{2} \right\rceil. \tag{2.47}$$

Theorem 2.24. *Let $k \geq 4$. Then the cycle graph C_k has*

- *If k is an even number, then C_k is a universal graph with*

$$\alpha(\mathsf{C}_k) = \Theta(\mathsf{C}_k) = \vartheta(\mathsf{C}_k) = \frac{k}{2}.$$

- *If $k \geq 5$ is an odd number, then*

$$\alpha(\mathsf{C}_k) = \left\lfloor \frac{k}{2} \right\rfloor \quad ; \quad \vartheta(\mathsf{C}_k) = \frac{k}{1 + \sec \frac{\pi}{k}}.$$

Lemma 2.2. *If G is a universal graph, then $\Theta(\mathsf{G}) = \alpha(\mathsf{G})$.*

Proof. If G is a universal graph, then by Theorems 2.6 and 2.21,

$$\Theta(\mathsf{G}) = \alpha(\mathsf{G}).$$

□

Remark 2.2. *The converse of Lemma 2.2 is in general false, see Example 2.2.*

Corollary 2.1. *Let $\mathsf{G} = \mathsf{K}(n, r)$ be a Kneser graph with $n \geq 2r$, then G is universal if and only if $r|n$.*

Proof. By Theorem 2.22,

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \frac{\binom{n}{r}}{\lfloor \frac{n}{r} \rfloor}.$$

For G to be a universal graph, the equality $\alpha(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$ has to hold (see Theorem 2.6), thus, G is universal if and only if:

$$\binom{n-1}{r-1} = \frac{\binom{n}{r}}{\lfloor \frac{n}{r} \rfloor},$$

which holds if and only if $r|n$, as required. □

Example 2.2. *The Petersen graph $\mathsf{G} = \mathsf{K}(5, 2)$ has $\alpha(\mathsf{G}) = \Theta(\mathsf{G})$ but is not universal.*

Chapter 3

The Shannon capacity of polynomials of graphs

Shannon conjectured that for any two graphs G and H, the capacity of their disjoint union is equal to the sum of their individual capacities [1]. This conjecture was later disproved by Alon [7], who showed that if G is the Schläfli graph, then

$$\Theta(\mathsf{G}) + \Theta(\overline{\mathsf{G}}) \leq 10 < 2\sqrt{27} \leq \Theta(\mathsf{G} + \overline{\mathsf{G}}).$$

In this section, we provide sufficient conditions on a sequence of graphs, such that the capacity of their disjoint union is equal to the sum of their individual capacities. The results of this section rely on a recent result by Schrijver [24]. Due to its importance, it is provided with a proof in the preliminaries (see Theorem 2.12), stating that for any two simple graphs G and H,

$$\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H}) \iff \Theta(\mathsf{G} + \mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H}).$$

This result was independently proved by Wigderson and Zuiddam [22], with credit to Holzman for personal communications.

Define the set $\mathbb{N}[x_1, \ldots, x_\ell]$ to be the set of all nonzero polynomials of variables $x_1, \ldots, x_\ell$ with nonnegative integral coefficients. A polynomial of graphs, $p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)$, is defined such that the plus operation of graphs stands for their disjoint union and a product of graphs stands for their strong product.

In this section, we derive sufficient conditions for a sequence of graphs $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ to satisfy the property that, for every polynomial $p \in \mathbb{N}[x_1, \ldots, x_\ell]$, the following equality holds:

$$\Theta(p(\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell)) = p(\Theta(\mathsf{G}_1), \Theta(\mathsf{G}_2), \ldots, \Theta(\mathsf{G}_\ell)). \tag{3.1}$$

By a corollary of the duality theorem (Theorem 2.12), which was proven by Schrijver [24, Theorem 3], the following surprising result holds.

Lemma 3.1. *Let* $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell$ *be a sequence of simple graphs. Then, equality* (3.1) *holds for every* $p \in \mathbb{N}[x_1, \ldots, x_\ell]$ *if and only if*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2 + \ldots + \mathsf{G}_\ell) = \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2) + \ldots + \Theta(\mathsf{G}_\ell). \tag{3.2}$$

By Lemma 3.1, we focus on finding sufficient conditions for the satisfiability of (3.2). In this section, we present two main results that provide such sufficient conditions, discuss the differences between them, and illustrate their application. We start by providing the following result.

Theorem 3.1. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ *be simple graphs, with* $\ell \in \mathbb{N}$*. Then for every* $p \in \mathbb{N}[x_1, \ldots, x_\ell]$,

$$p(\Theta(\mathsf{G}_1), \ldots, \Theta(\mathsf{G}_\ell)) \leq \Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)) \leq p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_\ell)). \tag{3.3}$$

Proof. By Theorems 2.10 and 2.11,

$$p(\Theta(\mathsf{G}_1), \ldots, \Theta(\mathsf{G}_\ell)) \leq \Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)),$$

and by Theorems 2.15 and 2.16,

$$\begin{aligned}\Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)) &\leq \vartheta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)) \\ &= p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_\ell)).\end{aligned}$$

□

Corollary 3.1. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ *be simple graphs, with* $\ell \in \mathbb{N}$*. If* $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ *for every* $i \in [\ell]$*, then*

$$\Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)) = p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_\ell)) \tag{3.4}$$

for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$.

Proof. By assumption, it follows that

$$p(\Theta(\mathsf{G}_1), \ldots, \Theta(\mathsf{G}_\ell)) = p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_\ell)).$$

Thus, by Theorem 3.1, equality (3.4) holds. □

We next provide our first sufficient condition.

Theorem 3.2. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ *be simple graphs, with* $\ell \in \mathbb{N}$. *If* $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ *for every* $i \in [\ell]$, *then equality* (3.1) *holds for every* $p \in \mathbb{N}[x_1, \ldots, x_\ell]$.

Proof. By the assumption and Theorem 2.16,

$$\begin{aligned}\vartheta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) &= \vartheta(\mathsf{G}_1) + \ldots + \vartheta(\mathsf{G}_\ell) \\ &= \Theta(\mathsf{G}_1) + \ldots + \Theta(\mathsf{G}_\ell) \\ &\leq \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) \\ &\leq \vartheta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell),\end{aligned}$$

which gives

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2 + \ldots + \mathsf{G}_\ell) = \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2) + \ldots + \Theta(\mathsf{G}_\ell).$$

Thus, by Lemma 3.1, equality (3.1) holds for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$. $\square$

Example 3.1. *If* $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ *are all Kneser graphs or self-complementary vertex-transitive graphs, then by Theorem 3.2, equality* (3.1) *holds for every polynomial* $p \in \mathbb{N}[x_1, \ldots, x_\ell]$. *In particular, if* $\mathsf{G}_i = \mathsf{K}(n_i, r_i)$ *(with* $n_i \geq 2r_i$*) for every* $i \in [\ell]$, *then*

$$\Theta(\mathsf{K}(n_1, r_1) + \ldots + \mathsf{K}(n_\ell, r_\ell)) = \binom{n_1 - 1}{r_1 - 1} + \ldots + \binom{n_\ell - 1}{r_\ell - 1}.$$

Similarly, if G_i *is a self-complementary and vertex-transitive graph for every* $i \in [\ell]$, *then by [34, Theorem 3.26]*

$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \sqrt{n_1} + \ldots + \sqrt{n_\ell},$$

where n_i *is the order of* G_i.

Next we give second sufficient conditions for equality (3.1) to hold.

Lemma 3.2. *Let* G_1 *and* G_2 *be simple graphs. If* $\Theta(\mathsf{G}_1) = \alpha_{\mathrm{f}}(\mathsf{G}_1)$, *then*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2) = \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2). \tag{3.5}$$

Proof. By Theorem 2.10,

$$\Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2) \le \Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2), \tag{3.6}$$

and by Theorem 2.5,

$$\begin{aligned}
\Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2) &= \lim_{k\to\infty} \sqrt[k]{\alpha((\mathsf{G}_1 \boxtimes \mathsf{G}_2)^k)} && (3.7)\\
&\le \lim_{k\to\infty} \sqrt[k]{\alpha_{\mathrm{f}}(\mathsf{G}_1^k)\alpha(\mathsf{G}_2^k)} && (3.8)\\
&= \alpha_{\mathrm{f}}(\mathsf{G}_1)\,\Theta(\mathsf{G}_2), && (3.9)
\end{aligned}$$

Combining inequalities (3.6) and (3.7) gives

$$\begin{aligned}
\Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2) &\le \Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2)\\
&\le \alpha_{\mathrm{f}}(\mathsf{G}_1)\,\Theta(\mathsf{G}_2).
\end{aligned}$$

Hence, by the assumption,

$$\Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2) = \Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2),$$

and equality (3.5) follows from Theorem 2.12. $\square$

Theorem 3.3. *Let $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell$ be simple graphs. If $\Theta(\mathsf{G}_i) = \alpha_{\mathrm{f}}(\mathsf{G}_i)$ for (at least) $\ell-1$ of these graphs, then equality* (3.1) *holds for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$.*

Proof. Without loss of generality, assume $\Theta(\mathsf{G}_i) = \alpha_{\mathrm{f}}(\mathsf{G}_i)$ for every $i \in [\ell - 1]$. By a recursive application of Lemma 3.2,

$$\Theta\left(\sum_{i=1}^{\ell} \mathsf{G}_i\right) = \Theta(\mathsf{G}_1) + \Theta\left(\sum_{i=2}^{\ell} \mathsf{G}_i\right) = \ldots = \sum_{i=1}^{\ell} \Theta(\mathsf{G}_i).$$

Thus, by Lemma 3.1, equality (3.1) holds for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$. $\square$

It is worth noting that our sufficient conditions for equality (3.1) to hold are less restrictive than those hinted in Shannon's paper [1], which are next presented. The following theorem is derived directly from Lemma 3.2 and the sandwich theorem. We provide the original proof in Appendix B.

Theorem 3.4. *Let G_1 and G_2 be simple graphs with $\alpha(\mathsf{G}_1) = \chi(\overline{\mathsf{G}}_1)$. Then*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2) = \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2).$$

Remark 3.1. *The sufficient condition in Theorem 3.4 is more restrictive than the one in Lemma 3.2. This holds since by Theorem 2.21,*

$$\alpha(\mathsf{G}) \le \Theta(\mathsf{G}) \le \alpha_{\mathrm{f}}(\mathsf{G}) \le \chi(\overline{\mathsf{G}}).$$

The next example suggests a family of graphs for which $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G}) < \chi(\overline{\mathsf{G}})$ holds for every graph G in that family, thus showing the possible applicability of Lemma 3.2 in cases where the conditions in Theorem 3.4 are not satisfied.

Example 3.2. *Let $\mathsf{G} = \overline{\mathsf{K}(n,r)}$ be the complement of a Kneser graph where $n > 2r$, $r > 1$, and $r|n$. It is claimed that the graph G satisfies*

$$\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G}) < \chi(\overline{\mathsf{G}}).$$

Indeed, the invariants of the Kneser graph and its complement are known, and by Theorem 2.22

$$\begin{aligned} \alpha(\mathsf{G}) &= \omega(\mathsf{K}(n,r)) = \frac{n}{r} \\ \alpha_{\mathrm{f}}(\mathsf{G}) &= \frac{n}{r} \\ \chi(\overline{\mathsf{G}}) &= \chi(\mathsf{K}(n,r)) = n - 2r + 2, \end{aligned}$$

so since the independence number and the fractional independence number are identical and equal to $\frac{n}{r}$ then $\Theta(\mathsf{G}) = \frac{n}{r}$. Hence, by the assumption that $n > 2r$, it follows that

$$\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G}) = \frac{n}{r} < n - 2r + 2 = \chi(\overline{\mathsf{G}}).$$

Remark 3.2. *The sufficient conditions provided by Theorems 3.2 and 3.3 do not supersede each other (see Examples 3.3 and 3.4). More explicitly, for every graph G,*

$$\Theta(\mathsf{G}) \le \vartheta(\mathsf{G}) \le \alpha_{\mathrm{f}}(\mathsf{G}),$$

so the condition $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$ is less restrictive than the condition $\Theta(\mathsf{G}) = \vartheta(\mathsf{G})$. Consequently, Theorem 3.3 imposes a stronger condition than the one in Theorem 3.2 on $\ell - 1$ of the graphs, while no condition is imposed on the ℓ-th graph. The following examples clarify the issue.

Example 3.3. *Let $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ be Kneser graphs, where*

$$\mathsf{G}_i = \mathsf{K}(n_i, r_i) \quad (n_i \geq 2r_i).$$

For these graphs we have

$$\alpha(\mathsf{G}_i) = \Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i) = \binom{n_i - 1}{r_i - 1} \quad , \quad \forall i \in [\ell].$$

On the other hand, by Theorem 2.22,

$$\alpha_{\mathrm{f}}(\mathsf{G}_i) = \frac{\binom{n_i}{r_i}}{\lfloor \frac{n_i}{r_i} \rfloor}. \tag{3.10}$$

If $r_i \nmid n_i$, then it follows that

$$\alpha_{\mathrm{f}}(\mathsf{G}_i) > \frac{\binom{n_i}{r_i}}{\frac{n_i}{r_i}} = \binom{n_i - 1}{r_i - 1} = \Theta(\mathsf{G}_i), \tag{3.11}$$

so, if $r_i \nmid n_i$, then $\alpha_{\mathrm{f}}(\mathsf{G}_i) > \Theta(\mathsf{G}_i)$.

Let the parameters $n_1, \ldots, n_\ell$, $r_1, \ldots, r_\ell$ be selected such that $r_i \nmid n_i$ for at least two of the graphs $\{\mathsf{G}_i\}_{i=1}^{\ell}$, we have $\alpha_{\mathrm{f}}(\mathsf{G}_i) > \Theta(\mathsf{G}_i)$, which then violates the satisfiability of the sufficient conditions in Theorem 3.3. Hence, in that case

$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \sum_{i=1}^{\ell} \binom{n_i - 1}{r_i - 1}$$

by Theorem 3.2, whereas this equality is not implied by Theorem 3.3.

Example 3.4. *If G_i is a perfect graph for every $i \in [\ell - 1]$ and G_ℓ is the complement of the Schläfli graph, then we have:*

- $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$ *for every $i \in [\ell - 1]$.*
- $\Theta(\mathsf{G}_\ell) < \vartheta(\mathsf{G}_\ell)$ *(By [14]).*

Thus, the sufficient conditions of Theorem 3.3 hold, whereas the sufficient conditions of Theorem 3.2 do not hold.

Chapter 4

The Shannon capacity of Tadpole graphs

The present section explores Tadpole graphs, where exact values and bounds on their Shannon capacity are derived.

Definition 4.1 (Tadpole graphs)**.** *Let $k, \ell \in \mathbb{N}$ with $k \geq 3$ and $\ell \geq 1$. The graph $\mathrm{T}(k, \ell)$, called the Tadpole graph of order (k, ℓ), is obtained by taking a cycle C_k of order k and a path P_ℓ of order ℓ, and then joining one pendant vertex of P_ℓ (i.e., one of its two vertices of degree 1) to a vertex of C_k by an edge. For completeness, if $\ell = 0$, the Tadpole graph is defined trivially as a cycle graph of order k, $\mathrm{T}(k, 0) = \mathsf{C}_k$.*

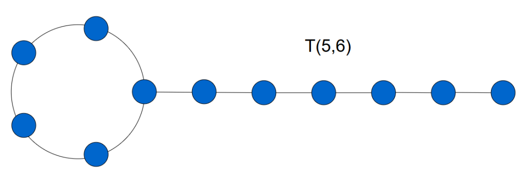

Figure 4.1: The Tadpole graph $\mathrm{T}(5, 6)$.

Every Tadpole graph is a connected graph with

$$|\,\mathsf{V}(\mathrm{T}(k, \ell))| = |\,\mathsf{E}(\mathrm{T}(k, \ell))| = k + \ell.$$

The Tadpole graph $\mathrm{T}(k, \ell)$ with $\ell \geq 1$ is irregular since it has one vertex of degree 3, $k + \ell - 2$ vertices of degree 2, and one vertex of degree 1 (see Figure 4.1).

The motivation to explore the Tadpole graphs comes from their similarity to the cycle graph, as graphs and as their equivalent DMCs, which is shown in Figure 4.2.

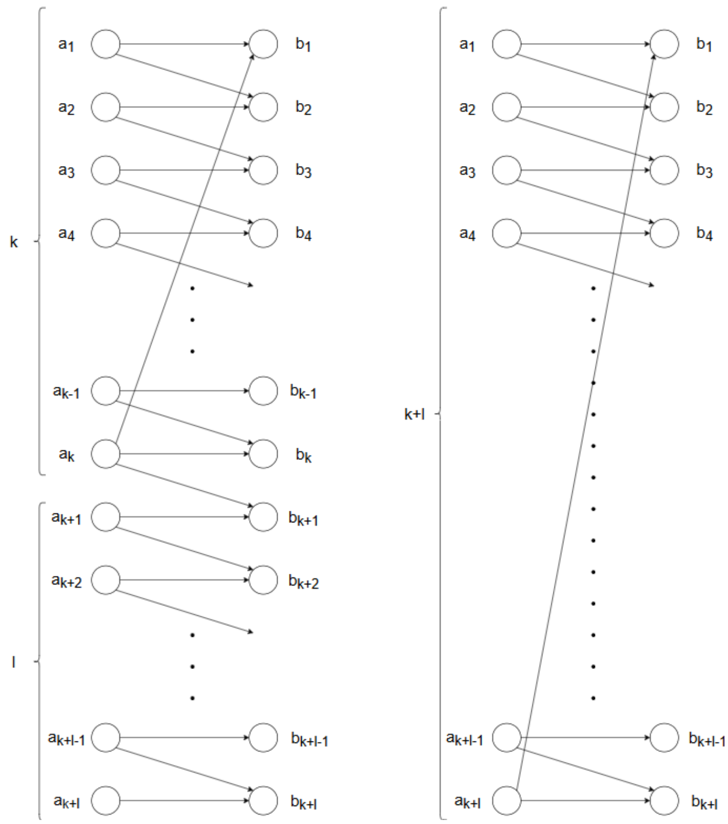

Figure 4.2: The DMCs of $\mathrm{T}(k, \ell)$ (Left plot) and $\mathsf{C}_{k+\ell}$ (Right plot).

Lemma 4.1. *Let $k \geq 3$ and $\ell \geq 0$. The independence number of the Tadpole graph $\mathrm{T}(k, \ell)$ is given by*

$$\alpha(\mathrm{T}(k, \ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.1}$$

Proof. The independence number of C_k is $\left\lfloor \frac{k}{2} \right\rfloor$, and the independence number of P_ℓ is $\left\lceil \frac{\ell}{2} \right\rceil$. One can select a maximal independent set by excluding the vertex of C_k that is adjacent to the vertex of P_ℓ, which gives (4.1). $\square$

Lemma 4.2. *Let $k \geq 3$ and $\ell \geq 0$.*

1. *If one of the following two conditions holds:*
   - *$k = 3$ or $k \geq 4$ is even,*
   - *$k \geq 5$ is odd and $\ell \geq 1$ is odd,*

   then
$$\vartheta(\mathrm{T}(k, \ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil.$$

2. *If $k \geq 5$ is odd and $\ell \geq 0$ is even, then*
$$\vartheta(\mathrm{T}(k, \ell)) = \frac{k}{1 + \sec \frac{\pi}{k}} + \frac{\ell}{2}.$$

Proof. First, if $k = 3$ then the cycle C_3 is a clique of order 3, and since the clique-cover number of P_ℓ is $\left\lceil \frac{\ell}{2} \right\rceil$ then

$$\sigma(\mathrm{T}(3, \ell)) = 1 + \left\lceil \frac{\ell}{2} \right\rceil.$$

By Lemma 4.1 and Theorem 2.21, a lower bound and an upper bound on $\vartheta(\mathrm{T}(k, \ell))$ coincide, which gives

$$\vartheta(\mathrm{T}(3, \ell)) = 1 + \left\lceil \frac{\ell}{2} \right\rceil.$$

Next, if $k > 3$, then the clique number of $\mathrm{T}(k, \ell)$ is 2 (since there are no triangles), thus, every clique in $\mathrm{T}(k, \ell)$ is either an edge or a single vertex. In this case, if k is even, it is possible to cover the cycle C_k by $\frac{k}{2}$ edges (cliques) and the path can be covered by $\left\lceil \frac{\ell}{2} \right\rceil$ edges, and thus $\sigma(\mathrm{T}(k, \ell)) = \frac{k}{2} + \left\lceil \frac{\ell}{2} \right\rceil$. And, if k is odd and ℓ is odd, then one can cover $\mathrm{T}(k, \ell)$ by $\frac{k+\ell}{2}$ edges (by covering C_k by $\frac{k-1}{2}$ edges, excluding the vertex of C_k that is adjacent to a pendant vertex of P_ℓ, and then covering the remaining vertices of $\mathrm{T}(k, \ell)$ by $\frac{\ell+1}{2}$ edges). Overall, in both instances we got

$$\sigma(\mathrm{T}(k, \ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil,$$

which gives (by using again the sandwich theorem in (2.46) and Lemma 4.1),

$$\vartheta(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil .$$

Finally, assume that $k \geq 5$ is odd and $\ell \geq 2$ is even (if $\ell = 0$ then $\mathrm{T}(k,0) = \mathsf{C}_k$ and the result follows from Theorem 2.24). By deleting the edge that connects the cycle C_k with the path P_ℓ, it follows that $\mathsf{C}_k + \mathsf{P}_\ell$ is a spanning subgraph of $\mathrm{T}(k,\ell)$. On the other hand, by deleting the leftmost vertex of P_ℓ (i.e., the vertex that is of distance 1 from the cycle C_k), and the two edges that are incident to it in $\mathrm{T}(k,\ell)$, it follows that $\mathsf{C}_k + \mathsf{P}_{\ell-1}$ is an induced subgraph of $\mathrm{T}(k,\ell)$. Hence, by Theorem 2.17,

$$\begin{aligned}\vartheta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) &\leq \vartheta(\mathrm{T}(k,\ell)) \\ &\leq \vartheta(\mathsf{C}_k + \mathsf{P}_\ell).\end{aligned}$$

Furthermore, by Theorems 2.23 and 2.24,

$$\begin{aligned}\vartheta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) &= \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2} \\ &= \vartheta(\mathsf{C}_k + \mathsf{P}_\ell).\end{aligned}$$

Thus,

$$\vartheta(\mathrm{T}(k,\ell)) = \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}.$$

□

Theorem 4.1. *Let $k \geq 3$ and $\ell \geq 0$ be integers.*

1. *If one of the following two conditions holds:*
   - *$k = 3$ or $k \geq 4$ is even,*
   - *$k \geq 5$ is odd and $\ell \geq 1$ is odd,*

   then

$$\Theta(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil . \tag{4.2}$$

2. *If $k \geq 5$ is odd and $\ell \geq 0$ is even, then*

$$\Theta(\mathrm{T}(k,\ell)) = \Theta(\mathsf{C}_k) + \frac{\ell}{2}. \tag{4.3}$$

Proof. First, if $k = 3$ or $k \geq 4$ is even, or $k \geq 5$ is odd and $\ell \geq 1$ is odd, then by Lemmas 4.1 and 4.2,

$$\Theta(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil.$$

Second, if $k \geq 5$ is odd and $\ell \geq 0$ is even, then by using the same subgraphs of $\mathrm{T}(k,\ell)$ from the proof of Lemma 4.2, and by Theorem 2.13,

$$\Theta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) \leq \Theta(\mathrm{T}(k,\ell)) \leq \Theta(\mathsf{C}_k + \mathsf{P}_\ell).$$

Furthermore, path graphs are universal, then, by Theorem 3.3,

$$\begin{aligned} \Theta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) &= \Theta(\mathsf{C}_k) + \Theta(\mathsf{P}_{\ell-1}) \\ &= \Theta(\mathsf{C}_k) + \frac{\ell}{2} \\ &= \Theta(\mathsf{C}_k + \mathsf{P}_\ell). \end{aligned}$$

Hence,

$$\Theta(\mathrm{T}(k,\ell)) = \Theta(\mathsf{C}_k) + \frac{\ell}{2}.$$

□

Corollary 4.1. *Let $\ell \geq 0$ be an even number and let $k \geq 5$ be an odd number.*

1. *If $k = 5$, then*

$$\Theta(\mathrm{T}(5,\ell)) = \sqrt{5} + \frac{\ell}{2}. \tag{4.4}$$

2. *If $k \geq 7$, then*

$$\frac{k+\ell-1}{2} \leq \Theta(\mathrm{T}(k,\ell)) \leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}. \tag{4.5}$$

Proof. If $k = 5$, then $\Theta(\mathsf{C}_5) = \sqrt{5}$ [11], and by Theorem 4.1,

$$\begin{aligned}\Theta(\mathrm{T}(5,\ell)) &= \Theta(\mathsf{C}_5) + \frac{\ell}{2} \\ &= \sqrt{5} + \frac{\ell}{2}.\end{aligned}$$

If $k \geq 7$, then by Theorem 2.24,

$$\begin{aligned}\frac{k-1}{2} &= \alpha(\mathsf{C}_k) \\ &\leq \Theta(\mathsf{C}_k) \\ &\leq \vartheta(\mathsf{C}_k) = \frac{k}{1+\sec\frac{\pi}{k}}.\end{aligned}$$

Thus, by Theorem 4.1,

$$\begin{aligned}\frac{k+\ell-1}{2} &\leq \Theta(\mathrm{T}(k,\ell)) \\ &\leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}.\end{aligned}$$

$\square$

Example 4.1. *Let $\mathsf{G}_1 = \mathrm{T}(5,6)$, then by Corollary 4.1,*

$$\Theta(\mathrm{T}(5,6)) = 3 + \sqrt{5} = 5.23607\ldots \tag{4.6}$$

For comparison, using the SageMath software [35] gives the values

$$\begin{aligned}\sqrt{\alpha(\mathsf{G}_1 \boxtimes \mathsf{G}_1)} &= \sqrt{26} = 5.09902, \\ \sqrt[3]{\alpha(\mathsf{G}_1 \boxtimes \mathsf{G}_1 \boxtimes \mathsf{G}_1)} &= \sqrt[3]{136} = 5.14256.\end{aligned}$$

Let $\mathsf{G}_2 = \mathrm{T}(7,6)$, then by Corollary 4.1 and the lower bound [36]

$$\Theta(\mathsf{C}_7) \geq \sqrt[5]{\alpha(\mathsf{C}_7^5)} \geq \sqrt[5]{367},$$

which yields

$$\begin{aligned}6.2578659\ldots = \sqrt[5]{367} + 3 &\leq \Theta(\mathsf{C}_7) + 3 \\ &= \Theta(\mathrm{T}(7,6)) \\ &\leq \frac{7}{1+\sec\frac{\pi}{7}} + 3 = 6.3176672.\end{aligned}$$

The above lower bound on $\Theta(\mathrm{T}(7,6))$ *improves the previous lower bound in the leftmost term of* (4.5)*, thus closing the gap between the upper and lower bounds from* 0.3176 *to* 0.0598*.*

The last example shows that the lower bound in Corollary 4.1 can be improved. The following result gives an improved lower bound.

Theorem 4.2. *If* $k \geq 5$ *is odd and* $\ell \geq 0$ *is even, then*

$$\sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2} \leq \Theta(\mathrm{T}(k,\ell)) \leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}. \tag{4.7}$$

Proof. The upper bound was proved in Corollary 4.1. We next prove the improved lower bound. By Hales' paper [28, Theorem 7.1], if $k \geq 3$ is odd, then

$$\alpha(\mathsf{C}_k^2) = \left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor,$$

which implies that

$$\begin{aligned}\Theta(\mathsf{C}_k) &\geq \sqrt{\alpha(\mathsf{C}_k^2)} \\ &= \sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor}.\end{aligned}$$

Thus, by combining the last inequality with Theorem 4.1, it follows that

$$\begin{aligned}\Theta(\mathrm{T}(k,\ell)) &= \Theta(\mathsf{C}_k) + \frac{\ell}{2} \\ &\geq \sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2}.\end{aligned}$$

□

The gap between the upper and lower bounds on $\Theta(\mathrm{T}(k,\ell))$ for $k \geq 7$ odd and $\ell \geq 0$ even satisfies,

$$\begin{aligned}&\left(\frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}\right) - \left(\sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2}\right) \\ &< \frac{k+\ell}{2} - \frac{k+\ell-1}{2} = \frac{1}{2}.\end{aligned} \tag{4.8}$$

Remark 4.1. *Theorem 4.2 gives an improved lower bound that is based on a lower bound of the capacity of odd cycles that was constructed by Hales (see [28]). Since hales' result, better lower bounds for odd cycles were constructed, and can be used to improve the lower bound in special cases. For better lower bounds on higher powers of odd cycles see [37, 38]. The next theorem shows an improvement to the bound in* (4.7)*, for special cases using a result from [37].*

Theorem 4.3. *Let $n, d \in \mathbb{N}$, let $\ell \geq 0$ be an even number, and define*

$$k = n2^d + 2^{d-1} + 1.$$

Then,

$$\sqrt[d]{nk^{d-1} + \left(\frac{k-1}{2}\right) k^{d-2}} + \frac{\ell}{2} \leq \Theta(\mathrm{T}(k, \ell)). \tag{4.9}$$

Proof. By Theorem 4.1,

$$\Theta(\mathrm{T}(k, \ell)) = \frac{\ell}{2} + \Theta(\mathsf{C}_k). \tag{4.10}$$

And, by [37, Theorem 1.4],

$$\begin{aligned} \Theta(\mathsf{C}_k) &\geq \sqrt[d]{\alpha(\mathsf{C}_k^d)} \\ &\geq \sqrt[d]{nk^{d-1} + \left(\frac{k-1}{2}\right) k^{d-2}}. \end{aligned} \tag{4.11}$$

Finally, combining (4.10) and (4.11) gives (4.9). □

Remark 4.2. *In Theorem 4.3 a bound from [37, Theorem 1.4] was used. It is worth noting that in his paper, Bohman conjectured that the lower bound was the exact value of the independence number (see [37, Conjecture 1.5]). In a later paper by Bohman, Holzman and Natarajan [38], a countably infinite subset of these values was confirmed, and by substituting $d = 3$, which gives $k = 8n + 5$, it was proven that if $k = 8n + 5$ is a prime number, then the lower bound was the exact value of the independence number, i.e.*

$$\alpha(\mathsf{C}_{8n+5}^3) = \tfrac{1}{2}(8n+5)\Big[(2n+1)(8n+5) - 1\Big]. \tag{4.12}$$

Chapter 5

When the graph capacity is not attained by the independence number of any finite power?

In Section 4, we proved that if $k = 3$, or if $k \geq 4$ is even, or if $k \geq 5$ and $\ell \geq 1$ are odd, then the capacity of the Tadpole graph $\mathrm{T}(k, \ell)$ coincides with its independence number. Additional families of graphs also share this property; for example, by [11], the capacities of all Kneser graphs are equal to their independence numbers. It is, however, well known that not all graphs posses this property; e.g., $\Theta(\mathsf{C}_5) = \sqrt{\alpha(\mathsf{C}_5^2)} = \sqrt{5}$ and $\alpha(\mathsf{C}_5) = 2$. The property that the Shannon capacity coincides with the square root of the independence number of the second (strong) power of the graph was proved, more generally for all self-complementary vertex transitive graphs [11] and all self-complementary strongly regular graphs [34], provided that the order of the graph is not a square of an integer. Families of graphs whose capacity is attained at a finite power that is strictly larger than 2 are yet to be found. In this section, we explore sufficient conditions on a graph G that makes its Shannon capacity be unattainable by the independence number of any finite power of G, i.e., we explore the conditions on G to have:

$$\Theta(\mathsf{G}) > \sqrt[k]{\alpha(\mathsf{G}^k)} \quad \forall\, k \in \mathbb{N}. \tag{5.1}$$

For example, the graph $\mathsf{G} = \mathsf{C}_5 + \mathsf{K}_1$ satisfies (5.1) (see Example 5.1).

This problem was explored by Guo and Watanabe [10], and a family of disconnected graphs that satisfies (5.1) was constructed. In this section, we start by presenting the method from [10], and then we provide two original approaches. The first original approach relies on a Dedekind's lemma in number theory (1858), and it uses a similar concept of proof to the one in Example 5.1. The second original approach uses the result from [10] to construct a countably infinite family of *connected* graphs whose capacity is strictly larger than the independence number of any finite (strong) power of the graph.

The following example provides a disconnected graph whose capacity is not attained at any finite power, the proof of this example uses an important concept that is later used in our first original approach. This example was shown by A. Wigderson and J. Zuiddam [22].

Example 5.1. *Let* $\mathsf{G} = \mathsf{C}_5 + \mathsf{K}_1$. *We have* $\mathsf{C}_5 \boxtimes \mathsf{K}_1 \cong \mathsf{C}_5$, *so*

$$\Theta(\mathsf{C}_5 \boxtimes \mathsf{K}_1) = \Theta(\mathsf{C}_5) = \Theta(\mathsf{C}_5)\,\Theta(\mathsf{K}_1).$$

Consequently, by Theorem 2.12,

$$\Theta(\mathsf{G}) = \Theta(\mathsf{C}_5) + \Theta(\mathsf{K}_1) = \sqrt{5} + 1,$$

and, for every $k \in \mathbb{N}$,

$$\begin{aligned}
\Theta(\mathsf{G})^k &= \left(\sqrt{5} + 1\right)^k \\
&= \sum_{i=0}^{k} \binom{k}{i} 5^{\frac{i}{2}} \\
&= \sum_{0 \le \ell \le \lfloor \frac{k}{2} \rfloor} \binom{k}{2\ell} 5^{\ell} + \sum_{0 \le \ell \le \lfloor \frac{k-1}{2} \rfloor} \binom{k}{2\ell+1} 5^{\ell+\frac{1}{2}} \\
&= c_k + d_k \sqrt{5} \notin \mathbb{N},
\end{aligned}$$

where

$$\begin{aligned}
c_k &= \sum_{\ell=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k}{2\ell} 5^{\ell} \in \mathbb{N} \\
d_k &= \sum_{\ell=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k}{2\ell+1} 5^{\ell} \in \mathbb{N}.
\end{aligned}$$

It therefore follows that, for all $k \in \mathbb{N}$,

$$\Theta(\mathsf{G})^k \neq \alpha(\mathsf{G}^k)$$

since the independence number is an integer. Consequently, (5.1) *holds.*

We next consider three approaches for the construction of families of graphs whose Shannon capacities satisfies the condition in (5.1). The first approach is due to Guo and Watanabe [10] with a simplified proof (the original proof from [10] is provided in Appendix C), and the second and third approaches of constructing such graph families are original.

5.1 First approach

Theorem 5.1. *[10] Let G be a universal graph (see Definition 2.21), and let H satisfy the inequality $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$. Then, the Shannon capacity of $\mathsf{K} \triangleq \mathsf{G} + \mathsf{H}$ is not attained at any finite power of K.*

Proof. Let $k \in \mathbb{N}$. By the universality of G,

$$\begin{aligned}
\alpha((\mathsf{G}+\mathsf{H})^k) &= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{G}^i \boxtimes \mathsf{H}^{k-i}) \\
&= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{G})^i \, \alpha(\mathsf{H}^{k-i}) \\
&= \sum_{i=0}^{k} \binom{k}{i} \Theta(\mathsf{G})^i \alpha(\mathsf{H}^{k-i}),
\end{aligned} \tag{5.2}$$

where the last equality holds by Lemma 2.2. Next, by the assumption that $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$ and since $\Theta(\mathsf{H}^m) = \Theta(\mathsf{H})^m \geq \alpha(\mathsf{H})^m$ for all $m \in \mathbb{N}$ with strict inequality if $m = 1$ (by assumption), it follows that for all $k \in \mathbb{N}$

$$\begin{aligned}
\sum_{i=0}^{k} \binom{k}{i} \Theta(\mathsf{G})^i \alpha(\mathsf{H}^{k-i}) &< \sum_{i=0}^{k} \binom{k}{i} \Theta(\mathsf{G})^i \Theta(\mathsf{H})^{k-i} \\
&= (\Theta(\mathsf{G}) + \Theta(\mathsf{H}))^k.
\end{aligned} \tag{5.3}$$

Finally, by Shannon's inequality (see Theorem 2.11),

$$\Theta(\mathsf{G}) + \Theta(\mathsf{H}) \leq \Theta(\mathsf{G}+\mathsf{H}). \tag{5.4}$$

Combining (5.2)–(5.4) and raising both sides of the resulting inequality to the power of $\frac{1}{k}$ gives

$$\sqrt[k]{\alpha((\mathsf{G}+\mathsf{H})^k)} < \Theta(\mathsf{G}+\mathsf{H}),$$

thus confirming the satisfiability of the condition in (5.1). $\square$

Corollary 5.1. *[10] Let* G *be a universal graph, and let* $\mathsf{K} = \mathsf{G} + \mathsf{C}_{2k+1}$ *with* $k \geq 2$ *be the disjoint union of a universal graph and an odd cycle of length at least* 5*. Then,* $\Theta(\mathsf{K})$ *is not attained at any finite (strong) power of* K*.*

Proof. This follows from Theorem 5.1 by showing that $\Theta(\mathsf{C}_{2k+1}) > \alpha(\mathsf{C}_{2k+1})$. For an odd cycle graph of length $2k+1$, $\alpha(\mathsf{C}_{2k+1}) = k$. By [28, Theorem 7.1], for every $j, k \in \mathbb{N}$ such that $2 \leq j \leq k$

$$\begin{aligned}
&\alpha(\mathsf{C}_{2j+1} \boxtimes \mathsf{C}_{2k+1}) = jk + \left\lfloor \frac{j}{2} \right\rfloor, \\
\Rightarrow &\alpha(\mathsf{C}_{2k+1}^2) = k^2 + \left\lfloor \frac{k}{2} \right\rfloor > k^2 = \alpha(\mathsf{C}_{2k+1})^2, \\
\Rightarrow &\Theta(\mathsf{C}_{2k+1}) \geq \sqrt{\alpha(\mathsf{C}_{2k+1}^2)} > \alpha(\mathsf{C}_{2k+1}).
\end{aligned}$$

$\square$

Remark 5.1. *This applies in particular to* $\mathsf{K}_1 + \mathsf{C}_5$ *since* K_1 *is universal.*

5.2 Second approach

The next approach relies on a lemma in number theory from 1858 by Dedekind. This lemma is provided as follows, whose simple and elegant proof from [39, p. 309] is presented here for completeness.

Lemma 5.1. *If the square-root of a natural number is rational, then it must be an integer; equivalently, the square-root of a natural number is either an integer or an irrational number.*

Proof. Let $m \in \mathbb{N}$ be a natural number such that $\sqrt{m} \in \mathbb{Q}$. Let $n_0 \in \mathbb{N}$ be the smallest natural number such that $n_0\sqrt{m} \in \mathbb{N}$. If $\sqrt{m} \notin \mathbb{N}$, then there exists $\ell \in \mathbb{N}$ such that $0 < \sqrt{m} - \ell < 1$. Setting $n_1 = n_0(\sqrt{m} - \ell)$, we get $n_1 \in \mathbb{N}$ and $0 < n_1 < n_0$. Also $n_1\sqrt{m} = n_0 m - \ell(n_0\sqrt{m}) \in \mathbb{N}$. This leads to a contradiction to the choice of n_0. $\square$

Next, we show the main result of this subsection.

Theorem 5.2. *Let $r \geq 2$, and let $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_r$ be graphs such that, for every $\ell \in [r]$,*

1. *The graph G_ℓ is either a Kneser graph or self-complementary strongly regular or self-complementary vertex-transitive.*

2. *There exists a single $\ell_0 \in [r]$ such that G_{ℓ_0} is self-complementary vertex-transitive or self-complementary strongly regular on n_{ℓ_0} vertices, where n_{ℓ_0} is not a square of an integer.*

Let $p \in \mathbb{N}[x_1, \ldots, x_r]$ be a polynomial whose nonzero coefficients are natural numbers such that

1. *There exists a monomial in $p(x_1, \ldots, x_r)$ whose degree in x_{ℓ_0} is odd.*

2. *The variable x_{ℓ_0} doesn't occur in (at least) one of the monomials in $p(x_1, \ldots, x_r)$.*

Let $\mathsf{G} = p(\mathsf{G}_1, \ldots, \mathsf{G}_r)$. Then, the Shannon capacity $\Theta(\mathsf{G})$ is not attained at any finite strong power of G (i.e., condition (5.1) *holds).*

Proof. For all $\ell \in [r]$, $\Theta(\mathsf{G}_\ell) = \vartheta(\mathsf{G}_\ell)$ (this follows from the assumptions on G_ℓ). Thus, by Corollary 3.1

$$\Theta(\mathsf{G}) = p(\vartheta(\mathsf{G}_1, \ldots, \vartheta(\mathsf{G}_r)).$$

Furthermore, if G_ℓ is a Kneser graph, then $\vartheta(\mathsf{G}_\ell) = \binom{n_\ell - 1}{r_\ell - 1} \in \mathbb{N}$ (by [11, Theorem 13]), and if G is self-complementary and vertex-transitive or self-complementary and strongly regular then $\vartheta(\mathsf{G}_\ell) = \sqrt{n_\ell}$ (by [34, Theorem 3.26]). So, for every $\ell \neq \ell_0$, $\vartheta(\mathsf{G}_\ell) \in \mathbb{N}$ by assumption. And, by assumption, $\vartheta(\mathsf{G}_{\ell_0}) \notin \mathbb{N}$. Next, by the assumptions of the theorem, there exists a monomial in p, whose degree in x_{ℓ_0} is odd, and there exists a monomial in p where the variable x_{ℓ_0} does not occur. Thus, there exists $a, b \in \mathbb{N}$ such that

$$p(\vartheta(\mathsf{G}_1, \ldots, \vartheta(\mathsf{G}_r)) = a\sqrt{n_{\ell_0}} + b.$$

Similarly to the concept of proof in Example 5.1, for every $k \in \mathbb{N}$, there exists $c_k, d_k \in \mathbb{N}$ such that

$$p(\vartheta(\mathsf{G}_1, \ldots, \vartheta(\mathsf{G}_r))^k = c_k\sqrt{n_{\ell_0}} + d_k.$$

Since $\sqrt{n_{\ell_0}} \notin \mathbb{N}$ (by assumption), $\sqrt{n_{\ell_0}} \notin \mathbb{Q}$ by Lemma 5.1, which implies that

$$\Theta(\mathsf{G})^k \notin \mathbb{N}.$$

Hence, for every $k \in \mathbb{N}$,

$$\Theta(\mathsf{G}) > \sqrt[k]{\alpha(\mathsf{G}^k)},$$

so the capacity of G is unattainable by any finite strong power of G. $\square$

Corollary 5.2. *Let $r \geq 2$, and let $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_r$ be graphs such that, for every $\ell \in [r]$,*

1. *The graph G_ℓ is self-complementary on n_ℓ vertices, and it is either strongly regular or vertex-transitive.*

2. *There exists a single $\ell_0 \in [r]$ such that n_{ℓ_0} is not a square of an integer.*

Then, the Shannon capacity of the disjoint union of these graphs is not attained at any finite strong power.

Proof. The result is achieved by applying Theorem 5.2 to the linear polynomial

$$p(x_1, \ldots, x_r) = \sum_{j=1}^{r} x_j.$$

$\square$

Example 5.2. *The special case of $\mathsf{G} = \mathsf{C}_5 + \mathsf{K}_1$ (presented earlier) is obtained by selecting in the previous corollary*

- $\mathsf{G}_1 = \mathsf{C}_5$*, which is a self-complementary and vertex-transitive graph.*
- $\mathsf{G}_2 = \mathsf{K}_1$*, which is a Kneser graph.*
- $r = 2$*.*

Hence, the Shannon capacity of G is not attained at any finite power of G.

Next we show another example, this time using Paley graphs.

Example 5.3. *Let $q_1, \ldots, q_\ell$ be integer powers of prime numbers, $p_i \equiv 1 \bmod 4$ for every $i \in [\ell]$, where only one of the q_i's is an odd power of the prime p_i. Define $\mathsf{G}_i = P(q_i)$ (where $P(q)$ is the Paley graph of order q) for every $i \in [\ell]$. Then, the Shannon capacity of the disjoint union of these graphs,*

$$\mathsf{G} = \mathsf{G}_1 + \mathsf{G}_2 + \ldots + \mathsf{G}_\ell$$

is not attained at any finite power of G. This follows from Corollary 5.2, as the Paley graphs are self-complementary and strongly regular, and the restrictions on $q_1, \ldots, q_\ell$ guaranty that the conditions of Corollary 5.2 hold.

5.3 Third approach

In the present last approach, we build a family of *connected* graphs, whose capacity is not attained at any of their finite powers. In particular, we prove that an infinite family of Tadpole graphs is not attained at a finite power.

Theorem 5.3. *Let H be a graph with $\alpha(\mathsf{H}) < \Theta(\mathsf{H})$, let $\ell \geq 2$ be an even number, and let $v \in \mathsf{V}(\mathsf{H})$ be an arbitrary vertex in H. Define the graph G as the disjoint union of H and P_ℓ with an extra edge between v and one of the two endpoints of P_ℓ. Then, the capacity of G is unattainable by any of its finite strong powers.*

Proof. Since $\mathsf{H} + \mathsf{P}_{\ell-1}$ is an induced subgraph of G, and $\mathsf{H} + \mathsf{P}_\ell$ is a spanning subgraph of G, it follows that

$$\Theta(\mathsf{H} + \mathsf{P}_{\ell-1}) \leq \Theta(\mathsf{G}) \leq \Theta(\mathsf{H} + \mathsf{P}_\ell). \tag{5.5}$$

Path graphs are universal, and for an even $\ell \geq 2$, the capacities of $\mathsf{P}_{\ell-1}$ and P_ℓ coincide (see Theorem 2.23). The latter holds because path graphs are bipartite and therefore perfect, so their Shannon capacities coincide with their independence numbers, and also $\alpha(\mathsf{P}_{\ell-1}) = \frac{\ell}{2} = \alpha(\mathsf{P}_\ell)$ if $\ell \geq 2$ is even. by universality of path graphs and since the independence number of P_ℓ and

$\mathsf{P}_{\ell-1}$ coincide for an even $\ell \geq 2$ (see Theorem 2.23), for every $k \in \mathbb{N}$,

$$\begin{aligned}
\alpha((\mathsf{H}+\mathsf{P}_{\ell-1})^k) &= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{H}^i \boxtimes \mathsf{P}_{\ell-1}^{k-i}) \\
&= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{H}^i)\, \alpha(\mathsf{P}_{\ell-1})^{k-i} \\
&= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{H}^i)\, \alpha(\mathsf{P}_{\ell})^{k-i} \\
&= \alpha\Big((\mathsf{H}+\mathsf{P}_{\ell})^k\Big). \qquad (5.6)
\end{aligned}$$

Consequently, by raising both sides of (5.6) to the power $\frac{1}{k}$, and letting $k \to \infty$, it follows that $\Theta(\mathsf{H}+\mathsf{P}_{\ell-1}) = \Theta(\mathsf{H}+\mathsf{P}_{\ell})$. Combining the last equality with (5.5), hence gives

$$\Theta(\mathsf{G}) = \Theta(\mathsf{H}+\mathsf{P}_{\ell}). \qquad (5.7)$$

By the same argument that yields (5.5), for all $k \in \mathbb{N}$, $(\mathsf{H}+\mathsf{P}_{\ell-1})^k$ is an induced subgraph of G^k, and $(\mathsf{H}+\mathsf{P}_{\ell})^k$ is a spanning subgraph of G^k, so

$$\alpha((\mathsf{H}+\mathsf{P}_{\ell-1})^k) \leq \alpha(\mathsf{G}^k) \leq \alpha((\mathsf{H}+\mathsf{P}_{\ell})^k). \qquad (5.8)$$

Thus, by combining (5.6) and (5.8), it follows that for every $k \in \mathbb{N}$

$$\alpha(\mathsf{G}^k) = \alpha\Big((\mathsf{H}+\mathsf{P}_{\ell})^k\Big). \qquad (5.9)$$

Finally, by Theorem 5.1 and equalities (5.7) and (5.9), for every $k \in \mathbb{N}$,

$$\Theta(\mathsf{G})^k = \Theta(\mathsf{H}+\mathsf{P}_{\ell})^k > \alpha\Big((\mathsf{H}+\mathsf{P}_{\ell})^k\Big) = \alpha(\mathsf{G}^k), \qquad (5.10)$$

so G is not attained at any of its finite strong powers. □

Corollary 5.3. *Let $k \geq 5$ be an odd number, and let $\ell \geq 2$ be an even number. Then, the Shannon capacity of the Tadpole graph $\mathrm{T}(k,\ell)$ is unattainable by any of its strong powers.*

Proof. This follows directly from Theorem 5.3 and Definition 4.1, by selecting $\mathsf{H} = \mathsf{C}_k$ for an odd $k \geq 5$. In the latter case, $\alpha(\mathsf{C}_k) < \Theta\mathsf{C}_k$ as required by Theorem 5.3. □

Remark 5.2. *Corollary 5.3 provides a countably infinite set of* connected *graphs whose Shannon capacities are unattainable by any of its strong powers. This is the first infinite family of connected graphs with that property. All previous constructions with that property were disconnected graphs.*

Chapter 6

The Shannon capacity of q-Kneser graphs

In this section we show the exact value of the Shannon capacity of a new family of graphs, containing the Kneser graphs.

Definition 6.1. *Let $n, k \in \mathbb{N}$ with $k \leq n$, let p be a prime, let $q = p^m$ with $m \in \mathbb{N}$, and let $\mathbb{F}_q$ be the Galois field of order q. The Gaussian coefficient, denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$, is given by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}. \tag{6.1}$$

The Gaussian coefficient can be given the following interpretation. Let V be an n-dimensional vector space over $\mathbb{F}_q$. The number of k-dimensional distinct subspaces that V possesses is equal to $\begin{bmatrix} n \\ k \end{bmatrix}_q$. We therefore define $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \triangleq 1$.

Passing to a continuous variable q and letting q tend to 1 gives

$$\begin{aligned} \lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-(k-1)}{k-(k-1)} \\ &= \binom{n}{k}, \end{aligned} \tag{6.2}$$

thus converging to the binomial coefficient.

Definition 6.2 (q-Kneser graphs)**.** *Let $V(n,q)$ be the n-dimensional vector space over the finite field $\mathbb{F}_q$, where q is a prime power. The q-Kneser graph $\mathsf{K}_q(n,k)$ is defined as follows:*

- *The vertices of $\mathsf{K}_q(n,k)$ are defined as the k-dimensional subspaces of $V(n,q)$.*
- *Two vertices are adjacent if their intersection contains only the zero vector.*

Some of the properties of the q-Kneser graphs are next presented (see [44, Proposition 3.4]).

Theorem 6.1. *Let $\mathsf{G} = \mathsf{K}_q(n,k)$ be a q-Kneser graph with $n \geq 2k$ and q a prime power. Then*

- *$\mathsf{K}_q(n,k)$ is of order*

$$|\mathsf{V}(\mathsf{K}_q(n,k))| = \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{6.3}$$

- *$\mathsf{K}_q(n,k)$ is of size*

$$|\mathsf{E}(\mathsf{K}_q(n,k))| = \frac{1}{2} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{6.4}$$

- *$\mathsf{K}_q(n,k)$ is $q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q$-regular.*
- *$\mathsf{K}_q(n,k)$ is vertex-transitive.*
- *$\mathsf{K}_q(n,k)$ is edge-transitive.*

We next calculate the Shannon capacity of $\mathsf{K}_q(n,k)$ in an analogous way to the calculation of the Shannon capacity of the Kneser graph $\mathsf{K}(n,k)$ by Lovász (see [11, Theorem 13]). To that end, we derive the independence number and Lovász ϑ-function of a q-Kneser graph, and show that they coincide, thus leading to the determination of the capacity of the graph.

Lemma 6.1. *[25] Let $\mathsf{G} = \mathsf{K}_q(n,k)$ be a q-Kneser graph with $n \geq 2k$, and let q be a prime power. Then,*

$$\alpha(\mathsf{K}_q(n,k)) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{6.5}$$

Proof. By the version of the Erdös-Ko-Rado Theorem for finite vector spaces (see [25, Theorem 9.8.1]), we get

$$\alpha(\mathsf{K}_q(n,k)) \le \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Moreover, we can construct a family of k-subspaces of $V(n,q)$ containing a fixed 1-dimensional subspace of $V(n,q)$. This family of subspaces has $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ subspaces (see Definition 6.1 and [25, Theorem 9.8.1]), and it is an independent set in $\mathsf{K}_q(n,k)$, thus

$$\alpha(\mathsf{K}_q(n,k)) \ge \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q,$$

which proves equality (6.5). □

Lemma 6.2. *Let* $\mathsf{G} = \mathsf{K}_q(n,k)$ *be a* q*-Kneser graph with* $n \ge 2k$ *and* q *a prime power. Then*

$$\vartheta(\mathsf{K}_q(n,k)) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{6.6}$$

Proof. By Theorem 2.19 and since the q-Kneser graphs are edge-transitive (see Theorem 6.1),

$$\vartheta(\mathsf{K}_q(n,k)) = -\frac{|\,\mathsf{V}(\mathsf{K}_q(n,k))|\,\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}, \tag{6.7}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of the adjacency matrix of $\mathsf{K}_q(n,k)$. By [26, Theorem 2], these eigenvalues are given by

$$\lambda_{\max} = q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \tag{6.8}$$

$$\lambda_{\min} = -q^{k^2-k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q. \tag{6.9}$$

Substituting (6.3), (6.8), and (6.9) into (6.7) gives

$$\begin{aligned}
\vartheta(\mathsf{K}_q(n,k)) &= \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot q^{k^2-k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}{q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q + q^{k^2-k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q} \\
&= \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}{q^{k} \begin{bmatrix} n-k \\ k \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}. \qquad (6.10)
\end{aligned}$$

This expression can be simplified, based on the following identity:

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{q^m-1}{q^r-1} \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q, \qquad (6.11)$$

where using (6.11) with $m = n - k$ and $r = k$ simplifies the denominator of (6.10) to

$$\begin{aligned}
& q^{k} \begin{bmatrix} n-k \\ k \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q \\
&= q^k \cdot \frac{q^{n-k}-1}{q^k-1} \cdot \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q \\
&= \frac{q^n-1}{q^k-1} \cdot \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q. \qquad (6.12)
\end{aligned}$$

Finally, combining (6.10) and (6.12) gives the simplified form

$$\begin{aligned}
\vartheta(\mathsf{K}_q(n,k)) &= \begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \frac{q^k-1}{q^n-1} \\
&= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.
\end{aligned}$$

□

Due to the coincidence of the independence number and the Lovász ϑ-function in Lemmata 6.1 and 6.2, their joint value is also equal to the Shannon capacity of the graph. This gives the following closed-form expression for the Shannon capacity of q-Kneser graphs.

Theorem 6.2. *The Shannon capacity of the q-Kneser graph, $\mathsf{K}_q(n,k)$, with $n \geq 2k$ and q a prime power, is given by*

$$\Theta(\mathsf{K}_q(n,k)) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{6.13}$$

Chapter 7

A new inequality for the capacity of graphs

The following result provides a relation between the Shannon capacity of any strong product of graphs, and the capacity of the disjoint union of the component graphs. If these component graphs are connected, then their strong product is a connected graph on a number of vertices that is equal to the product of the number of vertices in each component graph, whereas the disjoint union of these component graphs is a disconnected graph on a number of vertices that is equal to the sum of the orders of the component graphs (the latter order is typically much smaller than the former). The motivation of our inequality is due to the following result:

Theorem 7.1 (Unique Prime Factorization for Connected Graphs)**.** *Every connected graph has a unique prime factor decomposition with respect to the strong product.*

- The proof of this theorem was introduced by Dorfler and Imrich (1970), and Mckenzie (1971). See section 7.3 in the comprehensive book on graph products [27] (Theorem 7.14).
- In a paper by Feigenbaum and Schaffer [40], a polynomial-time algorithm was introduced for finding that unique prime factorization (with respect to strong products).

We next provide and prove the main result of this section.

Theorem 7.2. *Let $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell$ be simple graphs, then*

$$\Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell) \leq \left(\frac{\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)}{\ell}\right)^\ell. \tag{7.1}$$

Furthermore, if $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ for every $i \in [\ell]$, then inequality (7.1) holds with equality if and only if

$$\Theta(\mathsf{G}_1) = \Theta(\mathsf{G}_2) = \ldots = \Theta(\mathsf{G}_\ell). \tag{7.2}$$

In particular, if for every $i \in [\ell]$, one of the following statements hold:

- *G_i is a perfect graph,*
- *$\mathsf{G}_i = \mathsf{K}(n, r)$ for some $n, r \in \mathbb{N}$ with $n \geq 2r$,*
- *$\mathsf{G}_i = \mathsf{K}_q(n, r)$ for a prime factor q and some $n, r \in \mathbb{N}$ with $n \geq 2r$,*
- *G_i is vertex-transitive and self-complementary,*
- *G_i is strongly regular and self-complementary,*

then inequality (7.1) holds with equality if and only if the condition in (7.2) is satisfied.

Proof. Let $k \in \mathbb{N}$. By (2.29),

$$\begin{aligned}
&\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^{\ell k} \\
&= \Theta((\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^{\ell k}) \\
&= \Theta\left(\sum_{k_1, \ldots, k_\ell \colon k_1 + \ldots + k_\ell = \ell k} \binom{\ell k}{k_1, \ldots, k_\ell} \mathsf{G}_1^{k_1} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k_\ell}\right).
\end{aligned} \tag{7.3}$$

By (2.29)–(2.31),

$$\begin{aligned}
&\Theta\left(\sum_{k_1, \ldots, k_\ell \colon k_1 + \ldots + k_\ell = \ell k} \binom{\ell k}{k_1, \ldots, k_\ell} \mathsf{G}_1^{k_1} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k_\ell}\right) \\
&\geq \sum_{k_1, \ldots, k_\ell \colon k_1 + \ldots + k_\ell = \ell k} \binom{\ell k}{k_1, \ldots, k_\ell} \Theta\Big(\mathsf{G}_1^{k_1} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k_\ell}\Big) \\
&\geq \binom{\ell k}{k, \ldots, k} \Theta\Big(\mathsf{G}_1^{k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k}\Big) \\
&= \binom{\ell k}{k, \ldots, k} \Theta\Big(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell\Big)^k,
\end{aligned} \tag{7.4}$$

thus, by combining (7.3) and (7.4), it follows that

$$\binom{\ell k}{k, \ldots, k} \Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell)^k \leq \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^{\ell k}.$$

Raising both sides of the inequality to the power of $\frac{1}{k}$, and letting k tend to infinity gives

$$\Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell) \leq \left(\frac{\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)}{\ell}\right)^{\ell}, \tag{7.5}$$

which holds by the equality

$$\lim_{k \to \infty} \sqrt[k]{\binom{\ell k}{k, \ldots, k}} = \ell^{\ell}, \quad \forall\, \ell \in \mathbb{N}. \tag{7.6}$$

If $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ for every $i = 1, \ldots, \ell$, then by Theorem 3.2,

$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \Theta(\mathsf{G}_1) + \ldots + \Theta(\mathsf{G}_\ell). \tag{7.7}$$

By (7.7), inequality (7.1) is equivalent to

$$\sqrt[\ell]{\Theta(\mathsf{G}_1) \ldots \Theta(\mathsf{G}_\ell)} \leq \frac{\Theta(\mathsf{G}_1) + \ldots + \Theta(\mathsf{G}_\ell)}{\ell}, \tag{7.8}$$

and, by the conditions for equality in the AM-GM inequality, equality holds in (7.8) if and only if (7.2) holds. Finally, all the graphs that are listed in this theorem (Theorem 7.2) satisfy the equality $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ for $i \in [\ell]$. Thus, if G_i is one of these graphs for each $i \in [\ell]$, then inequality (7.1) holds with equality if and only if the condition in (7.2) is satisfied. $\square$

Corollary 7.1. *Let* G *be a graph on* n *vertices. Then,*

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{n}, \tag{7.9}$$

with equality in (7.9) *if the graph* G *is either self-complementary and vertex-transitive, or self-complementary and strongly regular, or either a conference graph, a Latin square graph or their complements.*

Proof. By Theorem 7.2,

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{\Theta(\mathsf{G} \boxtimes \overline{\mathsf{G}})}. \tag{7.10}$$

Next, choosing the "diagonal" vertices of $\mathsf{G} \boxtimes \overline{\mathsf{G}}$ gives an independent set of size n in $\mathsf{G} \boxtimes \overline{\mathsf{G}}$. Thus,

$$\Theta(\mathsf{G} \boxtimes \overline{\mathsf{G}}) \geq \alpha(\mathsf{G} \boxtimes \overline{\mathsf{G}}) \geq n. \tag{7.11}$$

Using (7.10) and (7.11) gives

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{n}.$$

Furthermore, let G be either (1) self-complementary and vertex-transitive or (2) strongly regular or (3) a conference graph, (4) a Latin square graph or (5) any one of the complements of the graphs in (1)–(4). Then, by Theorem 7.2 and [34, Theorems 3.23, 3.26 and 3.28], inequalities (7.10) and (7.11) hold with equality, thus inequality (7.9) holds with equality. □

Remark 7.1. *Inequality* (7.9)*, together with a subset of the sufficient conditions for its equality in Corollary 7.1, were proved by N. Alon in [7, Theorem 2.1]. Our proof is more simple, relying on a different approach, and the original proof in [7] is also presented in Appendix D. In [7, Theorem 2.1], Alon showed that inequality* (7.9) *holds with equality if* G *is self-complementary and vertex-transitive; some additional alternative sufficient conditions are provided in Corollary 7.1.*

By (7.1), some additional inequalities are derived in the next two corollaries.

Corollary 7.2. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ *be simple graphs, and let* $m_1, \ldots, m_\ell \in \mathbb{N}$*, and let* $m = \sqrt[\ell]{m_1 \ldots m_\ell}$*, then*

$$\Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell) \leq (m\ell)^{-\ell}\, \Theta(m_1\mathsf{G}_1 + \ldots + m_\ell\mathsf{G}_\ell)^\ell. \tag{7.12}$$

Proof. By (2.30), Inequality (7.12) is equivalent to

$$\Theta(m_1\mathsf{G}_1 \boxtimes \ldots \boxtimes m_\ell\mathsf{G}_\ell) \leq \ell^{-\ell}\, \Theta(m_1\mathsf{G}_1 + \ldots + m_\ell\mathsf{G}_\ell)^\ell.$$

Thus, by Theorem 7.2, Inequality (7.12) holds. □

Corollary 7.3. *Let $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ be simple graphs, with some $\ell \in \mathbb{N}$, and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$ be a probability vector with $\alpha_j \in \mathbb{Q}$ for all $j \in [\ell]$. Let*

$$K(\underline{\alpha}) = \{k \in \mathbb{N} : k\alpha_j \in \mathbb{N}, \forall j \in [\ell]\}.$$

Then, for all $k \in K(\underline{\alpha})$,

$$\Theta(\mathsf{G}_1^{\alpha_1 k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{\alpha_\ell k}) \le \exp\left(-kH(\underline{\alpha})\right) \, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k, \tag{7.13}$$

where the entropy function H is given by

$$H(\underline{\alpha}) \triangleq -\sum_{j=1}^{\ell} \alpha_j \log \alpha_j.$$

Proof. Let

$$A = \operatorname{lcm}(\alpha_1 k, \ldots, \alpha_\ell k).$$

Next, for every $1 \le j \le \ell$, define

$$n_j = \frac{A}{\alpha_j k}, \tag{7.14}$$

$$a_j = \sum_{i=1}^{j} \alpha_i k, \tag{7.15}$$

and for every i, $1 \le i \le k$, let j be the index that has $a_{j-1} + 1 \le i \le a_j$, and define

$$m_i = n_j.$$

Now, by Corollary 7.2 (Note that there are k graphs in the capacity)

$$\begin{aligned}
&\Theta\left(\mathsf{G}_1^{\alpha_1 k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{\alpha_\ell k}\right) \\
&\le (mk)^{-k} \, \Theta\left(m_1 \mathsf{G}_1 + \ldots + m_{a_1} \mathsf{G}_1 + m_{a_1+1} \mathsf{G}_2 + \ldots + m_k \mathsf{G}_\ell\right)^k \\
&= (mk)^{-k} \, \Theta\left((m_1 + \ldots + m_{a_1}) \mathsf{G}_1 + \ldots + (m_{a_{\ell-1}+1} + \ldots + m_k) \mathsf{G}_\ell\right)^k \\
&= (mk)^{-k} \, \Theta(n_1 \alpha_1 k \mathsf{G}_1 + \ldots + n_\ell \alpha_\ell k \mathsf{G}_\ell)^k,
\end{aligned}$$

where

$$
\begin{aligned}
m &= \sqrt[k]{m_1 \ldots m_k} \\
&= \sqrt[k]{n_1^{\alpha_1 k} \ldots n_\ell^{\alpha_\ell k}} \\
&= \sqrt[k]{\left(\frac{A}{\alpha_1 k}\right)^{\alpha_1 k} \ldots \left(\frac{A}{\alpha_\ell k}\right)^{\alpha_\ell k}} \\
&= \frac{A}{k} \left(\prod_{i=1}^{\ell} \alpha_i^{\alpha_i k}\right)^{-\frac{1}{k}}.
\end{aligned}
$$

By (7.14) and (2.30)

$$
\begin{aligned}
\Theta(n_1 \alpha_1 k \mathsf{G}_1 + \ldots + n_\ell \alpha_\ell k \mathsf{G}_\ell)^k &= \Theta(A\mathsf{G}_1 + \ldots + A\mathsf{G}_\ell)^k \\
&= A^k \, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\Theta(\mathsf{G}_1^{\alpha_1 k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{\alpha_\ell k}) &\leq (mk)^{-k} A^k \, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k \\
&= \left(A \left(\prod_{i=1}^{\ell} \alpha_i^{\alpha_i k}\right)^{-\frac{1}{k}} \right)^{-k} A^k \, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k \\
&= \left(\prod_{i=1}^{\ell} \alpha_i^{\alpha_i k}\right) \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k \\
&= \exp\left(-kH(\underline{\alpha})\right) \, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k.
\end{aligned}
$$

□

Remark 7.2. *Let* $\alpha_j = \frac{p_j}{q_j}$ *with* $(p_j, q_j) = 1$ *for all* $j \in [\ell]$. *Then,*

$$K(\underline{\alpha}) = \operatorname{lcm}(q_1, \ldots, q_\ell)\mathbb{N}.$$

This is true because if we choose k *that is not a multiple of* $\operatorname{lcm}(q_1, \ldots, q_\ell)$ *then there exists* $i \in [\ell]$ *such that* $\alpha_i \notin \mathbb{N}$.

In analogy to Corollary 7.1, we next derive upper and lower bounds on $\vartheta(\mathsf{G} + \overline{\mathsf{G}})$.

Theorem 7.3. *For every simple graph* G *on* n *vertices*

$$\vartheta(\mathsf{G}+\overline{\mathsf{G}}) \geq 2\sqrt{n} + \frac{(\vartheta(\mathsf{G})-\sqrt{n})^2}{\vartheta(\mathsf{G})}. \tag{7.16}$$

and for every d*-regular graph with spectrum* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$*,*

$$2 + \frac{n-d-1}{1+\lambda_2} - \frac{d}{\lambda_n} \leq \vartheta(\mathsf{G}+\overline{\mathsf{G}}) \leq \frac{n(1+\lambda_2)}{n-d+\lambda_2} - \frac{n\lambda_n}{d-\lambda_n}. \tag{7.17}$$

Furthermore, the following holds:

1. *If* G *is vertex-transitive or strongly regular, then inequality* (7.16) *holds with equality.*
2. *If both* G *and* $\overline{\mathsf{G}}$ *are edge-transitive, or if* G *is strongly regular, then the right inequality in* (7.17) *holds with equality.*
3. *If* G *is strongly regular, then the left inequality in* (7.17) *holds with equality.*

Proof. Using Theorems 2.16 and 2.18, we get

$$\begin{aligned}\vartheta(\mathsf{G}+\overline{\mathsf{G}}) &= \vartheta(\mathsf{G}) + \vartheta(\overline{\mathsf{G}}) \\ &\geq \vartheta(\mathsf{G}) + \frac{n}{\vartheta(\mathsf{G})} \\ &= 2\sqrt{n} + \frac{(\vartheta(\mathsf{G})-\sqrt{n})^2}{\vartheta(\mathsf{G})}.\end{aligned} \tag{7.18}$$

In addition, by Theorem 2.18, inequality (7.18) holds with equality if G is vertex-transitive or strongly regular. Furthermore, if G is a d-regular graph, then by [31, Proposiotion 1],

$$\begin{aligned}\vartheta(\mathsf{G}+\overline{\mathsf{G}}) &= \vartheta(\mathsf{G}) + \vartheta(\overline{\mathsf{G}}) \\ &\leq \frac{n(1+\lambda_2)}{n-d+\lambda_2} - \frac{n\lambda_n}{d-\lambda_n},\end{aligned} \tag{7.19}$$

and,

$$\begin{aligned}\vartheta(\mathsf{G}+\overline{\mathsf{G}}) &= \vartheta(\mathsf{G}) + \vartheta(\overline{\mathsf{G}}) \\ &\geq 2 + \frac{n-d-1}{1+\lambda_2} - \frac{d}{\lambda_n},\end{aligned} \tag{7.20}$$

which proves (7.17). Finally, by [31, Proposition 1], if both G and $\overline{\mathsf{G}}$ are edge-transitive, or if G is strongly regular, then the right inequality in (7.19) holds with equality, and if G is strongly regular, then the left inequality in (7.19) holds with equality. $\square$

Chapter 8

On disjunctive graph products and lossless compression

In the following chapter, we address a problem in lossless data compression that can be approached through graph theory, specifically by analyzing the chromatic number of disjunctive powers of graphs. We begin by defining the disjunctive product and outlining its main properties. Next, we present the problem together with the solution proposed in [32, Section 3.4], followed by a brief discussion of the classes of graphs to which this solution applies. We then introduce a new approach that accounts for the computational complexity of obtaining the optimal solution. Finally, we provide a specialized analysis of Kneser graphs in the context of this problem.

8.1 The disjunctive product

Definition 8.1. *Let* G *and* H *be simple graphs. The disjunctive product* $\mathsf{G} * \mathsf{H}$ *is a graph whose vertex set is* $\mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$*, and two distinct vertices* $(g_1, h_1), (g_2, h_2)$ *are adjacent in* $\mathsf{G} * \mathsf{H}$ *if* $\{g_1, g_2\} \in \mathsf{E}(\mathsf{G})$ *or* $\{h_1, h_2\} \in \mathsf{E}(\mathsf{H})$*.*

Define the k-fold disjunctive power of G as

$$\mathsf{G}^{*k} \triangleq \underbrace{\mathsf{G} * \ldots * \mathsf{G}}_{k-1 \text{ disjunctive products}} . \tag{8.1}$$

Next, we bring a few results regarding the disjunctive product of graphs from [32, Section 3.4].

Theorem 8.1. *Let* G *and* H *be simple graphs, their disjunctive product* $\mathsf{G} * \mathsf{H}$ *satisfies*

$$\mathcal{I}_{\max}(\mathsf{G} * \mathsf{H}) = \mathcal{I}_{\max}(\mathsf{G}) \times \mathcal{I}_{\max}(\mathsf{H}), \tag{8.2}$$

$$\alpha(\mathsf{G} * \mathsf{H}) = \alpha(\mathsf{G})\, \alpha(\mathsf{H}), \tag{8.3}$$

$$\chi_{\mathrm{f}}(\mathsf{G} * \mathsf{H}) = \chi_{\mathrm{f}}(\mathsf{G})\ \chi_{\mathrm{f}}(\mathsf{H}), \tag{8.4}$$

$$\chi_{\mathrm{f}}(\mathsf{G})\ \chi(\mathsf{H}) \leq \chi(\mathsf{G} * \mathsf{H}) \leq \chi(\mathsf{G})\ \chi(\mathsf{H}). \tag{8.5}$$

Furthermore, the following limit exists and it satisfies

$$\inf_{k \in \mathbb{N}} \sqrt[k]{\chi(\mathsf{G}^{*k})} = \lim_{k \to \infty} \sqrt[k]{\chi(\mathsf{G}^{*k})} = \chi_{\mathrm{f}}(\mathsf{G}). \tag{8.6}$$

Corollary 8.1. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_k$ *be simple graphs, then*

$$\chi_{\mathrm{f}}(\mathsf{G}_1 * \cdots * \mathsf{G}_k) = \prod_{\ell=1}^{k} \chi_{\mathrm{f}}(\mathsf{G}_\ell), \tag{8.7}$$

$$\chi(\mathsf{G}_1 * \cdots * \mathsf{G}_k) \leq \prod_{\ell=1}^{k} \chi(\mathsf{G}_\ell). \tag{8.8}$$

Proof. A recursive use of (8.4) gives (8.7), and a recursive use of (8.5) gives (8.8). □

Finally, we prove lower and upper bounds on the chromatic number of the disjunctive product of a series of graphs.

Theorem 8.2. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_k$ *be a sequence of graphs. Then, for all* $k \in \mathbb{N}$*,*

$$\max_{\ell \in [k]} \frac{\chi(\mathsf{G}_\ell)}{\chi_{\mathrm{f}}(\mathsf{G}_\ell)} \cdot \prod_{\ell=1}^{k} \chi_{\mathrm{f}}(\mathsf{G}_\ell) \leq \chi(\mathsf{G}_1 * \cdots * \mathsf{G}_k) \tag{8.9}$$

$$\leq \prod_{\ell=1}^{k} \chi_{\mathrm{f}}(\mathsf{G}_\ell) \cdot \left(1 + \sum_{\ell=1}^{k} \ln \alpha(\mathsf{G}_\ell)\right). \tag{8.10}$$

Specifically, for every graph G*,*

$$\chi_{\mathrm{f}}(\mathsf{G})^{k-1}\, \chi(\mathsf{G}) \leq \chi(\mathsf{G}^{*k}) \leq \chi_{\mathrm{f}}(\mathsf{G})^k \Big(1 + k\, \ln \alpha(\mathsf{G})\Big), \ \forall k \in \mathbb{N}. \tag{8.11}$$

Proof. In light of the leftmost inequality in (8.5), we get

$$\begin{aligned} \max\{\chi_{\mathrm{f}}(\mathsf{G}_1)\, \chi(\mathsf{G}_2), \chi(\mathsf{G}_1)\, \chi_{\mathrm{f}}(\mathsf{G}_2)\} &\leq \chi(\mathsf{G}_1 * \mathsf{G}_2) \\ &\leq \chi(\mathsf{G}_1)\, \chi(\mathsf{G}_2). \end{aligned}$$

Then, by induction on k, we get inequality (8.9):

$$\max_{\ell \in [k]} \frac{\chi(\mathsf{G}_\ell)}{\chi_{\mathrm{f}}(\mathsf{G}_\ell)} \cdot \prod_{\ell=1}^{k} \chi_{\mathrm{f}}(\mathsf{G}_\ell) \leq \chi(\mathsf{G}_1 * \cdots * \mathsf{G}_k).$$

In light of the inequality by Lovász (1975) [43], setting the graph G to be the k-fold disjunctive product $\mathsf{H}_k = \mathsf{G}_1 * \ldots * \mathsf{G}_k$ gives

$$\chi(\mathsf{H}_k) \leq \chi_{\mathrm{f}}(\mathsf{H}_k)(1 + \ln \alpha(\mathsf{H}_k)). \tag{8.12}$$

And, from (8.3),

$$\alpha(\mathsf{H}_k) = \prod_{\ell=1}^{k} \alpha(\mathsf{G}_\ell). \tag{8.13}$$

Combining (8.7), (8.12) and (8.13) gives inequality (8.10). □

8.2 A problem in lossless data compression with side information

We show an applications of the results presented above, and our contribution to the area.

To present the problem of lossless data compression with side information, we start with a story from [32, Section 3.4].

Alice and Bob teach at a school with n students. Each time at recess, exactly $2r$ of these students form two teams of equal size to play a game of Capture the Flag. Bob is on duty for the first half of the recess, so he knows the students in each team. However, he goes for lunch before recess is over. Alice is the gym teacher, and later that day she learns which team won. Alice knows the composition of the winning team. Bob knows, on the other hand, the students in both teams but he does not know which team won the game.

Question: what is the shortest message (minimum number of bits) that Alice should send to Bob to tell him the winning team?

To answer this question we first properly formulate the story. Alice and Bob share a graph G that is constructed as follows:

1. The n students at school are labeled by $1, \ldots, n$;

2. The vertices of the graph represent the $\binom{n}{r}$ possible teams of all the r-element subsets of $[n]$.

3. Two vertices are adjacent if they represent disjoint teams, so the two endpoints of an edge in G represent teams that may play together.

The construction of G yields that $\mathsf{G} = \mathsf{K}(n, r)$ is a Kneser graph with $n, r \in \mathbb{N}$ and $n \geq 2r$.

Alice knows the vertex $v \in \mathsf{V}(\mathsf{G})$, but she does not know the edge $e \in \mathsf{E}(\mathsf{G})$ whose endpoints refer to the two teams that played the game. Bob, on the other hand, knows the edge e but he does not know which of its endpoints v represents the winning team.

Under this graph-theoretic formulation, we ask what is the minimum number of bits that Alice should send to Bob to tell him the vertex v?

Alice may take the simplest approach of communicating to Bob the label (number) of her vertex v. It requires $\lceil \log_2 |\mathsf{V}(\mathsf{G})| \rceil$ bits. This does not take, however, the side information that is available to Bob where he knows the edge e in G.

In the book (see [32, Section 3.4]), an optimal solution is presented.

Proposition 8.1. *The minimum required number of bits to let Bob determine the vertex v is $\lceil \log_2 \chi(\mathsf{G}) \rceil$.*

Proof. [32, Section 3.4] First, Alice and Bob can precolor the vertices of the Kneser graph $\mathsf{G} = \mathsf{K}(n, r)$ using $\chi(\mathsf{G}) = n - 2r + 2$ colors. When Alice wants to inform Bob about the vertex v that she knows, she only needs to tell him the color of that vertex. This is sufficient for Bob to recover the vertex v since the edge e is known to Bob, and only one of its two endpoints can be assigned that given color.

If fewer than $\lceil \log_2 \chi(\mathsf{G}) \rceil$ bits are communicated from Alice to Bob, then there is a pair of adjacent vertices having the same representation of bits. Thus, if Bob holds that edge, he won't be able to decide to which vertex Alice refers. Hence, the minimum required number of bits to let Bob determine the vertex v is indeed $\lceil \log_2 \chi(\mathsf{G}) \rceil$. $\square$

Example 8.1. *Let $n = 250$ and $r = 100$. The simplest approach, where Alice sends the label of the vertex v, requires $\left\lceil \log_2 \binom{n}{r} \right\rceil = 239$ bits, while the optimal approach only requires $\lceil \log_2 (n - 2r + 2) \rceil = 6$ bits, which is a significant improvement.*

Next, suppose that Alice knows a list of vertices $v_1, \ldots, v_k$ for some $k \in \mathbb{N}$, and Bob knows a list of edges $e_1, \ldots, e_k$ with $v_i \in e_i$ for all $i \in [k]$. If Alice wishes to communicate her entire list of vertices to Bob, she can do so in k separate messages, for a total of $k\lceil \log_2 \chi(\mathsf{G})\rceil$ bits.

This poses another question: Is it possible for Alice to communicate her entire list of vertices to Bob with a smaller number of bits by a use of a single message (instead of k separate messages for each vertex)?

To answer this question, we make the following observation. Given a list of edges, $(e_1, \ldots, e_k)$, that Bob has, the list of vertices that Alice has, $(v_1, \ldots, v_k)$, can only be composed of the endpoints of $(e_1, \ldots, e_k)$, which correspond to 2^k optional list of vertices. If $(v_1, \ldots, v_k)$ and $(v'_1, \ldots, v'_k)$ are two such different vertices, then $\{v_i, v'_i\} = e_i \in \mathsf{E}(\mathsf{G})$ or $v_i = v'_i$ for every $i \in [k]$. Hence, the 2^k lists of vertices that Alice may hold form a clique in the k-fold strong power graph G^k. Thus, by precoloring the k-fold strong power of G and sharing it by Alice and Bob, it suffices for Alice to send the color of her vertex (list) to Bob. This will enable him to determine the vertex that Alice holds.

Using the strong power of G allows the reduction of required bits in a single message to Bob once every k days from $k\lceil \log_2 \chi(\mathsf{G})\rceil$ to $\lceil \log_2 \chi(\mathsf{G}^k)\rceil$ (To that end, Alice collects the list $(v_1, \ldots, v_k)$ that is a vertex of G^k, and a vertex precoloring of the power graph G^k is shared by Alice and Bob as common information). However, since G^k is a spanning subgraph of G^{*k}, then the 2^k possible vertices that Alice may hold also form a clique in the k-fold disjunctive power of G, and the precoloring of G^{*k} can be the common information between Alice and Bob.

Even though the disjunctive power of G offers a sub-optimal solution compared to the strong product of G, because

$$\lceil \log_2 \chi(\mathsf{G}^k)\rceil \leq \lceil \log_2 \chi(\mathsf{G}^{*k})\rceil, \tag{8.14}$$

in this analysis, we chose to use the G^{*k} as an approximation, because the disjunctive product has important tensorization properties, as can be seen in theorems 8.1 and 8.2. Thus, we use the disjunctive power of G and the required bits in a single message to Bob once every k days is $\lceil \log_2 \chi(\mathsf{G}^{*k})\rceil$.

In addition, by letting k tend to infinity, the number of required bits per game is reduced from $\lceil \log_2 \chi(\mathsf{G})\rceil$ to

$$\lim_{k\to\infty} \frac{\lceil \log_2 \chi(\mathsf{G}^{*k})\rceil}{k} = \log_2 \chi_{\mathrm{f}}(\mathsf{G}). \tag{8.15}$$

Example 8.2. *For $\mathsf{G} = \mathsf{K}(n,r)$, the amount of transmitted information per game is reduced from $\lceil \log_2(n-2r+2) \rceil$ to $\log_2 \frac{n}{r}$ bits.*

For example, for $n = 250$ and $r = 100$, the amount of information per game is reduced from 6 to 1.322 bits for $k = 1$ and $k \to \infty$, respectively.

Remark 8.1. *In (8.15), it was shown that asymptotically, using of the disjunctive power graph G^{*k} as the shared graph between Alice and Bob results in the value $\log_2 \chi_{\mathrm{f}}(\mathsf{G})$ as the minimal number of bits required for the transmission. Specifically, for $\mathsf{G} = \mathsf{K}(n,r)$ $(n \geq 2r > 1)$, the required number of bits is $\log_2 \frac{n}{r}$ (as seen in Example 8.2).*

If the strong power of $\mathsf{G} = \mathsf{K}(n,r)$ is used instead, then by Theorem 2.21, and since G^k is a spanning subgraph of G^{*k},

$$\begin{aligned} \frac{\lceil \log_2 \chi_{\mathrm{f}}(\mathsf{G}^k) \rceil}{k} &\leq \frac{\lceil \log_2 \chi(\mathsf{G}^k) \rceil}{k} \\ &\leq \frac{\lceil \log_2 \chi(\mathsf{G}^{*k}) \rceil}{k}. \end{aligned} \tag{8.16}$$

Kneser graphs are vertex-transitive, and the strong power of a vertex-transitive graph is also vertex transitive. So, by Theorem 2.20,

$$\chi_{\mathrm{f}}(\mathsf{G}^k) = \frac{|\,\mathsf{V}(\mathsf{G})|^k}{\alpha(\mathsf{G}^k)}, \tag{8.17}$$

which gives,

$$\begin{aligned} \lim_{k \to \infty} \frac{\lceil \log_2 \chi_{\mathrm{f}}(\mathsf{G}^k) \rceil}{k} &= \lim_{k \to \infty} \log_2 \sqrt[k]{\chi_{\mathrm{f}}(\mathsf{G}^k)} \\ &= \lim_{k \to \infty} \log_2 \sqrt[k]{\frac{|\,\mathsf{V}(\mathsf{G})|^k}{\alpha(\mathsf{G}^k)}} \\ &= \log_2 \frac{|\,\mathsf{V}(\mathsf{G})|}{\Theta(\mathsf{G})} = \log_2 \frac{n}{r}. \end{aligned} \tag{8.18}$$

Thus, combining (8.15), (8.16) and (8.18) gives

$$\lim_{k \to \infty} \frac{\lceil \log_2 \chi(\mathsf{G}^k) \rceil}{k} = \log_2 \frac{n}{r}. \tag{8.19}$$

Hence, if G is a Kneser graph, both the disjunctive power and the strong power result in the same asymptotic solution.

8.3 Graphs classification

Example 8.2 shows that Kneser graphs offer a significant gain in the reduction of the minimum amount of information bits per game that Alice needs to transmit to Bob by letting her communicate a list of vertices in a single message, while taking advantage of the fact that Bob knows a list of edges where each vertex in Alice's list is an endpoint of the corresponding edge in Bob's list.

This raises the question for which graphs such a reduction in the amount of delivered information per vertex is achievable?

We consider the family of vertex-transitive graphs, which includes the Kneser graphs.

Theorem 8.3. *Let G be a vertex-transitive graph. Then, by the proposed approach, there is no reduction in the amount of information (in bits) for the transmission of a list of vertices if and only if the vertex set of G can be partitioned into equal-sized color classes of size $\alpha(\mathsf{G})$.*

Proof. By the assumption that G is vertex-transitive

$$\chi_{\mathrm{f}}(\mathsf{G}) = \frac{|\,\mathsf{V}(\mathsf{G})|}{\alpha(\mathsf{G})} \leq \chi(\mathsf{G}),$$

with equality if and only if $\chi(\mathsf{G})\,\alpha(\mathsf{G}) = |\,\mathsf{V}(\mathsf{G})|$. This is equivalent to the condition that the vertex set $\mathsf{V}(\mathsf{G})$ can be partitioned into equal-sized color classes (independent sets) of cardinality $\alpha(\mathsf{G})$. □

8.4 Finite-length analysis

The approach presented so far to minimize the required amount of bits, allowed k to tend to infinity, but this adds a different problem, which is the computational complexity involved in the precoloring of G^{*k}. The information of the coloring of G^{*k} needs to be stored jointly by Alice and Bob, which complexity-wise, is impractical for large k (let alone $k \to \infty$).

We next provide a refined analysis, which is valid for all $k \in \mathbb{N}$, enabling to study the reduction in the required number of bits as a function of k, especially for (small) values of k for which the computational complexity at the preprocessing stage of precoloring G^{*k} is still feasible.

Theorem 8.4. *Let*

$$\mathrm{R}_k(\mathsf{G}) \triangleq \frac{\lceil \log_2 \chi(\mathsf{G}^{*k}) \rceil}{k}, \quad k \in \mathbb{N} \tag{8.20}$$

be the number of information bits per game, for a list of k games. Then,

$$\begin{aligned} \log_2 \chi_{\mathrm{f}}(\mathsf{G}) + \frac{1}{k} \log_2 \left(\frac{\chi(\mathsf{G})}{\chi_{\mathrm{f}}(\mathsf{G})} \right) &\leq \mathrm{R}_k(\mathsf{G}) && (8.21) \\ &\leq \log_2 \chi_{\mathrm{f}}(\mathsf{G}) + \frac{\log_2(1 + k \ln \alpha(\mathsf{G})) + 1}{k}, && (8.22) \end{aligned}$$

for every $k \in \mathbb{N}$.

Proof. This follows from the upper and lower bounds on $\chi(\mathsf{G}^{*k})$, with $k \in \mathbb{N}$, as they are given in Theorem 8.2. $\square$

Corollary 8.2. *If $\chi(\mathsf{G}) > \chi_{\mathrm{f}}(\mathsf{G})$, then*

$$\log_2 \chi_{\mathrm{f}}(\mathsf{G}) + O\Big(\frac{1}{k}\Big) \leq \mathrm{R}_k(\mathsf{G}) \leq \log_2 \chi_{\mathrm{f}}(\mathsf{G}) + O\Big(\frac{\log(k)}{k}\Big). \tag{8.23}$$

The following is a specialized analysis for Kneser graphs.

Example 8.3. *Let $\mathsf{G} = \mathsf{K}(n, r)$ be a Kneser graph with $n \geq 2r$, and let $\mathrm{R}_k(\mathsf{G})$ be defined as in Theorem 8.4. Then*

$$\begin{aligned} &\log_2 \binom{n}{r} + \frac{\log_2 \left(r \Big(1 - \frac{2(r-1)}{n} \Big) \right)}{k} \\ &\leq \mathrm{R}_k(\mathsf{G}) && (8.24) \\ &\leq \log_2 \binom{n}{r} + \frac{1}{k} \log_2 \left(k n H_b\Big(\frac{r}{n}\Big) + \frac{k}{2} \ln \left(\frac{r}{2\pi n (n-r)} \right) + 1 \right) + \frac{1}{k}, \end{aligned}$$

for all $k \in \mathbb{N}$, where $H_b \colon [0,1] \to [0, \ln 2]$ is the binary entropy function on base e, i.e.,

$$H_b(x) = \begin{cases} -x \ln x - (1-x) \ln(1-x), & 0 < x < 1, \\ 0, & x \in \{0, 1\}. \end{cases} \tag{8.25}$$

We now prove these bounds. For all $r, n \in \mathbb{N}$ such that $r < n$,

$$\begin{aligned}\sqrt{\frac{n}{8r(n-r)}}\exp\left(nH_b\Big(\frac{r}{n}\Big)\right) &\leq \binom{n}{r}\\ &\leq \sqrt{\frac{n}{2\pi r(n-r)}}\exp\left(nH_b\Big(\frac{r}{n}\Big)\right).\end{aligned}$$

The independence number of Kneser graphs is

$$\alpha(\mathsf{G}) = \binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}.$$

Thus, there exists $c \triangleq c_{n,r} \in \left(\frac{1}{8}, \frac{1}{2\pi}\right)$ such that

$$\ln \alpha(\mathsf{G}) = nH_b\Big(\frac{r}{n}\Big) + \frac{1}{2}\ln\Big(\frac{cr}{n(n-r)}\Big). \tag{8.26}$$

Furthermore, by Theorem 2.22,

$$\chi(\mathsf{G}) = n - 2r + 2, \quad \chi_{\mathrm{f}}(\mathsf{G}) = \frac{n}{r}. \tag{8.27}$$

Combining (8.11), (8.26) *and* (8.27) *implies that, for all $k \in \mathbb{N}$,*

$$\begin{aligned}\Big(\frac{n}{r}\Big)^k r\Big(1 - \frac{2(r-1)}{n}\Big) &\leq \chi(\mathsf{G}^{*k}) \qquad (8.28)\\ &\leq \Big(\frac{n}{r}\Big)^k\left[knH_b\Big(\frac{r}{n}\Big) + \frac{k}{2}\ln\Big(\frac{r}{2\pi n(n-r)}\Big) + 1\right].\end{aligned}$$

These exponential bounds (in k) quantify the convergence rate of the sequence $\{\sqrt[k]{\chi(\mathsf{G}^{*k})}\}_{k=1}^{\infty}$ to its asymptotic limit, $\chi_{\mathrm{f}}(\mathsf{G}) = \frac{n}{r}$, as we let k tend to infinity. Finally, substituting (8.28) *in* $\mathrm{R}_k(\mathsf{G}) \triangleq \frac{\lceil \log_2 \chi(\mathsf{G}^{*k}) \rceil}{k}$ *gives* (8.24).

Chapter 9

Summary and outlook

9.1 Summary

In this thesis, we study several research directions on the Shannon capacity of graphs. We establish sufficient conditions under which the Shannon capacity of a polynomial of graphs equals the corresponding polynomial of the individual capacities, thereby simplifying their evaluation. Exact values and new bounds are derived for two graph families: the q-Kneser graphs and the Tadpole graphs. We also construct graph families whose Shannon capacity is not attained by the independence number of any finite power, including a countably infinite family of connected graphs with this property. Furthermore, we prove an inequality relating the Shannon capacities of the strong product of graphs and their disjoint union, which yields streamlined proofs of known bounds.

In Chapter 3, we present two sufficient conditions on a sequence of graphs under which, for every polynomial in these graphs, the Shannon capacity of the polynomial equals the polynomial of the individual capacities. Building on Schrijver's recent result [24], we show that these conditions apply to all graph polynomials.

In Chapter 4, we study the Shannon capacity of Tadpole graphs. In Theorem 4.1, we determine the exact capacity in some cases and prove that, in the remaining cases, it equals the capacity of an odd cycle plus a natural number. This provides a formula that connects the capacity of Tadpole graphs to that of odd cycles, enabling sharper bounds by leveraging known results for the latter.

In Chapter 5, we address the attainability of the Shannon capacity, extending the work of Guo and Watanabe (1990). We construct two families of graphs whose Shannon capacity is not realized by the independence number of any finite strong power. In particular, we provide a countably infinite family of connected graphs with this property, which is the first known family of its kind.

In Chapter 6, we determine the exact Shannon capacity of q-Kneser graphs, using both a generalized Erdős–Ko–Rado theorem for finite vector spaces [25] and the known spectrum of q-Kneser graphs [26]. This yields a new class of graphs with explicitly determined capacity.

In Chapter 7, we prove an inequality between the Shannon capacity of the strong product of a sequence of graphs and the capacity of their disjoint union. This result enables a simpler proof of a lower bound for the capacity of the disjoint union of a graph and its complement, originally established by N. Alon (1998).

In Chapter 8, we propose a new approach to a problem in communication theory concerning lossless data compression, discussed in [32, Section 3.4]. Our method trades a slight loss in accuracy for a substantial reduction in computational complexity compared to the original solution, which relied on an infeasible precoloring step.

Finally, Chapter 9.2 suggests directions for further research that naturally build on the results of this thesis.

In summary, this thesis provides:

1. Sufficient conditions ensuring that the Shannon capacity of any graph polynomial equals the polynomial of the individual capacities.

2. Exact values and bounds for Tadpole graphs, including a formula linking a subfamily of them to odd cycles.

3. Two families of graphs—one infinite and connected—whose capacity is not attained by any finite power.

4. The exact capacity of q-Kneser graphs.

5. An inequality between the capacities of graph products and disjoint unions, yielding a simplified proof of a result of Alon.

6. Finite-length analysis of a graph-theoretic approach for solving a problem in lossless compression with side information. The novelty in the finite-length analysis is that it provides a quantitative tradeoff between

the computational complexity in the lossless compression scheme and the associated compression rate.

7. An outlook with several promising directions for further research directly related to these findings.

9.2 Outlook

This section suggests some potential directions for further research that are related to the findings in this thesis.

1. In Theorem 5.2, a construction of graphs was provided whose Shannon capacity is not attained by the independence number of any of their finite strong powers. The proof relied on Dedekind's lemma from number theory (see Lemma 5.1), showing that the capacity of this construction equals a value whose finite powers are not natural numbers, and hence cannot correspond to the finite root of an independence number. This method of proof also applies to rational numbers that are not integers, since none of their finite powers are natural numbers either. This raises an interesting question: *Does there exist a graph whose Shannon capacity is a rational number that is not an integer?*. At present, no such graphs are known. However, if the answer is positive, it would immediately follow that these graphs also possess the property of having a Shannon capacity that is not attained by the independence number of any of their finite strong powers (thus potentially leading to a fourth approach in Section 5).

2. It was proved in [17] that if the Shannon capacity of a graph is attained at some finite power, then the Shannon capacity of its Mycielskian is strictly larger than that of the original graph. In view of the constructions presented in Section 5, which yield graph families whose Shannon capacity is not attained at any finite power, it would be interesting to determine whether this property also holds in such cases. If it does not, the graph constructions from Section 5 could provide potential candidates for a counterexample.

3. By combining Theorems 2.5 and 2.6, the equality $\alpha(\mathsf{G} \boxtimes \mathsf{H}) = \alpha(\mathsf{G})\, \alpha(\mathsf{H})$ holds for every simple graph H if $\alpha(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$. Moreover, by Theorem 2.12 and Lemma 3.2, if $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$, then $\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\, \Theta(\mathsf{H})$

holds for all H. From these results, a natural question arises: *Is it true that the equality* $\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H})$ *holds for all* H *if and only if* $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$*?*. This question was already discussed to some extent in [45], where it was shown that

$$\sup_{\mathsf{H}} \frac{\Theta(\mathsf{G} \boxtimes \mathsf{H})}{\Theta(\mathsf{H})} \leq \alpha_{\mathrm{f}}(\mathsf{G}), \tag{9.1}$$

while raising the question about the possible gap between the left and right hand sides of (9.1).

Appendix A

An original proof of Theorem 2.16

Theorem A.1. *[16] Let* G *and* H *be simple graphs. Then*

$$\vartheta(\mathsf{G} + \mathsf{H}) = \vartheta(\mathsf{G}) + \vartheta(\mathsf{H}). \tag{A.1}$$

Proof. By Theorem 2.14, let $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an orthonormal representation of $\overline{\mathsf{G}}$ and let $\mathbf{c}$ be a unit vector such that

$$\vartheta(\mathsf{G}) = \sum_{i=1}^{n} (\mathbf{c}^{\mathrm{T}} \mathbf{u}_i)^2. \tag{A.2}$$

Likewise, let $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ be an orthonormal representation of $\overline{\mathsf{H}}$ and let $\mathbf{d}$ be a unit vector such that

$$\vartheta(\mathsf{H}) = \sum_{i=1}^{r} (\mathbf{d}^{\mathrm{T}} \mathbf{v}_i)^2. \tag{A.3}$$

Assume without loss of generality that the dimensions of $\mathbf{u}_i, \mathbf{c}, \mathbf{v}_j$ and $\mathbf{d}$ are m (if the dimensions are distinct, the vectors of the lower dimension can be padded by zeros). Next, let $\mathbf{A}$ as an orthogonal matrix of order $m \times m$ such that:

$$\mathbf{A}\mathbf{d} = \mathbf{c}. \tag{A.4}$$

Such a matrix $\mathbf{A}$ on the left-hand side of (A.4), satisfying $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}_n$, exists (e.g., the householder matrix defined as $\mathbf{A} = \mathbf{I}_n - \frac{2(\mathbf{c}-\mathbf{d})(\mathbf{c}-\mathbf{d})^{\mathrm{T}}}{\|\mathbf{c}-\mathbf{d}\|^2}$ provided that

$\mathbf{c} \neq \mathbf{d}$, and $\mathbf{A} = \mathbf{I}_n$ if $\mathbf{c} = \mathbf{d}$). Let $w = (\mathbf{w}_1, \ldots, \mathbf{w}_r)$ be defined as $\mathbf{w}_i = \mathbf{A}\mathbf{v}_i$ for every $i \in [r]$. Since the pairwise inner products are preserved under an orthogonal (orthonormal) transformation and $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is an orthonormal representation of $\overline{\mathsf{H}}$, so is $(\mathbf{w}_1, \ldots, \mathbf{w}_r)$. Likewise, we get

$$\sum_{i=1}^{r} (\mathbf{c}^{\mathrm{T}} \mathbf{w}_i)^2 = \sum_{i=1}^{r} ((\mathbf{A}\mathbf{d})^{\mathrm{T}} (\mathbf{A}\mathbf{v}_i))^2 = \sum_{i=1}^{r} (\mathbf{d}^{\mathrm{T}} \mathbf{v}_i)^2 = \vartheta(\mathsf{H}). \tag{A.5}$$

Next, the representation $(\mathbf{x}_1, \ldots, \mathbf{x}_{n+r}) = (\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_r)$ is an orthonormal representation of $\overline{\mathsf{G} + \mathsf{H}}$ (since there are no additional nonadjacencies in the graph $\overline{\mathsf{G} + \mathsf{H}}$ in comparison to the disjoint union of the pairs of nonadjacent vertices in $\overline{\mathsf{G}}$ and $\overline{\mathsf{H}}$). Hence, by Theorem 2.14, the equality

$$\vartheta(\mathsf{G}) + \vartheta(\mathsf{H}) = \sum_{i=1}^{n} (\mathbf{c}^{\mathrm{T}} \mathbf{u}_i)^2 + \sum_{i=1}^{r} (\mathbf{c}^{\mathrm{T}} \mathbf{w}_i)^2 = \sum_{i=1}^{n+r} (\mathbf{c}^{\mathrm{T}} \mathbf{x}_i)^2, \tag{A.6}$$

yields the inequality

$$\vartheta(\mathsf{G}) + \vartheta(\mathsf{H}) \leq \vartheta(\mathsf{G} + \mathsf{H}). \tag{A.7}$$

Next, by Theorem 2.14, let $(\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}_1, \ldots, \mathbf{v}_r)$ be an orthonormal representation of $\overline{\mathsf{G} + \mathsf{H}}$, where the vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ correspond to the vertices of G and the vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ correspond to the vertices of H, and let $\mathbf{c}$ be a unit vector such that

$$\vartheta(\mathsf{G} + \mathsf{H}) = \sum_{i=1}^{n} (\mathbf{c}^{\mathrm{T}} \mathbf{u}_i)^2 + \sum_{i=1}^{r} (\mathbf{c}^{\mathrm{T}} \mathbf{v}_i)^2. \tag{A.8}$$

By definition, since $(\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}_1, \ldots, \mathbf{v}_r)$ is an orthonormal representation of $\overline{\mathsf{G} + \mathsf{H}}$, it follows that if i and j are nonadjacent vertices in $\overline{\mathsf{G}}$, then they are nonadjacent in $\overline{\mathsf{G} + \mathsf{H}}$ and thus, $\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0$. Hence, $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ is an orthonormal representation of $\overline{\mathsf{G}}$. Similarly, it follows that $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is an orthonormal representation of $\overline{\mathsf{H}}$. Thus, by Theorem 2.14,

$$\vartheta(\mathsf{G} + \mathsf{H}) = \sum_{i=1}^{n} (\mathbf{c}^{\mathrm{T}} \mathbf{u}_i)^2 + \sum_{i=1}^{r} (\mathbf{c}^{\mathrm{T}} \mathbf{v}_i)^2 \leq \vartheta(\mathsf{G}) + \vartheta(\mathsf{H}). \tag{A.9}$$

Combining inequalities (A.7) and (A.9) gives the equality in (2.42). $\square$

Appendix B

The original proof of Theorem 3.4

Theorem 3.4 is a direct corollary of [1, Theorem 4]. To show this, we define an adjacency-reducing mapping.

Definition B.1. *Let* G *be a simple graph, and let* $f\colon \mathsf{V}(\mathsf{G}) \to \mathcal{U}$ *be a mapping from the vertices of* G *to a subset* $\mathcal{A}$ *of* $\mathsf{V}(\mathsf{G})$*.* f *is called an adjacency-reducing mapping if for every nonadjacent vertices* $u, v \in \mathsf{V}(\mathsf{G})$*, the vertices* $f(u)$ *and* $f(v)$ *are nonadjacent as well.*

Proof. Let $\mathcal{U} = \{u_1, u_2, \ldots, u_k\}$ be a maximal independent set of vertices in G_1. By assumption, $\mathsf{V}(\mathsf{G}_1)$ can be partitioned into k cliques. Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be such cliques. Obviously every $u_i, u_j \in \mathcal{U}$ cannot be in the same clique, because they are nonadjacent. So, every clique in $\mathcal{C}$ contains exactly one vertex from $\mathcal{U}$. For simplicity, assume that $\mathcal{U}$ and $\mathcal{C}$ are ordered in a way that every $j \in [k]$ has $u_j \in C_j$. Next, define the mapping $f : \mathsf{V}(\mathsf{G}_1) \to \mathcal{U}$ as follows. Let $v \in \mathsf{V}(\mathsf{G}_1)$ be a vertex, and let C_j be the clique that has $v \in C_j$, then $f(v) = u_j$. If v_1 and v_2 are nonadjacent, then they belong to different cliques in $\mathcal{C}$. Thus $f(v_1)$ and $f(v_2)$ are mapped to different vertices in $\mathcal{U}$. Since $\mathcal{U}$ is an independent set, $f(v_1)$ and $f(v_2)$ are nonadjacent. Thus, f is an adjacency-reducing mapping of G_1 into $\mathcal{U}$. Then, by [1, Theorem 4], equality (3.5) holds. $\square$

Appendix C

The original proof of Theorem 5.1

Theorem C.1. *[10] Let* G *be a universal graph, and let* H *satisfy* $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$*. The Shannon capacity of* $\mathsf{K} \triangleq \mathsf{G} + \mathsf{H}$ *is not attained at any finite power of* K*.*

Proof. Let G be a universal graph and let $k \in \mathbb{N}$. Then, since G is universal,

$$\begin{aligned}
\alpha(\mathsf{K}^{2k}) &= \sum_{\ell=0}^{2k} \binom{2k}{\ell} \alpha(\mathsf{G}^{2k-\ell})\, \alpha(\mathsf{H}^{\ell}) \\
&= \sum_{\ell=0}^{2k} \sum_{0 \le i,j \le k; i+j=\ell} \binom{k}{i} \binom{k}{j} \alpha(\mathsf{G}^{2k-(i+j)})\, \alpha(\mathsf{H}^{i+j}) \\
&= \sum_{\ell=0}^{2k} \sum_{0 \le i,j \le k; i+j=\ell} \binom{k}{i} \binom{k}{j} \alpha(\mathsf{G})^{2k-(i+j)} \alpha(\mathsf{H}^{i+j}) \\
&= \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i} \binom{k}{j} \alpha(\mathsf{G})^{2k-i-j} \alpha(\mathsf{H}^{i+j}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\alpha(\mathsf{K}^{k})^2 &= \left(\sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{G})^{k-i} \alpha(\mathsf{H}^{i}) \right)^2 \\
&= \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i} \binom{k}{j} \alpha(\mathsf{G})^{2k-i-j} \alpha(\mathsf{H}^{i})\, \alpha(\mathsf{G}^{j}).
\end{aligned}$$

Subtracting both equalities gives

$$\alpha(\mathsf{K}^{2k}) - \alpha(\mathsf{K}^k)^2 = \sum_{i=0}^{k}\sum_{j=0}^{k}\left(\binom{k}{i}\binom{k}{j}\alpha(\mathsf{G})^{2k-i-j}\left(\alpha(\mathsf{H}^{i+j}) - \alpha(\mathsf{H}^i)\,\alpha(\mathsf{G}^j)\right)\right).$$

Since by assumption $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$, there exists $i_0, j_0 \in \mathbb{N}$ such that

$$\alpha(\mathsf{H}^{i+j}) - \alpha(\mathsf{H}^i)\,\alpha(\mathsf{G}^j) > 0.$$

Otherwise, for all $m \in \mathbb{N}$, $\alpha(\mathsf{H}^m) = \alpha(\mathsf{H})^m$, so $\Theta(\mathsf{H}) = \alpha(\mathsf{H})$ in contradiction to our assumption. Hence $\alpha(\mathsf{K}^{2k}) > \alpha(\mathsf{K}^k)^2$ for all $k \geq \max\{i_0, j_0\}$.

Suppose by contradiction that $\Theta(\mathsf{K}) = \sqrt[k_0]{\alpha(\mathsf{K}^{k_0})}$ for some $k_0 \geq \max\{i_0, j_0\}$. Then $\alpha(\mathsf{K}^{2k_0}) > \alpha(\mathsf{K}^{k_0})^2$, which gives

$$\Theta(\mathsf{K}) \geq \sqrt[2k_0]{\alpha(\mathsf{K}^{2k_0})} > \sqrt[k_0]{\alpha(\mathsf{K}^{k_0})} = \Theta(\mathsf{K}),$$

thus leading to a contradiction. Hence, it is not attained for any $k \geq \max\{i_0, j_0\}$.

Note that if we assume that $\Theta(\mathsf{K}) = \sqrt[k]{\alpha(\mathsf{K}^k)}$ for some $k < \max\{i_0, j_0\}$, then for every $n \in \mathbb{N}$

$$\Theta(\mathsf{G}) \geq \sqrt[nk]{\alpha(\mathsf{K}^{nk})} \geq \sqrt[k]{\alpha(\mathsf{K}^k)} = \Theta(\mathsf{G}),$$

and then, using the same argument for $nk \geq \max\{i_0, j_0\}$, we get a contradiction. Hence, the Shannon capacity of K is not attained for any $k \in \mathbb{N}$. $\square$

Appendix D

The original proof of Inequality (7.9)

Theorem D.1. *[7] Let* G *be a simple graph with* n *vertices. Then*

$$\Theta(\mathsf{G}+\overline{\mathsf{G}}) \geq 2\sqrt{n}. \tag{D.1}$$

In addition, inequality (D.1) *holds with equality if* G *is self-complementary and vertex-transitive.*

Proof. Let $\mathcal{A} \cup \mathcal{B} = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be the vertex set of $\mathsf{G}+\overline{\mathsf{G}}$, where $\mathcal{A}$ represents the vertices of G and $\mathcal{B}$ represents the matching vertices of $\overline{\mathsf{G}}$. Thus, if $\{a_i, a_j\} \in \mathsf{E}(\mathsf{G})$, then $\{b_i, b_j\} \notin \mathsf{E}(\overline{\mathsf{G}})$. Next, we construct an independent set in the graph $(\mathsf{G}+\overline{\mathsf{G}})^{2k}$ for every k, and we use that set to bound the capacity of $\mathsf{G}+\overline{\mathsf{G}}$. For every k, define the set $\mathcal{S}$ as the set of vectors $\mathbf{v} = (v_1, v_2, \ldots, v_{2k})$ that follows the following two rules:

- $|\{i : v_i \in \mathcal{A}\}| = |\{j : v_j \in \mathcal{B}\}|$,
- For every $1 \leq i \leq k$, if in $\mathbf{v}$, a_r is the i-th coordinate from the left that belongs to $\mathcal{A}$, and b_s is the i-th coordinate from the left that belongs to $\mathcal{B}$, then $r = s$.

Now we prove that $\mathcal{S}$ is an independent set in $(\mathsf{G}+\overline{\mathsf{G}})^{2k}$. Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vectors from $\mathcal{S}$, we consider two cases:

Case 1, if there is an index t that has $u_t \in \mathcal{A}$ and $v_t \in \mathcal{B}$ then obviously the two vertices $\mathbf{u}$ and $\mathbf{v}$ are not adjacent in $(\mathsf{G}+\overline{\mathsf{G}})^{2k}$.

Case 2, if there isn't such an index, then there are two indices $1 \le i, j \le k$, and two indices r, s between 1 and $2k$ that have $u_r = a_i$, $u_s = b_i$, and $v_r = a_j$, $v_s = b_j$. Since $\{a_i, a_j\}$ is an edge in G if and only if $\{b_i, b_j\}$ is not an edge in $\overline{\mathsf{G}}$, then the vertices $\mathbf{u}$ and $\mathbf{v}$ are not adjacent in $(\mathsf{G} + \overline{\mathsf{G}})^{2k}$. Hence, $\mathcal{S}$ is an independent set in $(\mathsf{G} + \overline{\mathsf{G}})^{2k}$. Next, the size(cardinality) of $\mathcal{S}$ is $\binom{2k}{k} n^k$, because there are $\binom{2k}{k}$ ways to choose the locations of the coordinates from $\mathcal{A}$ and that dictates the location of the coordinates from $\mathcal{B}$ as well, and, for every coordinate there are n options to choose from, which gives $|\mathcal{S}| = \binom{2k}{k} n^k$. Finally, we get

$$\alpha((\mathsf{G} + \overline{\mathsf{G}})^{2k}) \ge |\mathcal{S}| = \binom{2k}{k} n^k,$$

which gives

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \ge \lim_{k \to \infty} \sqrt[2k]{\alpha((\mathsf{G} + \overline{\mathsf{G}})^{2k})} \ge \lim_{k \to \infty} \sqrt[2k]{\binom{2k}{k} n^k} = 2\sqrt{n}.$$

In addition, from [11, Theorem 12], inequality (D.1) indeed holds with equality when G is a self-complementary and vertex-transitive graph. $\square$

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לשם מילוי חלקי של הדרישות לקבלת תואר

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הוגש לסנט הטכניון – מכון טכנולוגי לישראל

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המחקר נעשה בפקולטה להנדסת חשמל ומחשבים

בהנחיית פרופ׳ יגאל ששון

מחבר חיבור זה מצהיר כי המחקר, כולל איסוף הנתונים, עיבודם והצגתם, התייחסות והשוואה לחקרים קודמים וכו׳, נעשה כולו בצורה ישרה, כמצופה ממחקר מדעי המבוצע לפי אמות המידה האתיות של העולם האקדמי. כמו כן, הדיווח על המחקר ותוצאותיו בחיבור זה נעשה בצורה ישרה ומלאה, לפי אותן אמות מידה.

הכרת תודה

תודה רבה למנחה שלי, פרופ׳ יגאל ששון, על הנחייתו ותמיכתו הצמודה לכל אורך הדרך. למדתי ממנו הרבה מאוד על עולם המחקר ועל הכתיבה האקדמית, ובזכותו נחשפתי לתחומים מרתקים רבים בקומבינטוריקה ותורת האינפורמציה. בנוסף, אני רוצה להודות לפרופ׳ רון הולצמן ופרופ׳ יובל קסוטו שהיו בועדת הבחינה שלי ועברו על המחקר שלי ביסודיות. ולבסוף, אני רוצה להודות למשפחתי על תמיכתם בי לאורך המחקר.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי

תקציר

קיבול שאנון של גרפים (Shannon capacity) הוגדר לראשונה במאמר פורץ הדרך של שאנון (C. Shannon) בשנת 1956, ערך זה מוגדר על גרף שמייצג ערוץ, והוא שווה לקצב האינפורמציה הגדול ביותר שניתן לשלוח דרך הערוץ, באופן שמאפשר לשחזר את ההודעה במוצא ללא שגיאה. בניגוד לקיבול הקלאסי, שמאפשר שגיאה קטנה ששואפת לאפס עם גודל ההודעה, קיבול שאנון של גרף דורש תקשורת ללא שגיאה. על מנת לקשור בין קיבול של ערוץ לבין תורת הגרפים, שאנון הגדיר לכל ערוץ דיסקרטי חסר-זיכרון (discrete memoryless channel) גרף בלבול (confusion graph), שהצמתים שלו הם סימבולי הכניסה של הערוץ, ושני צמתים בגרף הבלבול מחוברים בקשת אם ורק אם הסימבולים המתאימים שלהם בערוץ יכולים להוביל לאותו פלט במוצא הערוץ בהסתברות חיובית. לכל ערוץ דיסקרטי חסר-זיכרון, ניתן להגדיר גרף בלבול מתאים, וזה מספק קשר חשוב בין בעיית הקיבול לתורת הגרפים, דבר שמאפשר לנו לחשב את הקיבול בעזרת סט הכלים הרחב של תורת הגרפים.

חישוב הקיבול שאנון של גרף מהווה בעייה קשה מאוד, ונכון להיום, נמצאו רק משפחות אחדות של גרפים שהקיבול שלהם ידוע, כמו גרפי קנסר (Kneser graphs) או גרפים מושלמים (perfect graphs). הסיבה שחישוב הקיבול היא משימה כל כך קשה היא שהקיבול מוגדר כגבול של סדרה אינסופית של מספרי יציבות (independence numbers) של גרפים בעלי גודל עולה אקספוננציאלית. בשביל לחשב מספר יציבות צריך למצוא את הקבוצה הבלתי תלויה (independent set) הגדולה ביותר בגרף, שזאת בעיה NP-hard, כלומר, שפתרונה דורש מספר פעולות אקספוננציאלי לפי גודל הגרף. אז בשביל לחשב את הקיבול של גרף, צריך לפתור סדרה אינסופית של בעיות שדורשות מספר פעולות אקספוננציאלי כתלות בגודל הגרף שגדל אקספוננציאלית עם הסדרה. לכן, מבחינת סיבוכיות החישוב של הקיבול, זאת בעיה שלא ניתן לפתור עם אלגוריתמים פשוטים או מחשבים. לכן, חוקרים רבים ניסו למצוא חסמים עבור הקיבול, ולחשב אותו בעזרתם.

במאמר של שאנון משנת 1956, הוכח שמספר היציבות הפרקטיאונלי (fractional independence number) מהווה חסם עליון עבור הקיבול, ובעזרת החסם העליון הזה ומספר היציבות שמהווה חסם תחתון לקיבול, שאנון הוכיח שלכל גרף במשפחת הגרפים המושלמים, הקיבול שווה למספר היציבות. לאחר מכן, במאמר פורץ הדרך של לובאס (Lovasz) משנת 1979, הוגדרה הפונקציית לובאס של גרפים, והוכח שהפונקצייה הזאת מהווה חסם עליון לקיבול שאנון, ועוד יותר מזה, הוכח שהיא בהכרח קטנה או שווה למספר היציבות

הפרקטיאונלי, ולכן היא מהווה חסם עליון טוב יותר עבור הקיבול. היתרון הכי גדול של פונקציית לובאס של גרפים, היא שניתן לחשב אותה בזמן פולינומיאלי כתלות בגודל הגרף, דבר שמספק לנו חסם שניתן לחישוב על הקיבול. בנוסף לחסמים על הקיבול באופן כללי, נמצאו חסמים על בניות מסוימות של גרפים, לדוגמה, אלון (N. Alon) מצא חסם התלוי בגודל בגרף, על האיחוד המופרד (disjoint union) של גרף עם הגרף המשלים שלו במאמר משנת 1998. בנוסף לקיבול שאנון, ניתן לקשור בין בעיות נוספות בתקשורת לבין בעיות בתורת הגרפים, בחיבור זה נעסוק בבעיה נוספות לקיבול, שנוגעת בדחיסה ללא איבוד מידע.

חיבור זה עוסק במגוון בעיות הנוגעות בקיבול שאנון של גרפים ותכונותיו, ובקשר נוסף בין בעיה בתקשורת לבין תורת הגרפים. ראשית, נעסוק בפולינומים של גרפים, ונמצא תנאים מספיקים על סדרה של גרפים, עבורם הקיבול של כל פולינום של סדרת הגרפים שווה לפולינום של הקיבולים של הגרפים בהתאם, דבר שמאפשר חישוב פשוט יותר של בניות גדולות של גרפים. שנית, נעסוק במשפחה חדשה של גרפים, גרפי טאדפול (Tadpole graphs), נמצא את הקיבול המדויק שלהם במקרים מסוימים, ונמצא חסמים עבור הקיבול במקרים אחרים, בנוסף, נוכיח נוסחה הקושרת בין הקיבול של תת-משפחה של גרפי טאדפול לבין הקיבול של גרפי לולאה מסדר אי-זוגי. לאחר מכן, נבנה שתי משפחות רחבות חדשות של גרפים, שהקיבול שלהם לא מתקבל על ידי מספר היציבות של אף אחד מהחזקות החזקות (strong powers) הסופיות שלהם, ובתוך זה, נמצא משפחה אינסופית של גרפים קשירים המקיימים תכונה זו, שלמיטב ידיעתנו, זאת המשפחה הראשונה מהסוג הזה שנמצאה. בנוסף, נמצא את הקיבול המדויק של משפחת הגרפים q-Kneser, ובכך נרחיב את קבוצת הגרפים שקיבולם ידוע. לאחר מכן נוכיח אי-שוויון בין מכפלה חזקה (strong product) של סדרת גרפים ובין האיחוד המופרד שלהם, שבעזרתו ניתן הוכחה פשוטה יותר לחסם תחתון ידוע של קיבול של גרפים מובנים. לבסוף, נתמקד בבעייה בתקשורת הנוגעת בדחיסה ללא איבוד מידע, ונציג גישה חדשה לפתרונה בעזרת תורת הגרפים, המתחשבת בסיבוכיות החישוב של אלגוריתם הפענוח.