Aspects of convex optimization and concentration in coding

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Presentation outline

Main Topics

1. Concentration of measures in LDPC code ensembles
   1. Background (Doob's martingale and Azuma's inequality)
   2. concentration of conditional entropy
   3. concentration of message-passing error probability for ISI channels
2. LP decoding using convex optimization
   1. Background (LP decoding and optimization)
   2. Bounds on interior-point and Newton's method's iterations.
   3. Application of bounds to an IPM-based LP decoding
3. Summary

Concentration of measures in LDPC code ensembles

Doob's martingale

Definition - [Doob's Martingale]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$ be a monotonic sequence of sub σ-algebras of $\mathcal{F}$. A sequence $X_0, X_1, \ldots$ of random variables (RVs) is a martingale if:

1. $X_i : \Omega \to \mathbb{R}$.
2. $\{\omega \in \Omega : X_i(\omega) \leq t\} \in \mathcal{F}_i \quad \forall i, \forall t \in \mathbb{R}$.
3. $\mathbb{E}[|X_i|] < \infty$.
4. $X_i = \mathbb{E}[X_{i+1}|\mathcal{F}_i]$ almost surely.

Example - Random walk

$X_n = \sum_{i=0}^{n} U_i$ where U_i is an i.i.d. sequence of RVs with $\mathbb{E}[U_i] = 0$.

Doob's martingale- Remarks

Remark 1

Given a RV $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and an arbitrary filtration of sub σ-algebras $\{\mathcal{F}_i\}$, let

$$X_i = \mathbb{E}[X|\mathcal{F}_i] \quad i = 0, 1, \ldots .$$

Then, the sequence $X_0, X_1, \ldots$ forms a martingale.

Remark 2

One can choose

$$\mathcal{F}_0 = \{\Omega, \emptyset\}, \quad \mathcal{F}_n = \mathcal{F}$$

so that $X_0, X_1, \ldots, X_n$ is a martingale sequence where

$X_0 = \mathbb{E}[X|\mathcal{F}_0] = \mathbb{E}[X]$ (since $\mathcal{F}_0$ doesn't provide information about X).
$X_n = \mathbb{E}[X|\mathcal{F}_n] = X$ a.s. (since $\mathcal{F}_n$ provides full information about X).

Azuma-Hoeffding inequality

Theorem - **[Azuma-Hoeffding inequality]**

Let $X_0, \ldots, X_n$ be a martingale. If the sequence of differences are bounded, i.e.,

$$|X_i - X_{i-1}| \le d_i \quad \forall\, i = 1, 2, \ldots, n \quad \text{a.s.}$$

then

$$\mathbb{P}(|X_n - X_0| \ge r) \le 2 \exp\left(- \frac{r^2}{2 \sum_{i=1}^n d_i^2} \right), \quad \forall\, r > 0.$$

Theorem I - [Concentration of Conditional Entropy of LDPC code ensembles (Méasson et al. 2008)]

Let $\mathcal{C}$ be chosen uniformly at random from the ensemble LDPC(n, λ, ρ). Assume that the transmission of the code $\mathcal{C}$ takes place over an MBIOS channel. Let $H(\mathbf{X}|\mathbf{Y})$ designate the conditional entropy of the transmitted codeword $\mathbf{X}$ given the received sequence $\mathbf{Y}$ from the channel. Then for any $\xi > 0$,

$$Pr\left(|H(\mathbf{X}|\mathbf{Y}) - \mathbb{E}_{\mathsf{LDPC}(n,\lambda,\rho)}[H(\mathbf{X}|\mathbf{Y})]| \geq \sqrt{n}\,\xi\right) \leq 2\exp(-B\xi^2)$$

where $B \triangleq \frac{1}{2(d_{\mathsf{c}}^{\max}+1)^2(1-R_d)}$, $d_{\mathsf{c}}^{\max}$ is the maximal check-node degree, and R_{d} is the design rate of the ensemble.

Proof - **[outline]**

1. Introduction of a martingale sequence with bounded differences
   - Define the RV $Z = H_{\mathcal{G}}(\mathbf{X}|\mathbf{Y})$, where $\mathcal{G}$ is a graph of a code chosen uniformly at random from the ensemble LDPC(n, λ, ρ)
   - Define the martingale sequence $Z_t = \mathbb{E}[Z|\text{first } t \text{ parity check equations are revealed}] \quad t \in \{0, 1, \ldots, m\}$.
2. Upper bounds on the differences $|Z_{t+1} - Z_t|$
   - Show that $|Z_{t+1} - Z_t| \leq (r+1)\, H(\tilde{X}|\mathbf{Y})$, where r is the degree of the parity-check equation revealed at time t, and $\tilde{X} = X_{i_1} \oplus \ldots \oplus X_{i_r}$ is the modulo-2 sum of some r bits in the codeword $\mathbf{X}$.
   - Bound r by $d_{\mathrm{c}}^{\max}$ (the maximal parity-check node degree).
   - Bound $H(\tilde{X}|\mathbf{Y})$ by 1 (since $\tilde{X}$ is a bit).
3. Apply Azuma's inequality by using $|Z_{t+1} - Z_t| \leq d_{\mathrm{c}}^{\max} + 1$ for every $t = 0, \ldots, m-1$ where $m = n(1 - R_{\mathrm{c}})$ is the number of parity-check nodes, and R_{c} is the code rate.

Improvement 1 - A tightened upper bound on the conditional entropy

Instead of upper bounding $H_{\mathcal{G}}(\tilde{X}|\mathbf{Y})$ by 1, we rely on the inequality $H_{\mathcal{G}}(\tilde{X}|\mathbf{Y}) \leq h_2\left(\frac{1-C^{\frac{r}{2}}}{2}\right)$. Further, for a BSC or BEC, this bound can be improved to $h_2\left(\frac{1-\left[1-2h_2^{-1}(1-C)\right]^r}{2}\right)$ and $1-C^r$, respectively.

proof -**[outline]**

1. Upper bound $H(\tilde{X}\big|\mathbf{Y})$ with $H(\tilde{X}\big|\mathbf{Y}) \leq H(\tilde{X}\big|Y_{i_1},\ldots,Y_{i_r}) \leq 1 - \frac{1}{2\ln 2}\sum_{p=1}^{\infty}\frac{(g_p)^r}{p(2p-1)}$ where $g_p \triangleq \int_0^{\infty} a(l)(1+e^{-l})\tanh^{2p}\left(\frac{l}{2}\right)dl, \quad \forall\, p \in \mathbb{N}$ and a denotes the symmetric *pdf* of the LLR (Sason 2009).
2. Substitute the bound $g_p \geq C^p$ and use the power series expansion of $h_2(x)$ to get an explicit bound. For the case of BSC and BEC it is known that $g_p = \left(1-2h_2^{-1}(1-C)\right)^{2p}, \quad$ and $g_p = C$ respectfully for all $p \in \mathbb{N}$, thus leading to tighter upper bounds.

Improvement 2 - A more careful consideration of the parity-check degree distribution

Instead of taking the trivial bound $r \leq d_{\mathsf{c}}^{\mathsf{max}}$ for all m terms in the Azuma's inequality, one can rely on the degree distribution of the parity-check nodes from the edge perspective. The number of parity-check nodes of degree r is $n(1-R_{\mathsf{d}})\Gamma_r$.

Theorem II - **[Tightened Expressions for B]**

Considering the terms of Theorem I. Applying the two improvements yields a tighter expressions for B.

- General MBIOS - $B \triangleq \frac{1}{2(1-R_{\mathsf{d}})\sum_{i=1}^{d_{\mathsf{c}}^{\max}}(i+1)^2\Gamma_i\left[h_2\left(\frac{1-C^{\frac{i}{2}}}{2}\right)\right]^2}$
- BSC - $B \triangleq \frac{1}{2(1-R_{\mathsf{d}})\sum_{i=1}^{d_{\mathsf{c}}^{\max}}(i+1)^2\Gamma_i\left[h_2\left(\frac{1-[1-2h_2^{-1}(1-C)]^i}{2}\right)\right]^2}$
- BEC - $B \triangleq \frac{1}{2(1-R_{\mathsf{d}})\sum_{i=1}^{d_{\mathsf{c}}^{\max}}(i+1)^2\Gamma_i\,(1-C^i)^2}$

Comparison of Theorem I Vs. Theorem II

Comparison for the limit where $C \to 1$ bit per channel use

We consider two cases

- $d_{\mathsf{c}}^{\mathsf{max}} < \infty$ - Theorem II yields $B \to \infty$ which is in contrast to Theorem I where the parameter B does not depend on C and is finite. Note that B should be indeed infinity for a perfect channel, and therefore Theorem II is tight in this case.
- $d_{\mathsf{c}}^{\mathsf{max}} = \infty$ (i.e., tornado codes)- The Value of B in Theorem I vanishes when $d_{\mathsf{c}}^{\mathsf{max}} = \infty$ and therefore is useless. On the other hand using the value of B in Theorem II , it can be shown that if $\rho'(1) < \infty$ then $B \to \infty$.

Numerical comparison for BEC and BIAWGN

Consider the $(2, 20)$ regular LDPC code ensemble and communication over a BEC or BIAWGNC with capacity of 0.98 per channel use. Compared to Theorem I applying Theorem II results in tighter expressions for B

- BIAWGN - Improvement by factor $\left[h_2\left(\frac{1-C^{\frac{d_c}{2}}}{2}\right)\right]^{-2} = 5.134$
- BEC - Improvement by factor $\frac{1}{\left(1-C^{d_c}\right)^2} = 9.051$

Comparison for Heavy-Tail Poisson Distribution (Tornado Codes)

Consider the capacity-achieving Tornado LDPC code ensemble for a BEC with erasure probability p. We wish to design a code ensemble that achieves a fraction $1-\epsilon$ of the capacity.

- Theorem I- Since $d_{\mathrm{c}}^{\mathrm{max}} = \infty$, then $B = 0$. Therefore this result is useless.
- Theorem II- B scales at least like $O\left(\frac{1}{\log^2\left(\frac{1}{\varepsilon}\right)}\right)$. This follows from
  - Stability condition - $\rho'(1)\lambda'(0)p \leq 1$.
  - $d_{\mathrm{c}}^{\mathrm{avg}}$ and $1/\lambda_2$ scales at least like $O\left(\log\left(\frac{1}{\varepsilon}\right)\right)$ (Sason 2009).

  The parameter B tends to zero slowly as we let the fractional gap ϵ tend to zero. This demonstrates a rather fast concentration.

Theorem - [Message-passing error probability for ISI channels]

Consider an ensemble of regular $(n, d_{\mathsf{v}}, d_{\mathsf{c}})$ LDPC codes transmitted over an ISI channel $y_t = \sum_{i=0}^{I} h_i x_{t-i} + n_t$.

The decoder uses the windowed sum-product algorithm with width W.

Over the probability space of all graphs and channel realizations, assume ℓ iterations passed and let

- $Z^{(\ell)}$ - Number of incorrect variable-to-check node messages.
- $p^{(\ell)}$ - Expected probability of incorrect messages passed along an edge with a tree-like directed neighborhood of depth ℓ.
- $\mathcal{N}_{\vec{e}}^{(\ell)}$ - The neighborhood of depth ℓ of an edge $\vec{e} = (\mathsf{v}, \mathsf{c})$.

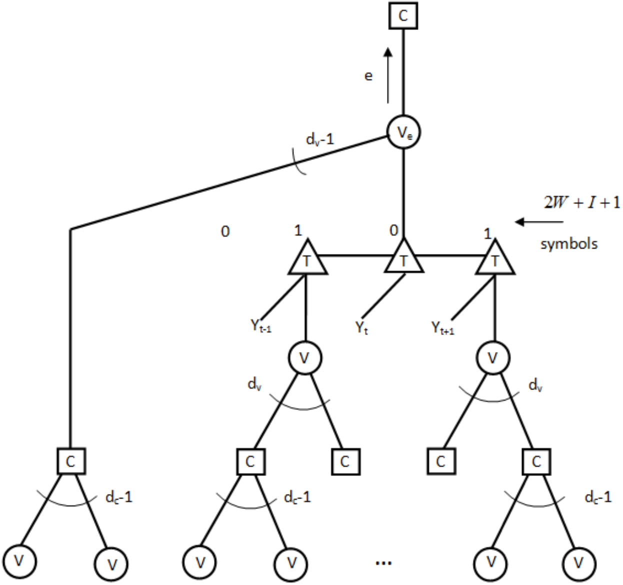
Then, there exist constants $\beta, \gamma > 0$ such that

- $\Pr\left(\mathcal{N}_{\vec{e}}^{(\ell)} \text{ not tree-like}\right) \le \frac{\gamma}{n}$
- For any $\epsilon > 0$ and $n > \frac{2\gamma}{\epsilon}$, $\Pr\left(\left|Z^{(\ell)} - n d_{\mathsf{v}} p^{(\ell)}\right| > n d_{\mathsf{v}} \epsilon\right) \le e^{-\beta \epsilon^2 n}$

Expressions for γ and β

Denote $\alpha \equiv (d_{\mathsf{v}} - 1 + 2Wd_{\mathsf{v}})(d_{\mathsf{c}} - 1)$ as the expansion factor of the graph then

1. $\gamma(d_{\mathsf{v}}, d_{\mathsf{c}}, I, W, \ell) = {N_{\mathsf{v}}^{(\ell)}}^2 + \frac{d_{\mathsf{c}}}{d_{\mathsf{v}}} {N_{\mathsf{c}}^{(\ell)}}^2$ where
   - $N_{\mathsf{v}}^{(\ell)} = 1 + [(d_{\mathsf{v}} - 1)(d_{\mathsf{c}} - 1) + 2W(1 + d_{\mathsf{v}}(d_{\mathsf{c}} - 1))] \sum_{i=0}^{\ell-1} \alpha^i$
   - $N_{\mathsf{c}}^{(\ell)} = 1 + (d_{\mathsf{v}} - 1 + 2Wd_{\mathsf{v}}) \sum_{i=0}^{\ell-1} \alpha^i$
2. $\frac{1}{\beta} = 8\left(16 d_{\mathsf{v}} (N_e^{(\ell)})^2 + (N_Y^{(\ell)})^2\right) / d_{\mathsf{v}}^2$ where
   - $N_Y^{(\ell)} = d_{\mathsf{v}}(2W + 1) \sum_{i=0}^{\ell-1} \alpha^i$
   - $N_e^{(\ell)} = 1 + d_{\mathsf{c}}(d_{\mathsf{v}} - 1 + 2Wd_{\mathsf{v}}) \sum_{i=0}^{\ell-1} \alpha^i$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Proof outline

1. [Neighborhood is tree-like with high probability] - $P_{\bar{t}}^{(\ell)} \equiv \Pr\left(\mathcal{N}_{\vec{e}}^{(\ell)} \text{not tree-like}\right) \leq \frac{\gamma}{n}$
   - We upper bound $P_t^{(\ell)} = 1 - P_{\bar{t}}^{(\ell)}$ by factorizing it as

$$P_t^{(\ell)} = \Pr\left\{\mathcal{N}_{\vec{e}}^{(0)} \text{ is tree}\right\} \prod_{\ell^*=0}^{\ell-1} \Pr\left\{\mathcal{N}_{\vec{e}}^{(\ell^*+1)} \text{ is tree} | \mathcal{N}_{\vec{e}}^{(\ell^*)} \text{ is tree}\right\}$$

   - For each factor we reveal the edges one by one and bound the probability that an exposed edge creates a cycle.
2. [Convergence of expectation to cycle-free case] - $\left|\mathbb{E}[Z^{(\ell)}] - nd_{\mathsf{v}}p^{(\ell)}\right| < nd_{\mathsf{v}}\epsilon/2.$
   - Use $Pr\left(\mathcal{N}_{\vec{e}}^{(\ell)} \text{not tree}\right) \leq \frac{\gamma}{n}$ and conditional expectation to upper bound $\left|\mathbb{E}[Z^{(\ell)}] - nd_{\mathsf{v}}p^{(\ell)}\right|$

Proof outline - (cont.)

1. [Concentration around expected value] -
   $Pr\left(\left|Z^{(\ell)} - \mathbb{E}[Z^{(\ell)}]\right| > nd_{\mathsf{v}}\epsilon/2\right) \leq e^{-\beta\epsilon^2 n}$.
   - Define a martingale sequence based on $Z^{(\ell)}$ and the revelation of the graph and the channel realization.
   - Show that revealing an edge of the graph or a received value at a particular message node has an effect on a bounded number of messages. Thus the sequence of martingale differences is bounded.
   - Apply Azuma's inequality

Remark

By setting $W = I = 0$ we can compare the results to the results for the memoryless case (Richardson and Urbanke, 2001 [3]) :

- γ - Exactly the same expression.
- β - Considerably tightened. However, the bound remains very pessimistic.

LP decoding using convex optimization

Theorem - [The ML decoder as a min-sum problem]

For any binary-input memoryless channel, the codeword of minimum cost is the Maximum-Likelihood codeword.

$$\mathbf{x}_{\mathsf{ML}} = \arg\max_{\mathbf{x}\in\mathcal{C}} \mathsf{Pr}\left[\mathbf{y}|\mathbf{x}\right] = \arg\min_{\mathbf{x}\in\mathcal{C}} \left(\sum_{i=1}^{n} \ell_i x_i\right)$$

where $\ell_i(y_i) = \ln\left(\frac{\mathsf{Pr}[y_i|x_i=0]}{\mathsf{Pr}[y_i|x_i=1]}\right)$ is the log-likelihood ratio of a code bit x_i, given the received word $\mathbf{y}$.

Complexity

The algorithm's complexity is NP, since in general, the calculation of the cost function for all 2^k possible codewords is required. Next, a relaxed presentation of a linear code presented.

Definition - [The fundamental (relaxed) polytope]

$$\mathcal{P}(H) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \text{ satisfies box and parity constraints}\}$$

Box constraints

$$\forall j \in [1, n], \quad 0 \leq x_j \leq 1$$

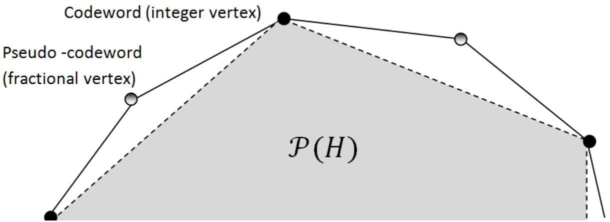
Parity constraints

$$\forall i \in [1, m], \quad \forall S \in T_i, \quad \sum_{t \in S} (1 - x_t) + \sum_{t \in A_i \smallsetminus S} x_t \geq 1$$

- $A_i = \{j \in [1, n] : h_{ij} = 1\}$, for $i \in [1, m]$, where h_{ij} is the (i, j)-element of H.
- $T_i (i \in [1, m])$ is the set of all subsets of odd size in A_i, namely $T_i = \{S \subset A_i : |S| \text{ is odd}\}$

Fundamental polytope's basic properties

- The considered fundamental polytope is a **proper polytope** (i.e., $\mathcal{P} \cap \{0,1\}^n = \mathcal{C}$), thus the **ML certificate** property holds.
- For a general linear code with distribution $\rho(x)$, the total number of inequalities is $M = 2n + m d_c^{\text{avg}} \sum_{i=1}^{d_c^{\max}} \frac{\rho_i 2^{i-1}}{i}$.
- If the row distribution $\rho(x)$ satisfies $\rho_2 = 0$, then the point $\mathbf{x}^{(0)} = (1/2, 1/2, ..., 1/2)$ is a feasible point (i.e., $\mathbf{x} \in \mathcal{P}(H)$).
- The polytope's fractional vertices result in pseudo-codewords.



Interior-point method

Convex problem (Inequality constrained)

$$\begin{cases} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, M \end{cases}$$

where $f_i(x)$ are convex

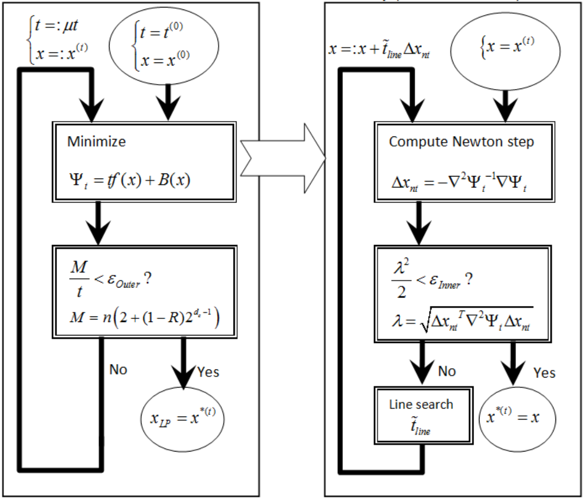
Define

$$\begin{cases} f_0(x) = \sum_{i=1}^{n} \ell_i x_i \\ B(x) = -\sum_{i=1}^{M} \log\left(-f_i(x)\right) \\ M = n(2 + (1-R)2^{d_c - 1}) \\ x^{(0)} = (1/2, 1/2, \dots, 1/2) \end{cases}$$

Search parameters

$$\begin{cases} \text{Outer}: t^{(0)}, \mu, \varepsilon_{Outer} \\ \text{Inner}: \varepsilon_{Inner}, \textit{line-search method} \end{cases}$$

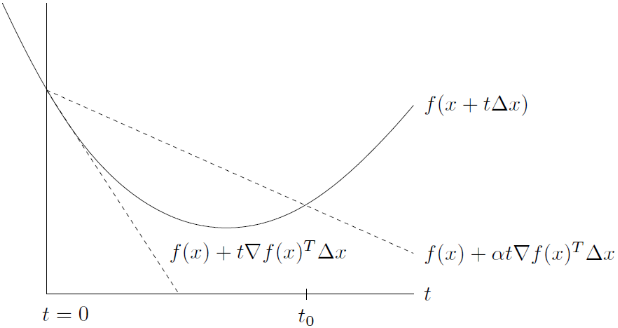
Outer loop (Logarithmic barrier method) **Inner loop** (Newton's method)



Definition - [Backtracking line-search]

Given the line-search parameters $\alpha \in (0, 0.5), \beta \in (0, 1)$, and a descent direction Δx for $f(x)$ at $x \in \textbf{dom}(f)$, initialize t_{line} with $t_{\text{line}} := 1$ and perform the following iterative algorithm :

1. If $f(x + t_{\text{line}}\Delta x) \leq f(x) + \alpha t_{\text{line}} \nabla f(x)^T \Delta x$, quit.
2. Update $t_{\text{line}} := \beta t_{\text{line}}$



Complexity analysis of the IPM

Definition - [Self-concordant function]

A convex function $f : \mathbb{R} \to \mathbb{R}$ is self-concordant (s.c.) if

$$|f^{(3)}(x)| \leq 2f''(x)^{3/2}$$

for all $x \in \mathrm{dom}(f)$.

Assuming the objective function is s.c. (which holds for our LP problem), we provide analytic bounds on the number of Newton iterations.

First step Un-constrained problems

Second step Extension to inequality-constrained problems

Theorem- **[complexity bound for un-constrained s.c. problems solved with Newton's method and backtracking line-search]**

Consider an un-constrained s.c. problem solved using Newton's method and backtracking line-search. Let $\alpha \in (0, 1/2)$ and $\beta \in (0, 1)$ be the parameters of the backtracking line-search. Let $\eta_{\max} \in (0, \frac{3-\sqrt{5}}{2})$ be a free parameter and define $\eta \equiv \min\left(\frac{1}{2}\frac{1-2\alpha}{1-\alpha}, \eta_{\max}\right)$.
Then the number of Newton iterations is upper bounded by

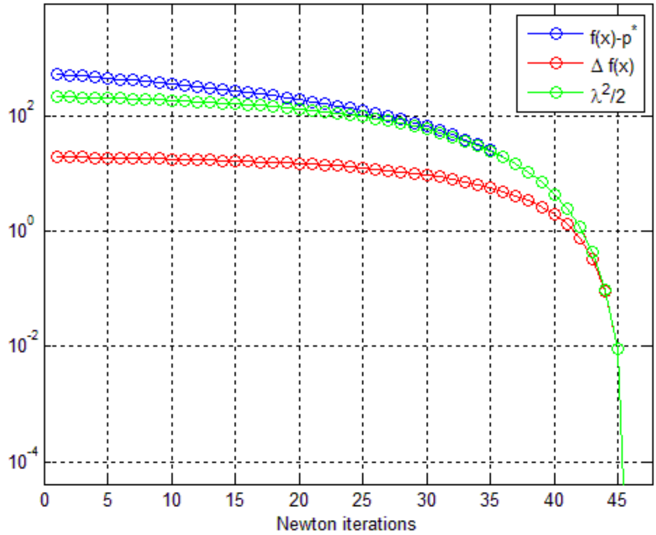
$$N_{\text{total}} \leq N_{\text{Damped}}^{\text{bound}} + N_{\text{Quad}}^{\text{bound}} = \frac{f\left(x^{(0)}\right) - p^*}{\gamma} + c$$

where

$$\gamma = \alpha\beta\eta^2$$

$$c = \left\lceil \log_2\left(\frac{\log_2(\sqrt{\varepsilon}/(1-\eta)^2)}{\log_2(\eta/(1-\eta)^2)}\right)\right\rceil.$$

Graph of typical convergence



Proof outline

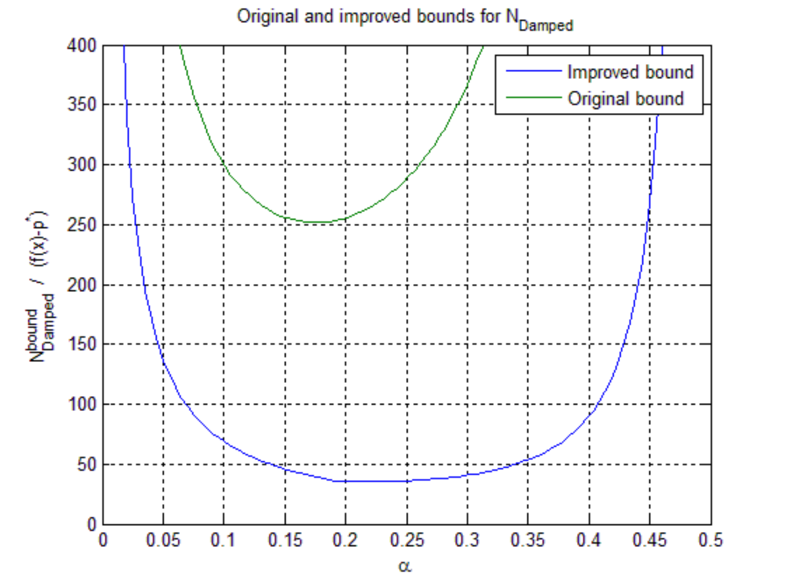
Damped phase $\lambda^{(n)} > \eta$ with slow convergence.

- Use the s.c. definition to show that $\Delta f^{(n)} \geq t^{(n)} \left(\lambda^{(n)}\right)^2 + t^{(n)}\lambda^{(n)} + \log\left(1 - t^{(n)}\lambda^{(n)}\right)$.
- Show that $t_{\text{bk}} \geq \beta \min\left(1, \frac{2(1-\alpha)}{1+2\lambda(1-\alpha)}\right)$.
- Use the backtracking condition $\Delta f^{(n)} \geq \alpha\left(\lambda^{(n)}\right)^2 t_{\text{bk}}^{(n)}$ to bound the convergence rate $\Delta f^{(n)} \geq \gamma^{(n)}(\alpha, \beta, \lambda^{(n)})$.
- Use $\lambda^{(n)} > \eta$ to globally bound $\Delta f^{(n)} \geq \gamma(\alpha, \beta, \eta)$

Quadratic phase $\lambda^{(n)} \leq \eta$, $t = 1$ with fast convergence.

- Show that if $\lambda \leq \frac{1}{2}\frac{1-2\alpha}{1-\alpha}$ then $t = 1$.
- Recursively use the bound $\lambda^{(n+1)} \leq \frac{(\lambda^{(n)})^2}{(1-\lambda^{(n)})^2} \quad t = 1, \lambda < 1.$
- Upper bound $\lambda \leq \eta_{\text{max}} < \frac{3-\sqrt{5}}{2}$ to insure monotonicity of the recursive bound.
- Count the iterations from $\lambda = \eta$ until $\lambda = \sqrt{\epsilon}$.

Comparison with previously reported results [1]



Theorem- **[complexity bound for un-constrained s.c. problems solved with Newton's method and pre-determined step-size]**

Consider an un-constrained s.c. problem solved using Newton's method. Let $\eta \in (0,1)$ be chosen arbitrarily, and consider the following pre-determined choice of the step size $t^{(n)}$:

- If $\lambda^{(n)} \geq \eta$, then $t^{(n)} = \frac{1}{1+\lambda^{(n)}}$.
- Otherwise, if $\lambda^{(n)} < \eta$, let $t^{(n)} = \arg\min_{t \in (0,1]} G(t, \lambda^{(n)})$.

Then the number of Newton iterations is upper bounded by

$$N_{\text{Total}} \leq \left(f(x^{(0)}) - p^*\right)/\gamma + c.$$

η	0.250	0.381	0.700	0.900	0.990	1.000
$1/\gamma$	37.25	17.18	5.90	3.87	3.31	3.26
c	4	5	10	35	392	∞

The coefficients are given for $\varepsilon = 10^{-10}$.

Proof outline

Damped phase $\lambda^{(n)} > \eta$ with slow convergence.

- Use the s.c. definition to show that $\Delta f^{(n)} \geq t^{(n)} \left(\lambda^{(n)}\right)^2 + t^{(n)}\lambda^{(n)} + \log\left(1 - t^{(n)}\lambda^{(n)}\right)$.
- Show that $t^{*(n)} = \frac{1}{1+\lambda^{(n)}}$ optimize the bound on Δf. $\Rightarrow \Delta f^{*(n)} \geq \lambda^{(n)} - \log\left(1 + \lambda^{(n)}\right)$.
- Use $\lambda^{(n)} > \eta$ to globally bound $\Delta f^{(n)} \geq \eta - \log(1 + \eta)$

Quadratic phase $\lambda^{(n)} \leq \eta$, with fast convergence.

- Show that $\lambda^{(n+1)} \leq G(t^{(n)}, \lambda^{(n)})\lambda^{(n)}, \lambda t < 1$.
- Optimize the recursive bound using $t^{(n)} = \arg\min_{t\in(0,1]} G(t, \lambda^{(n)})$.
- Count the iterations from $\lambda = \eta$ until $\lambda = \sqrt{\epsilon}$.

Remark : The counting reveals a new transition phase between the damped and the quadratic phases.

Bounds compared to numerical results (Backtracking + Pre-determined)

- Simulated results indicate that the number of iterations scales like $\frac{f\left(x^{(0)}\right)-p^*}{\gamma}+c$. However, the scaling factor for $f\left(x^{(0)}\right)-p^*$ is much smaller then predicted by the bounds.
- The bounds are mainly loose during the damped phase. The is mainly because $\lambda^{(n)}$ was globally bounded (i.e., $\lambda^{(n)} \geq \eta$).
- The pre-determined step-size optimize the bound, but practically it is less efficient compared to backtracking line-search. This is mainly an artifact of the domain of the bounds which is $\lambda t < 1$.

Extension of complexity bounds to inequality-constrained problems

Consider an inequality-constrained s.c. optimization problem. Assuming the problem is solved using an interior-point method (IPM) with logarithmic barrier (where the parameters of the outer iterations are set to $t^{(0)}, \mu$, and the inner iterations are performed using Newton's method), then the number of Newton iterations is upper bounded by

$$\begin{aligned} N_{\text{Total}}^{\text{Inequality}} &= N_{\text{outer}} N_{\text{inner}} + N_{\text{initial}} \\ &\leq \left\lceil \frac{\log\left(M/(\varepsilon t^{(0)})\right)}{\log \mu} \right\rceil \left(\frac{M\left(\mu - 1 - \log \mu\right)}{\gamma} + c \right) + N_{\text{initial}} \end{aligned}$$

The numbers γ and c are extracted from the bounds on the un-constrained problem (according to the line-search method used).

Complexity bound of the IPM-based LP decoder

Consider the IPM-based LP decoding algorithm. Denote $\ell_{\max}$ as an upper bound on $|\ell_i(y_i)|$. The number of Newton iterations is upper bounded by

$$N_{\mathsf{tot}} \leq \left\lceil \frac{\log\left(M/(\varepsilon t^{(0)})\right)}{\log \mu} \right\rceil \left(\frac{M\left(\mu - 1 - \log \mu\right)}{\gamma} + c \right) + \frac{1/2t^{(0)}\ell_{\max} n}{\gamma} + c$$

The numbers γ and c are chosen according to the line-search method.

The number of inequalities is given by $M = 2n + md_c^{\mathsf{avg}} \sum_{i=1}^{d_{\mathsf{c}}^{\mathsf{max}}} \frac{\rho_i 2^{i-1}}{i}$, or $M = 2n + m2^{d_{\mathsf{c}}-1} = n(2 + (1-R)2^{d_{\mathsf{c}}-1})$ for regular codes.

Optimized LP bound

An optimized bound is obtained by choosing the IPM search parameters as:

$$\mu^* = 1 + \sqrt{\frac{2c\gamma}{M}}$$
$$t^{(0)*} = \frac{\sqrt{32Mc\gamma}}{n\ell_{\max}}.$$

Assuming $M \geq \frac{c\gamma}{20}$, the bound can be simplified to

$$N^*_{total} \leq \sqrt{\frac{8cM}{\gamma}} \left[\ln \left(\sqrt{\frac{M}{32c\gamma}} \frac{\ell_{\max} n}{\epsilon} \right) + 1 \right] + c$$

Remark on the choice of parameters

Practically, good values of μ lie in the range 2-100. We would not use the value $\mu^* = 1 + \sqrt{\frac{2c\gamma}{M}}$ which is far too small.

Parametric behavior of the optimized bound

- **[Number of Inequalities -** M **]** :
  - The bound scales like $O(\sqrt{M}\ln M)$ as opposed to $O(M\ln M)$ without the optimization of t and μ.
  - Trade-off between decoding performance and decoding complexity.
- **[Block length -** n **]** :
  - **General linear codes -**
    - M scales like $O(2^{d_c})$ assuming d_c scales like $O(n)$.
    - In general the complexity is exponential in $\mathbf{n}$.
  - **LDPC codes -**
    - The bound scales like $O(\sqrt{n}\ln n)$ (since d_c is fixed).
    - The hessian matrix $\nabla^2\Psi_t(\mathbf{x})$ is sparse.
    - The total complexity scales like $\mathbf{O}(\mathbf{n}^{\mathbf{1.5}}\ln\mathbf{n})$.
- **[Check node degree -** d_c **]** :
  - Bound on the number of iterations is exponential in d_c. Therefore the complexity of capacity approaching codes (where $d_c^{\max}$ is large) is high.
  - Alternative polytopes yield lower complexity with respect to d_c.

Numerical comparison

We consider an LDPC(1008,3,6) code transmitted through an AWGN channel with SNR 2.0 dB. Moreover, for the IPM, the following parameters are assumed :
$\epsilon_{\text{Inner}} = 10^{-3}, \epsilon_{\text{Outer}} = 10^{-3}, \alpha = 0.3, \beta = 0.5, t^{(0)} = 20$ and $\mu = 20$.

Source	IPM parameters	Search method	Iterations
Simulations	$(\mu, t^{(0)}) = (20, 20)$	$(\alpha, \beta) = (0.3, 0.5)$	10^2
Original bound [2]	$(\mu, t^{(0)}) = (20, 20)$	$(\alpha, \beta) = (0.3, 0.5)$	10^8
Tightened bound	$(\mu, t^{(0)}) = (20, 20)$	$(\alpha, \beta) = (0.3, 0.5)$	10^7
Tightened bound	$(\mu, t^{(0)}) = (20, 20)$	Pre-determined t	10^6
Tightened bound	Optimized	Pre-determined t	10^4

Summary

Concentration

- We tightened a concentration inequality for the conditional entropy.
- The improved inequality enables to prove concentration in the case of Tornado codes, where the original bound was useless.
- We provided explicit expressions for the concentration rate of erroneous messages for ISI channels.
- The new bounds, particularized to MBIOS channels tighten known results.

LP decoding

- Tightened complexity bounds on the number of Newton iterations are given for several line-search methods.
- The bounds were applied to an IPM-based LP decoder.
- The behavior of the LP decoder's complexity is investigated.

Related papers

1. I. Sason and R. Eshel “On concentration of measures for LDPC code ensembles,” *Proceedings 2011 IEEE International Symposium on Information Theory (ISIT 2011)*, pp. 1268–1272, St. Petersburg, Russia, July 31–Aug 5, 2011
2. C. Méasson, A. Montanari and R. Urbanke, "Maxwell construction: The hidden bridge between iterative and maximum a-posteriori decoding," *IEEE Trans. on Information Theory*, vol. 54, pp. 5277–5307, December 2008.
3. T. Richardson and R. Urbanke, "The capacity of low-density parity check codes under message-passing decoding." *IEEE Trans. on Information Theory*, vol. 47, pp. 599–618, February 2001.
4. A. Kavcic, X. Ma and M. Mitzenmacher, "Binary intersymbol interference channels: Gallager bounds, density evolution, and code performance bounds," *IEEE Trans. on Information Theory*, vol. 49, no. 7, pp. 1636-1652, July 2003.
5. S. Boyd and L. Vanderberghe, *Convex Optimization*, Cambridge Press, 2004.
6. T. Wadayama, "An LP decoding algorithm based on primal path-following interior point method," *Proceedings 2009 IEEE International Symposium on Information Theory*, pp. 389–394, Seoul, South Korea, June 28 - July 3, 2009.