Gallager-Type Bounds for Parallel Channels with Applications to Modern Coding Techniques

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Outline

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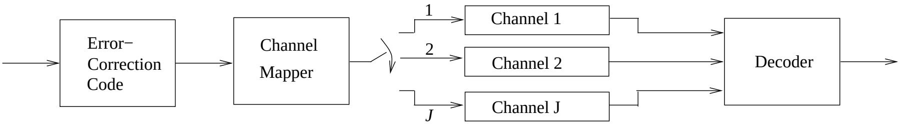
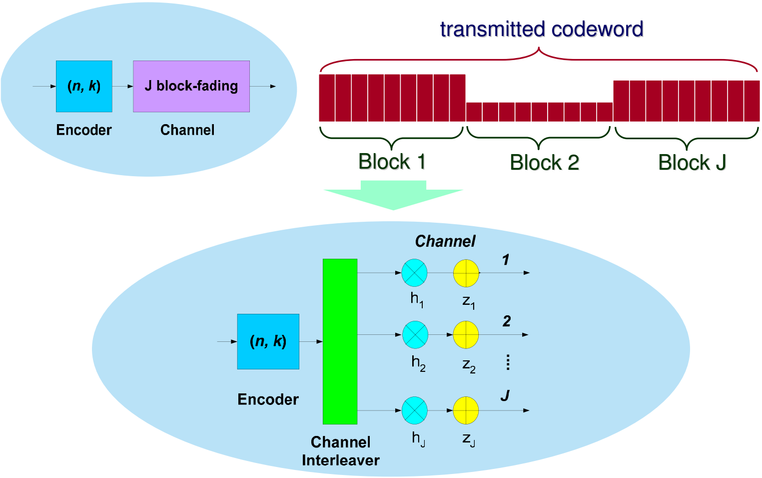
General Model of Parallel Channels

- Transmission takes place over a set of J independent parallel channels.
- *Channel mapper* maps encoded bits into a set of parallel channels.
- Each bit is mapped to one and only one of these channels.



Example: Block Fading



Other Applications of the Parallel Channel Model

- Codes employing non-uniform error protection.
- Bit-interleaved coded modulation.
- Multi-carrier systems (OFDM).
- Rate-compatible puncturing of turbo-like codes.

etc.

Upper Bounds on ML Decoding Error Probability

Single channel:

- Bounding techniques for the performance of error-correcting codes under maximum-likelihood (ML) decoding were addressed by various researchers (see, e.g., Sason-Shamai, FnT Tutorial, 2006).
- These bounds rely on the *distance spectrum* of the code, and in the asymptotic case where the block length tends to infinity, they provide bounds on decoding thresholds.

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Parallel channels:

- Performance bounds enable to assess the ML decoding error probability of error-correcting codes over these channels.
- In the asymptotic case, they provide attainable channel regions for reliable communications over parallel channels.

Outline

Concept of Derivation of the DS2 Bound (1)

- The classic 1965 Gallager bound is given by

$$P_{e|m} \leq \sum_{\underline{y}} p_n\left(\underline{y}|\underline{x}^m\right)\left(\sum_{m' \neq m}\left(\frac{p_n(\underline{y}|\underline{x}^{m'})}{p_n(\underline{y}|\underline{x}^{m})}\right)^{\lambda}\right)^{\rho} \qquad \lambda, \rho \geq 0$$

Concept of Derivation of the DS2 Bound (2)

- From the 1965 Gallager bound, multiplying and dividing by an arbitrary probability tilting measure $\Psi_n^{(m)}(\underline{y})$ yields

$$P_{e|m} \leq \sum_{\underline{y}} \Psi_n^{(m)}(\underline{y}) \left(\Psi_n^{(m)}(\underline{y})^{-\frac{1}{\rho}} p_n \left(\underline{y}|\underline{x}^m\right)^{\frac{1}{\rho}} \sum_{m' \neq m} \left(\frac{p_n(\underline{y}|\underline{x}^{m'})}{p_n(\underline{y}|\underline{x}^m)} \right)^{\lambda} \right)^{\rho}$$

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- Applying Jensen's inequality gives the DS2 bound

$$P_{e|m} \leq \left(\sum_{m' \neq m} \sum_{\underline{y}} \Psi_n^{(m)}(\underline{y})^{1-\frac{1}{\rho}} p_n(\underline{y}|\underline{x}^m)^{\frac{1}{\rho}} \left(\frac{p_n(\underline{y}|\underline{x}^{m'})}{p_n(\underline{y}|\underline{x}^m)} \right)^{\lambda} \right)^{\rho}, \quad \begin{array}{c} 0 \leq \rho \leq 1 \\ \lambda \geq 0 \end{array}$$

Concept of Derivation of the DS2 Bound (3)

- For memoryless binary-input output-symmetric (MBIOS) channels, the probability density function satisfies

$$p_n(\underline{y}|\underline{x}) = \prod_{i=1}^{n} p(y_i|x_i)$$

and we also assume a probability tilting measure which can be factorized as follows:

$$\Psi_n(\underline{y}) = \prod_{i=1}^{n} \psi(y_i).$$

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- As a result, the DS2 bound is expressed in terms of single-letter functions and the distance spectrum of the code.
- Let $\{A_h\}_{h=0}^{n}$ be the distance spectrum where A_h denotes the number of codewords of Hamming weight h.

The DS2 Bound for a Single MBIOS Channel

Theorem (Sason and Shamai, IEEE IT. Dec. 2002)

The ML decoding error probability of a binary linear block code, transmitted over a single MBIOS channel is upper bounded by

$$P_{\mathrm{e}} \leq \left\{ \sum_{h=1}^{n} A_h \left(\sum_{y} \psi(y)^{1-\frac{1}{\rho}} p(y|0)^{\frac{1}{\rho}} \right)^{n-h} \left(\sum_{y} \psi(y)^{1-\frac{1}{\rho}} p(y|0)^{\frac{1-\lambda\rho}{\rho}} p(y|1)^{\lambda} \right)^{h} \right\}^{\rho} \qquad \begin{array}{c} 0 \leq \rho \leq 1 \\ \lambda \geq 0 \end{array} .$$

- The function $\psi(\cdot)$ is a probability tilting measure, subject to optimization so as to minimize the upper bound.

Applying the DS2 Bound to Code Ensembles

- The DS2 bound applies also to structured ensembles of codes.
- Replace the distance spectrum $\{A_h\}$ with the *average* distance spectrum in the bound.
- This result applies due to Jensen's inequality

$$\mathbf{E}(x^\rho) \leq (\mathbf{E}x)^\rho \quad 0 \leq \rho \leq 1$$

On the Calculation of the DS2 Bound

- When the DS2 bound is taken over the whole code, the optimized tilting measure depends on the *entire* distance spectrum.
- When channel conditions are good, the bound *can be tightened* by adding degrees of freedom to the optimization.
- This is accomplished by
  - Recognizing that for a linear code transmitted over a symmetric channel, $P_{\mathrm{e}} = P_{\mathrm{e}|0}$.
  - Partitioning the code into constant Hamming-weight subcodes and calculate the optimized tilting measure for the DS2 bound of each subcode separately.
  - Using union bound over these subcodes, i.e.,

$$P_{\mathrm{e}} \leq \sum_{h=0}^{n} P_{\mathrm{e}|0}(h).$$

The 1961 Gallager Bound and its Connection to the DS2 Bound

- A similar bounding technique was used by Gallager (in 1961) to obtain an upper bound on the ML decoding error probability.
- Divsalar (JPL 1999) and Sason & Shamai (IEEE IT. Dec. 2002) and have demonstrated that this bound is a *special case* of the DS2 bound; thus *the DS2 bound is inherently tighter*.
- Optimized tilting measures for constant Hamming-weight subcodes obtained by Sason and Shamai: DS2 bound (IEEE IT. Dec. 2002), 1961 Gallager bound (Oct. 1998), both using calculus of variations.

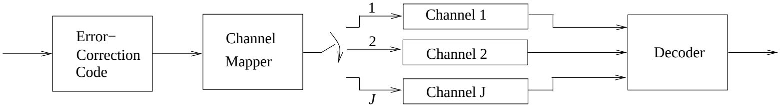
Outline

Generalizing the 1961 Gallager and DS2 Bound to Parallel Channels

- Liu et al. (IEEE IT. 2006) generalized the 1961 Gallager bound for the setting of parallel channels. In their analysis they rely on sub-optimal tilting measures.
  - On one hand, simplify the computational complexity of the bound.
  - On the other hand, these bounds are not tight.
- Our motivation is two-fold:
  - First, we are interested in finding optimized tilting measures for the 1961 Gallager bound.
  - Second, due to the superiority of the DS2 bound over the 1961 Gallager bound for a single channel, we generalize the DS2 bound to parallel channels.

Random Channel Mapper

- Following the analysis of Liu et al., we assume a *random* channel mapper.
- The random channel mapper assigns a coded bit to channel j with probability α_j, independent of other bits.



- Let n_j denote the number of bits mapped to channel j. The vector $\underline{n} = (n_1, n_2, \ldots, n_J)$ has a multinomial distribution, i.e.,

$$P_{\underline{N}}(\underline{n}) = \binom{n}{n_1, n_2, \ldots, n_J} \alpha_1^{n_1} \alpha_2^{n_2} \ldots \alpha_J^{n_J}$$

Random Channel Mapper: Split Weight Enumerator

- For a *specific* choice of the codeword and channel mapping, let h_j denote the Hamming weight of that part of the codeword which is transmitted over channel j.

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Random Channel Mapper: Split Weight Enumerator

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- For a specific channel mapping, the *split weight enumerator* (SWE) $A_{h_1,h_2,\ldots,h_J}$ is defined as the number of codewords with partial Hamming weights $\{h_j\}_{j=1}^{J}$.
- The SWE is in general not available. However, the SWE *averaged over all channel mappings* assigning n_j bits to channel j is given by

$$\mathbf{E}(A_{h_1,h_2,\ldots,h_J}) = A_h \cdot \frac{\binom{h}{h_1,\ldots,h_J}\binom{n-h}{n_1-h_1,\ldots,n_J-h_J}}{\binom{n}{n_1,\ldots,n_J}}$$

where

$$h_1 + h_2 + \ldots + h_J = h, \qquad 0 \le h_j \le n_j \ \ (1 \le j \le J).$$

Generalization of the DS2 Bound for Parallel Channels

- Based on the random mapper approach, we obtain a generalization of the DS2 bound for parallel channels.
- General concept:
  - Calculate DS2 bound for a *specific channel mapping*.
  - Average the result *over all channel mappings*.
- Resulting averaged bound is expressed in terms of the distance spectrum of the code (or ensemble). This is the dividend gained by using the random channel mapper.

Statement of the Bound

Theorem (Generalization of the DS2 Bound for Parallel Channels)

Under the above assumptions on the channel model, let $p(y|x;j)$ designate the transition probability of the j-th channel ($1 \leq j \leq J$). Then, the ML decoding error probability is upper bounded by

$$P_{\mathrm{e}} \leq \left\{ \sum_{h=0}^{n} A_h \left(\sum_{j=1}^{J} \alpha_j A(\lambda, \rho; j, \psi(\cdot; j)) \right)^h \left(\sum_{j=1}^{J} \alpha_j B(\rho; j, \psi(\cdot; j)) \right)^{n-h} \right\}^{\rho}$$

Statement of the Bound (Cont.)

where

$$A(\lambda, \rho; j, \psi(\cdot; j)) \triangleq \sum_y \psi(y; j)^{1-\frac{1}{\rho}} p(y|0; j)^{\frac{1-\lambda\rho}{\rho}} p(y|1; j)^{\lambda}$$

$$B(\rho; j, \psi(\cdot; j)) \triangleq \sum_y \psi(y; j)^{1-\frac{1}{\rho}} p(y|0; j)^{\frac{1}{\rho}}$$

- $0 \leq \rho \leq 1, \ \lambda \geq 0$.
- The functions $\psi(\cdot; j)$ are J probability tilting measures, subject to optimization.

Optimization of the tilting measures for the DS2 Bound

The optimal tilting measures are obtained by variational calculus, and are given by

$$\begin{aligned} \psi(y;j) &= \beta_j \, p(y|0;j) \, C(y;j)^{\rho} \\ C(y;j) &\triangleq 1 + k \left(\frac{p(y|1;j)}{p(y|0;j)} \right)^{\lambda}. \end{aligned}$$

The optimal parameters k and β_j are related by the implicit equations

$$k = \frac{\delta}{1-\delta} \frac{\sum_{j=1}^{J} \sum_{y \in \mathcal{Y}} \left\{ \alpha_j \beta_j^{1-\frac{1}{\rho}} p(y|0;j) C(y;j)^{\rho-1} \right\}}{\sum_{j=1}^{J} \sum_{y \in \mathcal{Y}} \left\{ \alpha_j \beta_j^{1-\frac{1}{\rho}} p(y|0;j) C(y;j)^{\rho-1} \left(\frac{p(y|1;j)}{p(y|0;j)} \right)^{\lambda} \right\}}, \quad \delta \triangleq \frac{h}{n}$$

$$\beta_j = \left[\sum_{y \in \mathcal{Y}} p(y|0;j) C(y;j)^{\rho} \right]^{-1}.$$

Interim Notes and Observations

- When the block length tends to infinity, this bound extends the attainable channel regions as predicted by Liu et al. (as they relied on *special cases* of the 1961 Gallager bound).
- This bound makes use of Jensen's inequality twice: Once for the vector tilting measure $\Psi_n^{(m)}(\underline{y})$ in the beginning of the derivation, and a second time in the process of averaging over all possible mappings.
- Consequently, this version does not attain the random coding exponent.

$\Rightarrow$ Hence, the motivation to derive another alternative of the DS2 bound which attains the random coding exponent (in this respect, a stimulating discussion with Shlomo Shamai is acknowledged).

Generalizing the DS2 Bound to Parallel Channels: 2nd version

- Recall that the random channel mapper assigns a coded bit to channel j with probability α_j, independent of other bits.

$\Rightarrow$ j may be interpreted as the internal state of a state-dependent channel with output $b = (y, j)$. We call this the *channel state information at the receiver* (CSIR) model.

- Resulting transition probability $p_B(b|x)$ is equivalent to an MBIOS channel, so the DS2 bound may be applied directly.

Statement of the Bound

Theorem (Generalized DS2 Bound for Parallel Channels, 2nd Version)

The ML decoding error probability of a binary linear block code, transmitted over a set of J parallel MBIOS channels with transition probabilities $p(y|x;j)$, with a random mapping of bits to channels w.p. $\alpha_j, j = 1, \ldots, J$ is upper bounded by

$$P_e \leq \left\{ \sum_{h=0}^{n} A_h \left(\sum_{j=1}^{J} \alpha_j^{\frac{1}{\rho}} A(\lambda, \rho; j, \tilde{\psi}(y;j)) \right)^h \left(\sum_{j=1}^{J} \alpha_j^{\frac{1}{\rho}} B(\rho; j, \tilde{\psi}(y;j)) \right)^{n-h} \right\}^{\rho}$$

Statement of the Bound (cont.)

$$A(\lambda, \rho; j, \tilde{\psi}(y;j)) \triangleq \sum_{y} \left(\tilde{\psi}(y;j)^{1-\frac{1}{\rho}} p(y|0;j)^{\frac{1-\lambda\rho}{\rho}} p(y|1;j)^{\lambda} \right)$$

$$B(\rho; j, \tilde{\psi}(y;j)) \triangleq \sum_{y} \tilde{\psi}(y;j)^{1-\frac{1}{\rho}} p(y|0;j)^{\frac{1}{\rho}}$$

- $0 \leq \rho \leq 1, \lambda \geq 0$.
- The function $\tilde{\psi}(y;j)$ is a probability tilting measure, subject to optimization.

Interim Notes and Observations (cont'd)

- Obtaining optimized bound requires optimization over a single tilting measure $\tilde{\psi}(y;j)$ (as opposed to J tilting measures in the first version).
- The 2nd version of the generalized DS2 bound attains the random coding exponent while the first version does not !
- However, neither of these two bounds is uniformly tighter than the other for general ensembles.

Generalizing the 1961 Gallager Bound to Parallel Channels

- Liu *et al.* (IEEE IT, April 2006) have generalized the 1961 Gallager bound to the case of parallel channels.
- They use the technique of obtaining a bound for a specific channel mapping, and then averaging over all possible mappings (same as 1st version of DS2 bound).
- Applying the CSIR model in this case yields the same bound.
- The 1961 Gallager bound for parallel channels is less tight than the 2nd version of the DS2 bound (similar to the single-channel case).

Statement of the Bound

Theorem (Generalized 1961 Gallager Bound for Parallel Channels, Liu *et al.* (IEEE IT, April 2006))

The ML decoding error probability of a binary linear block code, transmitted over a set of J parallel MBIOS channels with transition probabilities $p(y|x;j)$, with a random mapping of bits to channels w.p. α_j where $j \in \{1, \ldots, J\}$ is upper bounded by

$$P_e \le 2^{h(\rho)} \left\{ \sum_{h=1}^{n} A_h \left[\sum_{j=1}^{J} \alpha_j Z(r;j) \right]^h \left[\sum_{j=1}^{J} \alpha_j G(r;j) \right]^{n-h} \right\}^{\rho} \cdot \left\{ \sum_{j=1}^{J} \alpha_j G(s;j) \right\}^{n(1-\rho)}$$

Statement of the Bound (cont.)

where

$$r \leq 0, \quad s \geq 0$$

and

$$G(r;j) \triangleq \sum_y p(y|0;j)^{1-r} f(y;j)^r$$

$$Z(r;j) \triangleq \sum_y \left[p(y|0;j)p(y|1;j)\right]^{\frac{1-r}{2}} f(y;j)^r$$

$$\rho \triangleq \frac{s}{s-r}, \quad 0 \leq \rho \leq 1$$

and $f(\cdot;j)$ is an arbitrary tilting measure, constrained to be non-negative and even (i.e., $f(y;j) = f(-y;j)$).

Statement of the Bound (cont.)

- By partitioning the code into constant Hamming-weight subcodes, we obtain that the optimized tilting measures $f(y;j)$ are given by

$$f(y;j) = \left\{ \frac{(1-c)\left(p(y|0;j)^{\frac{1-s(1-\rho^{-1})}{2}} - p(y|1;j)^{\frac{1-s(1-\rho^{-1})}{2}}\right)^2}{p(y|0;j)^{1-s} + p(y|1;j)^{1-s}} + \frac{2c\,\left(p(y|0;j)p(y|1;j)\right)^{\frac{1-s(1-\rho^{-1})}{2}}}{p(y|0;j)^{1-s} + p(y|1;j)^{1-s}} \right\}^{\frac{\rho}{s}}, \quad (\rho, s, c) \in [0,1]^3.$$

- The parameters ρ, s, c are optimized numerically.

Special Cases of the Bounds for Parallel Channels

- The union bound for Parallel Channels. A special case of both versions of the DS2 bound and the 1961 Gallager bound.
- Simplified sphere bound for Parallel Channels (extension of Divsalar 1999 to parallel channels). A special case of both versions of the DS2 bound and the 1961 Gallager bound.
- Shulman-Feder (SF) bound for Parallel Channels. A special case of the DS2 bound and the 1961 Gallager bound.

Modified Shulman-Feder (MSF) bound

- The Shulman-Feder bound contains a multiplication by the quantity

$$\max_{1 \le h \le n} \left(\frac{A_h}{2^{-n(1-R)} \binom{n}{h}} \right)^{\rho}$$

- Problem: This quantity measures the maximum distance between the distance spectrum of the code ensemble and that of the random ensemble and can be considerably large.
- Consequently, the bound is not so tight for some structured ensembles of codes whose distance spectra largely deviate from the binomial distribution for low Hamming weights (see, e.g., Twitto et al., IEEE IT 2007).
- General idea behind MSF bound (Miller-Burshtein, IEEE IT 2001): Use combination of SF bound with union bound.

Modified Shulman-Feder (MSF) bound (cont'd)

- Divide the set of normalized Hamming weights $\Psi_n \triangleq \left\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$ into two disjoint subsets, $\Psi_n = \Psi_n^+ \cup \Psi_n^-$.
- Apply the union bound for all codewords of normalized Hamming weight $\in \Psi_n^+$ and SF bound for codewords of normalized Hamming weight $\in \Psi_n^-$.
- The partition of Ψ_n may be optimized, yielding the largest MSF (LMSF) region. Nickname: LMSF bound.
- Typically, union bound is used for low and high Hamming weights where the distance spectrum of typical ensembles deviates considerably from that of the random code ensemble.
- The LMSF bound is a special case of the 2nd version of the DS2 bound and the 1961 Gallager bound for parallel channels.

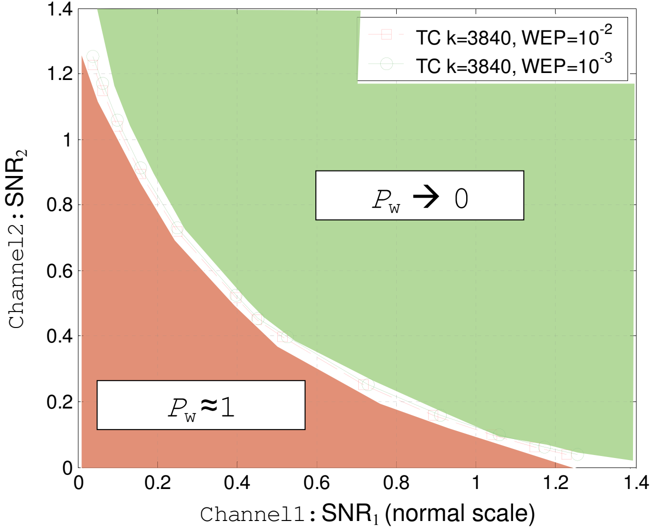
Outline

Attainable Channel Regions

- Single channel (single parameter $\longrightarrow$ noise threshold)
- J parallel channels
  - Communication is reliable if the average ML decoding word error probability (bound) approaches 0.
  - Attainable channel region: a set of J-tuples of channel parameters for which the communication over these set of parallel channels in reliable.
  - Tightening existing upper bounds $\longrightarrow$ extension of known attainable channel regions.

Attainable Channel Regions - example

Turbo codes with R=1/3, two parallel AWGN channel, $\alpha_1 = \alpha_2 = 1/2$



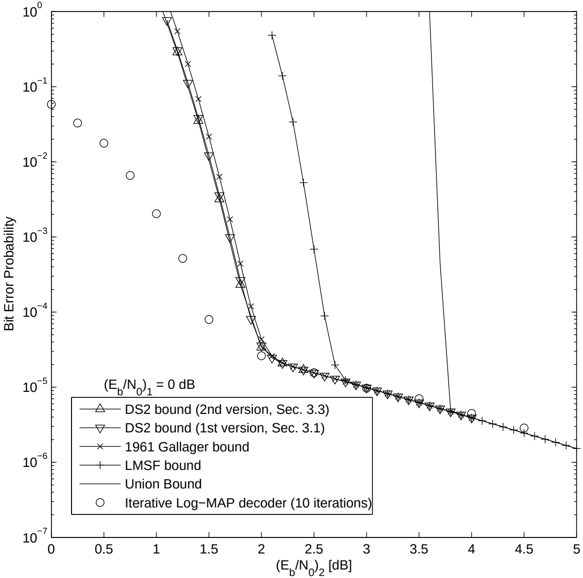
Asymptotics

- Let $[\mathcal{C}(n)]$ denote an ensemble of codes of length n.
- The *average distance spectrum* of $[\mathcal{C}(n)]$ the ensemble is denoted by $A_h^{[\mathcal{C}(n)]}$.
- $r^{[\mathcal{C}(n)]}(\delta) \triangleq \frac{\ln A_h^{[\mathcal{C}(n)]}}{n}$ is the *normalized exponent of the distance spectrum*, where $\delta \triangleq \frac{h}{n}$ is the *normalized distance*
- The *asymptotic growth rate* of the distance spectrum is defined by $r^{[\mathcal{C}]}(\delta) \triangleq \lim_{n\to\infty} r^{[\mathcal{C}(n)]}(\delta)$
- Techniques for calculating $r^{[\mathcal{C}]}(\delta)$ have been addressed in the literature, e.g., RA codes (McElice, 1998), LDPC codes (Burshtein-Miller, IEEE IT. 2004) and others.
- Attainable channel regions are evaluated using the asymptotic growth rate of the distance spectrum and the generalized 1961 and DS2 bounds.

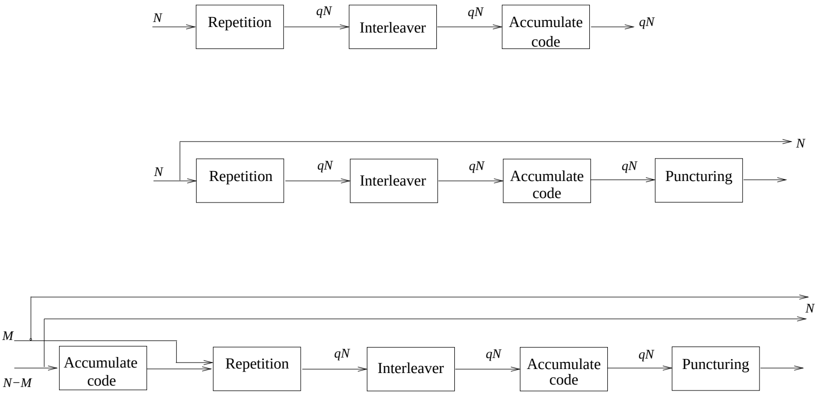
Outline

Performance Bounds for Uniformly Interleaved Turbo Codes

- Consider the ensemble of uniformly interleaved turbo codes of rate $R = \frac{1}{3}$.
- The encoder consists of two convolutional encoders with polynomials $G(D) = \left[1, \frac{1+D^4}{1+D+D^2+D^3+D^4}\right]$.
- Interleaver between the encoders is of length 1000.
- Transmission takes place over parallel binary input AWGN channels.
- Each bit is equally likely to be assigned to one of the channels $(\alpha_1 = \alpha_2 = \frac{1}{2})$.
- $\frac{E_b}{N_0}$ on the first channel is set to 0 dB.


Bit Error Probability
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
$(E_b/N_0)_2$ [dB]
$(E_b/N_0)_1$ = 0 dB
DS2 bound (2nd version, Sec. 3.3)
DS2 bound (1st version, Sec. 3.1)
1961 Gallager bound
LMSF bound
Union Bound
Iterative Log−MAP decoder (10 iterations)

Repeat-Accumulate Codes and Variations



Repeat-Accumulate Codes and Variations (cont.)

- All ensembles: we choose a common rate of $\frac{1}{3}$, input block length is N.
- First ensemble: non-systematic repeat-accumulate codes (McElice *et al.*, Allerton 1998) - NSRA codes
- Second ensemble: systematic and punctured repeat-accumulate codes - SPRA codes.
- Third ensemble: systematic and punctured accumulate-repeat-accumulate codes - SPARA codes. Denote $\alpha \triangleq \frac{M}{3N}$ to be the fraction of bits which do not pass through the outer accumulator, but pass directly to the repetition code.

Distance Spectra of Considered Ensembles

- First ensemble (NSRA): distance spectrum derived by McElice *et al.* (Allerton 1998):

$$r^{[\mathcal{C}]}(\delta) = \max_{0 \le u \le \min(2\delta, 2-2\delta)} \left\{ -\frac{2}{3}H(u) + (1-\delta)H\left(\frac{u}{2(1-\delta)}\right) + \delta H\left(\frac{u}{2\delta}\right) \right\}$$

  where H is the binary entropy function.

- Second and third ensembles: distance spectrum and asymptotic growth rate derived using the techniques by Abbasfar *et al.* (GLOBECOM 2004).

Distance Spectra of Considered Ensembles

- For SPRA codes:

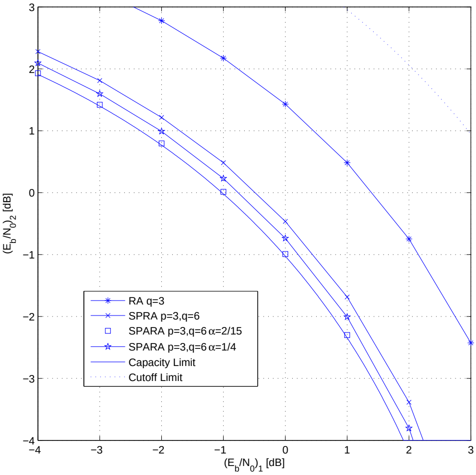
$$r^{[\mathcal{C}]}(\delta) =$$

$$\max_{\eta,\rho_1,\rho_2}\left\{-\frac{5}{3}\,H\left(\frac{2\rho_2+\eta}{2}\right)+\eta\,H\left(\frac{\rho_1}{\eta}\right)+\left(\frac{2}{3}-\eta\right)\,H\left(\frac{\rho_2-\rho_1}{\frac{2}{3}-\eta}\right)\right.$$

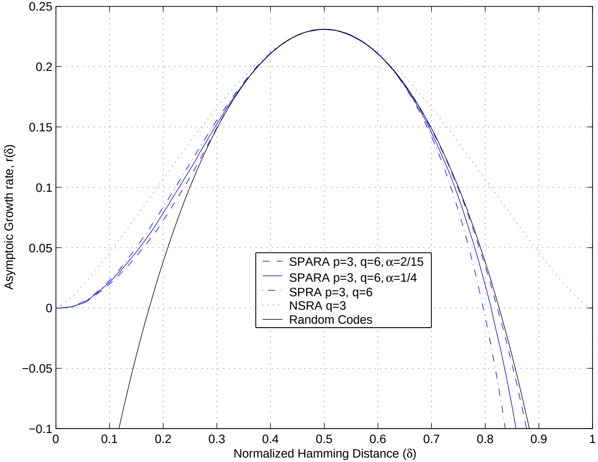
$$+\left(\frac{2}{3}-\delta+\frac{2\rho_2+\eta}{6}\right)H\left(\frac{\eta}{2\left(\frac{2}{3}-\delta+\frac{2\rho_2+\eta}{6}\right)}\right)$$

$$\left.+\left(\delta-\frac{2\rho_2+\eta}{6}\right)H\left(\frac{\eta}{2\left(\delta-\frac{2\rho_2+\eta}{6}\right)}\right)+(\eta+\rho_2-2\rho_1)\ln 3\right\}$$

Attainable Channel Regions



Distance Spectra of Considered Ensembles



Discussion

- Ensemble of SPARA codes approaches capacity very closely. This is attributed to the spectral thinning effect (Perez *et al.*, IEEE IT Nov. 1996).
- Computer simulation results indicate
  - the superiority of SPARA codes over SPRA and NSRA codes (similarly to their performance under ML decoding).
  - For ensembles of SPARA codes, if we let α tend to zero, then the iterative decoding cannot start $\Rightarrow$ Unlike ML decoding, there is an intermediate value of α for maximizing the performance of the iterative decoder (whereas for ML decoding, $\alpha \rightarrow 0$ is optimal).
  - Possible solution for the iterative decoder: setting a small fraction of the bits to be pilot bits which enable the iterative decoder to start.

Outline

Summary

- Improved upper bounds on the ML decoding error probability for parallel channels have been derived.
- These upper bounds considerably improve on previously-known attainable channel regions.
- The interconnections and relations between the bounds are established.
- Simpler upper bounds were derived as special cases of the improved bounds (Liu et al., IEEE IT 2006).

Summary (cont'd)

- Analysis of asymptotic growth rates of the distance spectra of some turbo-like ensembles, and in particular repeat-accumulate codes and their modern variants (systematic and punctured RA and ARA ensembles).
- The above bounds for the repeat-accumulate codes and their modern variants indicate that better performance are obtained even under iterative decoding for ensembles whose growth rate of their distance spectra resembles the binomial distribution of the random code ensemble.

Further Reading

This talk is based on the paper:

- I. Sason and I. Goldenberg, “Coding for parallel channels: Gallager bounds for binary linear codes with applications to repeat-accumulate codes and variations,” *accepted to IEEE Trans. on Information Theory*, February 2007.
- Available at:

  `http://www.ee.technion.ac.il/people/sason/`.

Thank you for your attention !