Scaling Laws of Wake-Fields in Optical Structures

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Outline

- **Background & Motivation**
  - Optical Accelerators
  - Wake-Fields
- **Assumptions & approach to solution**
- **Wake-field in the vicinity of dielectric body**
  - Radiated energy
  - Frequency dependence of dielectric coefficient
- **Wake-field due to geometric discontinuity**
  - Frequency-domain versus time-domain solution
  - Radiated energy
- **Wake-field due to surface roughness**
  - Simple model: quasi-periodic structure with random perturbation
  - Radiated energy: average value and standard-deviation
- **Summary**

Background: Accelerator

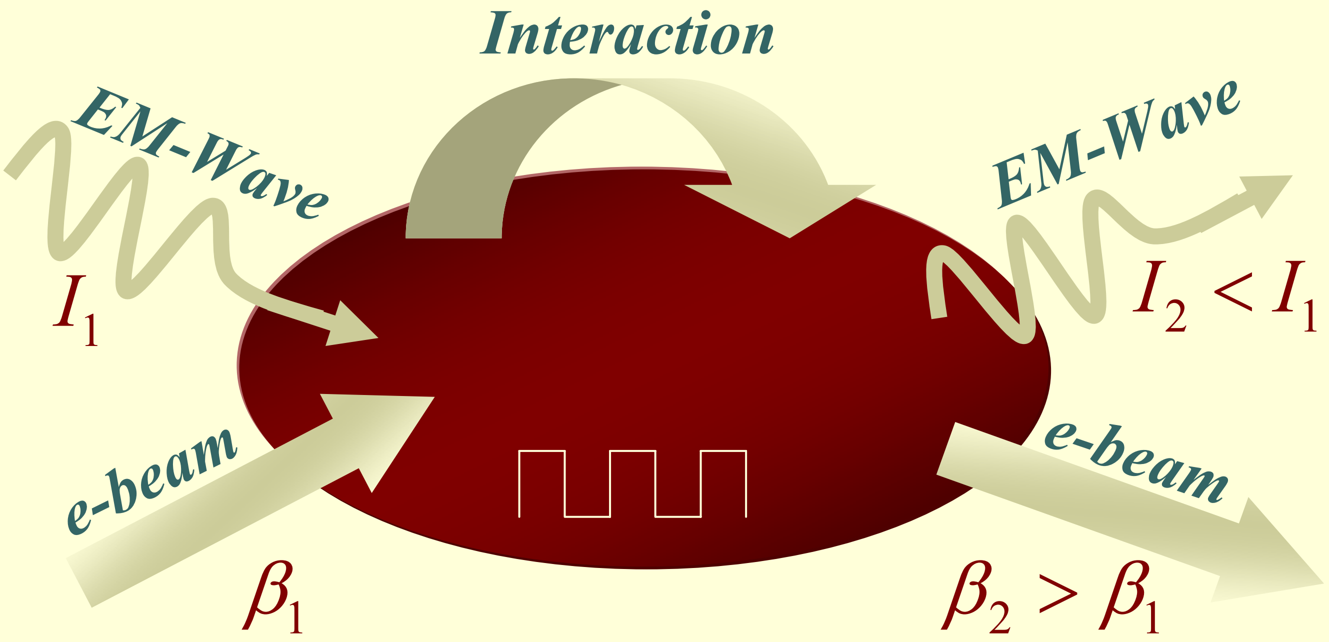
Accelerator: A device used to produce high-energy (high-speed) particles by converting EM energy into kinetic energy.

Fermilab Accelerator

Applications:

- Foundation of nature (constituents of matter)
- Particle physics
- X – ray sources
- Medical Diagnostics and treatment

Background: Beam-Wave Interaction



The EM wave's phase velocity = The electrons' average velocity

Motivation: Optical Accelerators

High-energy requirement

Compactness

Cost consideration

→ *High-gradient* → *High frequency* → *Compact accelerators based on optical structures*

$$E \propto \sqrt{P\omega^2}$$

Motivation: Lawson-Woodward Theorem

No net energy gain in vacuum using optical fields

- **Infinite interaction region.**
- **No walls or boundaries are present.**
- **Highly relativistic electron on the acceleration path.**
- **No static electric or magnetic fields present.**
- **Non-linear effects are neglected.**

Motivation: Lawson-Woodward Theorem

Lawson-Woodward theorem conditions

No acceleration in vacuum

Acceleration: interaction should be limited in space or time

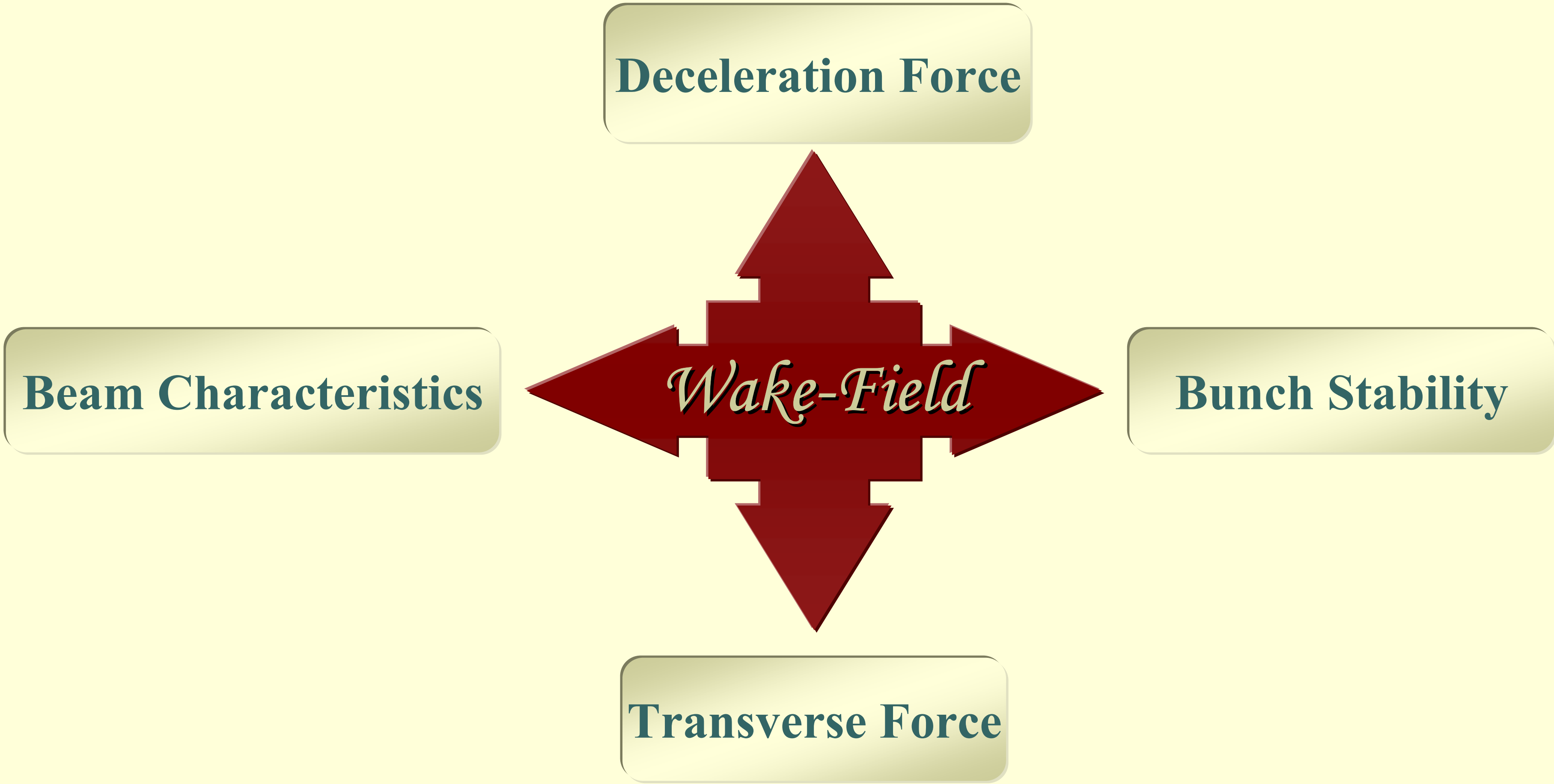
***Optical* acceleration structures**

Motivation: Wake-Fields

- ***Optical Acceleration Structures***: Laser acceleration of electrons in vacuum entails relativistic motion of electrons in the vicinity of metallic or dielectric structures of sub-micron size.

- It is important to determine:
  - Characteristics of the ***wake-field*.**
  - The impact of the ***wake-field*** on the bunch properties.
  - Impact of radius of curvature.
  - ***Wake-field*** of bunches moving/leaving geometric discontinuities.
  - ***Wake-field*** due to surface roughness in optical structures.
  - Temperature consideration.
  - Nonlinear effects.

Motivation: Wake-Fields



Motivation: Wake-Fields

- We focus on an ***analytic*** or ***quasi-analytic*** solutions, in order to have better understanding of the wake-field properties.

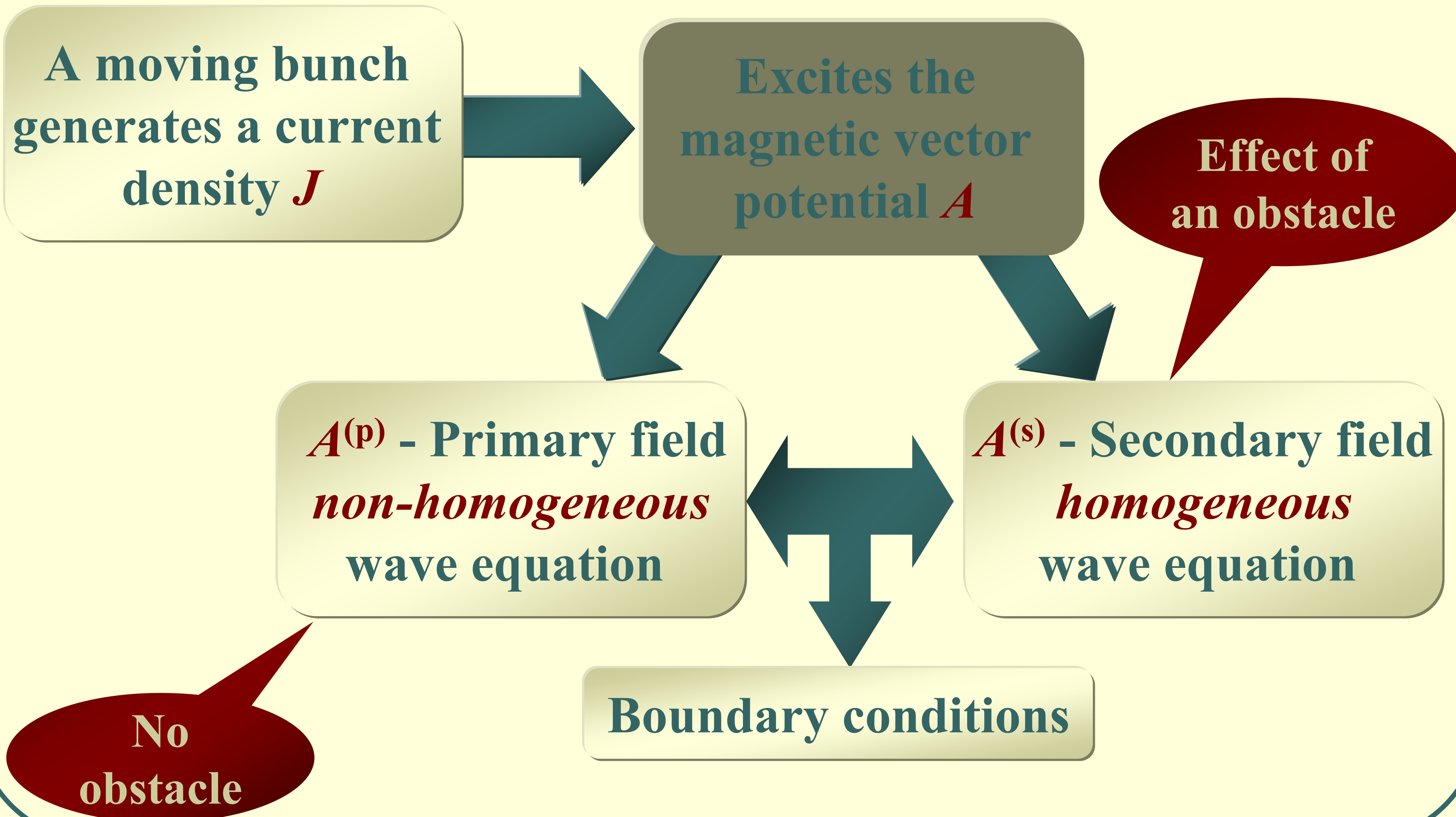
- A bunch moving in a free space generates a broad spectrum of ***evanescent waves***.

- In the presence of an obstacle the evanescent waves are ***diffracted***, carrying away energy.

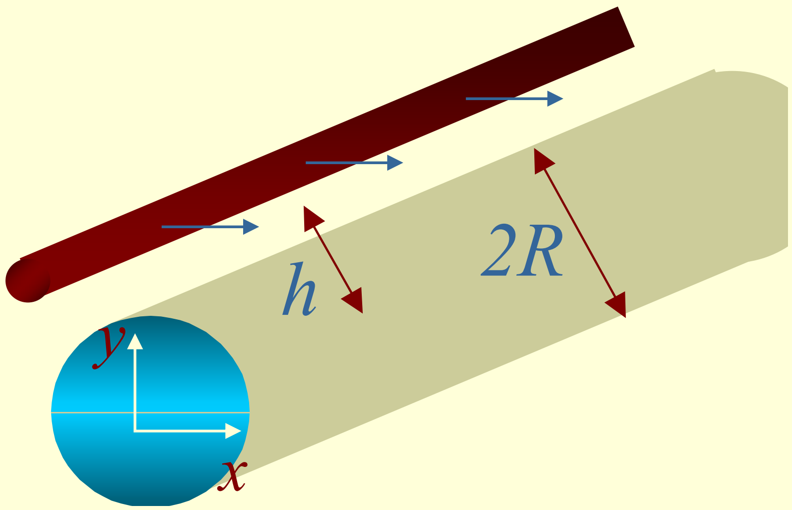
Assumptions

- An ***external force*** is acting on the moving bunch keeping its velocity ***constant***.
- ***Transverse motion*** of the moving bunch is neglected.

Approach to Solution



Line Charge Moving in the Vicinity of a Dielectric Cylinder



- Free space (ε_0, μ_0)
- Infinite cylinder
- Constant velocity

Primary Field

$$J_x(x,y;\omega) = -\frac{\lambda}{2\pi}\delta(y-h)\exp(-j\frac{\omega}{c\beta}x)$$

$$E_\varphi^{(p)} = \frac{-\lambda\mu_0}{4\pi\,\Gamma}\exp\left(-j\frac{\omega}{c\beta}x - \Gamma|y-h|\right)\left[\frac{-j\omega}{\gamma^2\beta^2}\sin\varphi + \frac{\omega}{\gamma\beta^2}\operatorname{sgn}(y-h)\cos\varphi\right]$$

$$H_z^{(p)} = \frac{-\lambda}{4\pi}\exp\left(-j\frac{\omega}{c\beta}x - \Gamma|y-h|\right)\operatorname{sgn}(y-h)$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}; \qquad \Gamma \equiv \frac{\omega}{c}\frac{1}{\gamma\beta}$$

$$x = r\cos\varphi; \qquad y = r\sin\varphi$$

Secondary Field

$$H_z^{(s)}(r,\varphi;\omega) = \sum_{n=-\infty}^{\infty} e^{jn\varphi} \begin{cases} A_n \mathrm{H}_n^{(2)}\left(\dfrac{\omega}{c} r\right) & r > R \\ B_n \mathrm{J}_n\left(\dfrac{\omega}{c}\sqrt{\varepsilon_r}\, r\right) & r < R \end{cases}$$

$$E_\varphi^{(s)}(r,\varphi;\omega) = j\eta_0 \sum_{n=-\infty}^{\infty} e^{jn\varphi} \begin{cases} A_n \dot{\mathrm{H}}_n^{(2)}\left(\dfrac{\omega}{c} r\right) & r > R \\ B_n \sqrt{\varepsilon_r}\, \dot{\mathrm{J}}_n\left(\dfrac{\omega}{c}\sqrt{\varepsilon_r}\, r\right) & r < R \end{cases}$$

$\eta_0 \equiv \sqrt{\mu_0 / \varepsilon_0}$

Longitudinal Impedance - Spectrum

$$Z_{\parallel} = -\frac{1}{\lambda}\int_{-\infty}^{\infty} dx\; E_x(x,h;\omega)\exp(j\frac{\omega}{c\beta}x)$$

$$W/\Delta_z = -\frac{\lambda}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dx\; E_x(x,h;\omega)\exp(j\frac{\omega}{c\beta}x)$$

$$\Rightarrow \qquad \frac{1}{\Delta_z}\frac{dW}{d\omega} = \frac{\lambda^2}{2\pi}Z_{\parallel}$$

$$\overline{W} = \frac{W/\Delta_z}{\lambda^2/\pi\varepsilon_0}$$

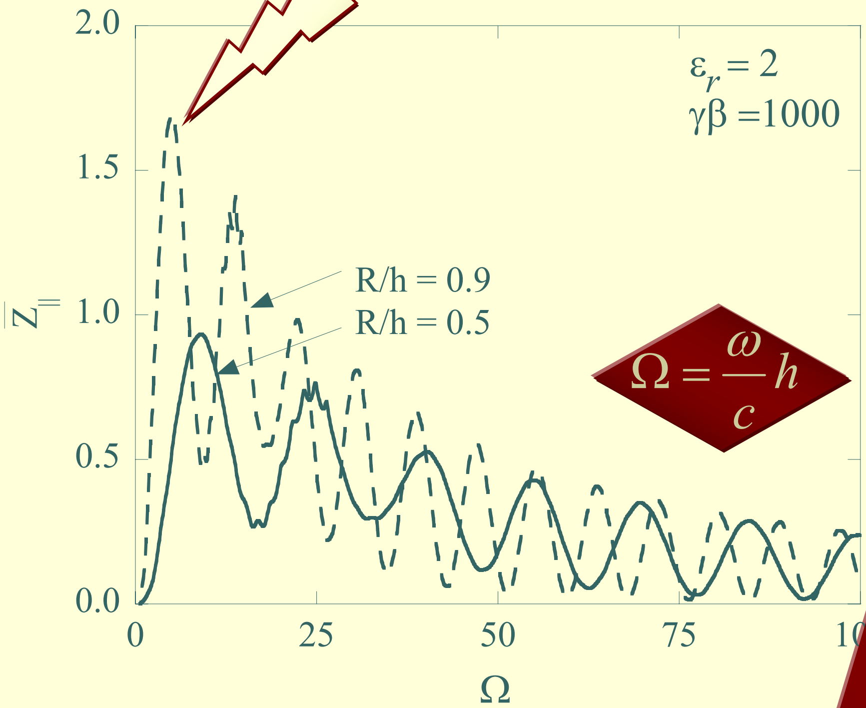
$$\overline{Z}_{\parallel} \equiv \frac{1}{2\eta_0 h}Z_{\parallel}$$

$$\Rightarrow \qquad \frac{d\overline{W}}{d\Omega} = \overline{Z}_{\parallel}$$

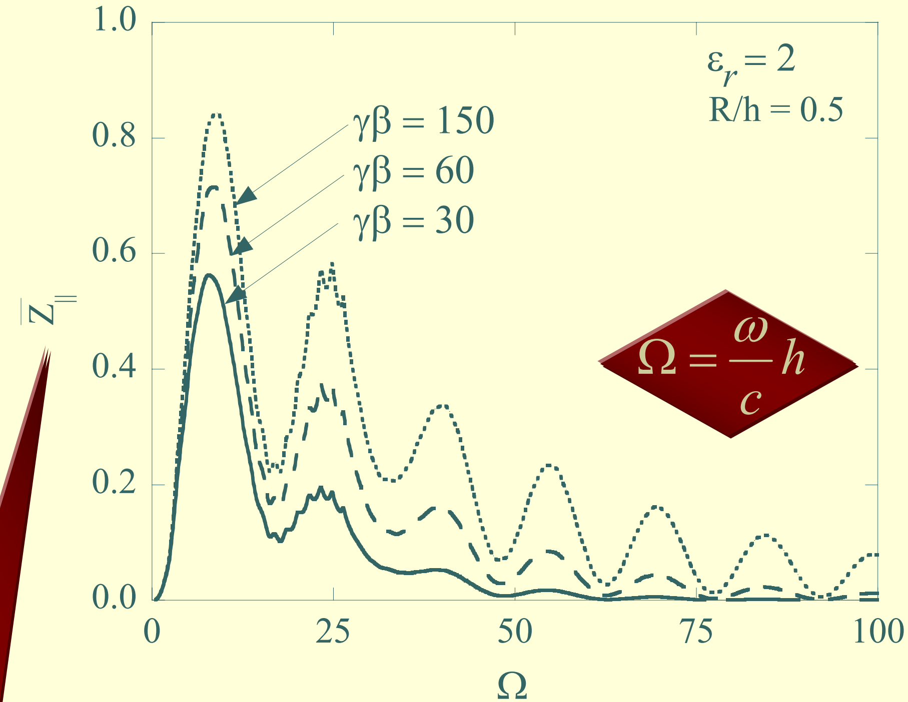
$$\Omega = \frac{\omega}{c}h$$

Longitudinal Impedance - Spectrum

$\left(\Omega_{peak}, \overline{Z}_{peak}\right)$



$\Omega = \dfrac{\omega}{c} h$



$\Omega = \dfrac{\omega}{c} h$

$$\overline{Z}_{\parallel} \equiv \frac{1}{2\eta_0 h} Z_{\parallel}$$

Longitudinal Impedance - Spectrum

Scaling Laws

$$\overline{Z}_{peak} \propto \frac{R}{h}$$

$$\overline{Z}_{peak} \propto (\gamma\beta)^{1/3}$$

$$\Omega_{peak} \propto \sqrt{\frac{h}{R}}$$

$$\Omega R/h >> 1$$

$$\overline{Z}_{\parallel} \propto \frac{1}{\Omega}$$

Radiated Energy

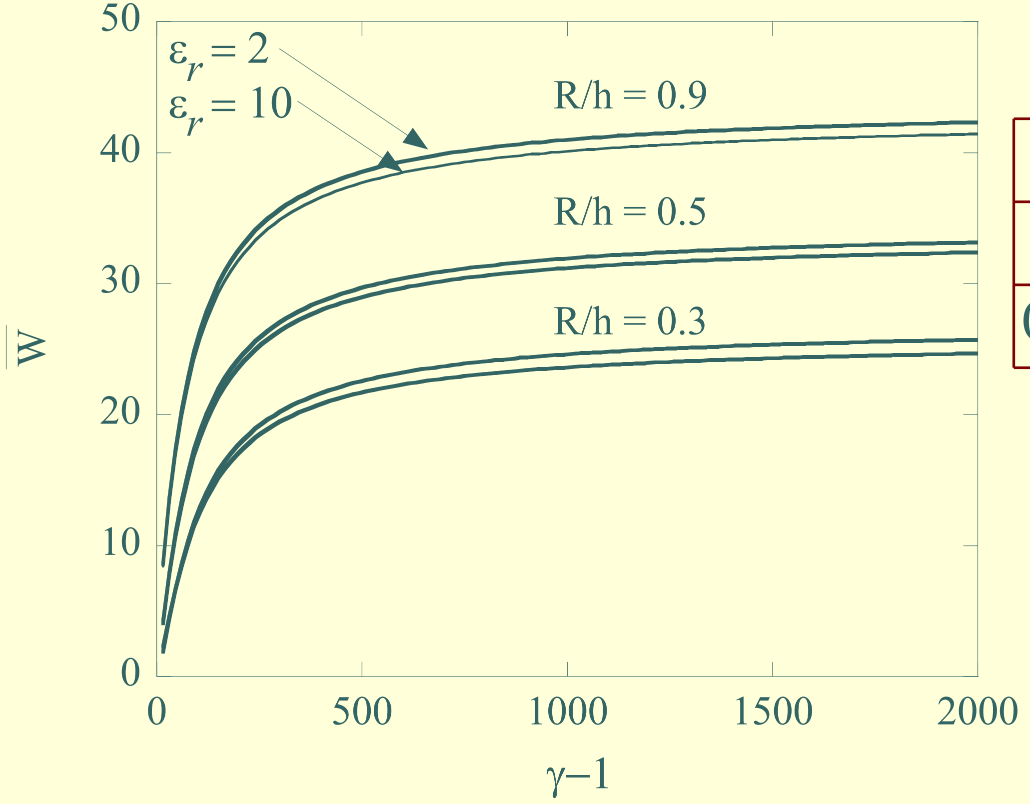
$$\overline{W} = \int_0^\infty d\Omega \frac{1}{\Omega} \sum_{n=-\infty}^{\infty} \left| \overline{A}_n(\Omega) \right|^2 \qquad \frac{d\overline{W}}{d\Omega} = \frac{1}{\Omega} \sum_{n=-\infty}^{\infty} \left| \overline{A}_n(\Omega) \right|^2$$

$$\overline{A}_n(\Omega) = \frac{\left[\frac{V_n}{j\beta} \sqrt{\varepsilon_r} J_n\left(\sqrt{\varepsilon_r} \Omega \frac{R}{h} \right) - U_n \dot{J}_n\left(\sqrt{\varepsilon_r} \Omega \frac{R}{h} \right) \right] e^{-\frac{\Omega}{\gamma\beta}}}{H_n^{(2)}\left(\Omega \frac{R}{h} \right) \dot{J}_n\left(\sqrt{\varepsilon_r} \Omega \frac{R}{h} \right) - \sqrt{\varepsilon_r} \dot{H}_n^{(2)}\left(\Omega \frac{R}{h} \right) J_n\left(\sqrt{\varepsilon_r} \Omega \frac{R}{h} \right)}$$

Analytic Functions

$$U_n = \left(\frac{\gamma + 1}{\gamma - 1} \right)^{n/2} J_n\left(\Omega \frac{R}{h} \right) e^{-jn\pi/2}; \qquad V_n = \frac{1}{2}(1 + \frac{j}{\gamma}) U_{n-1} + \frac{1}{2}(1 - \frac{j}{\gamma}) U_{n+1}$$

Radiated Energy



$$\overline{W} \approx a_1 \ln\left[b_1(\gamma-1)\right]$$

$\varepsilon_r > 2$

$0.1 < R/h < 0.9$

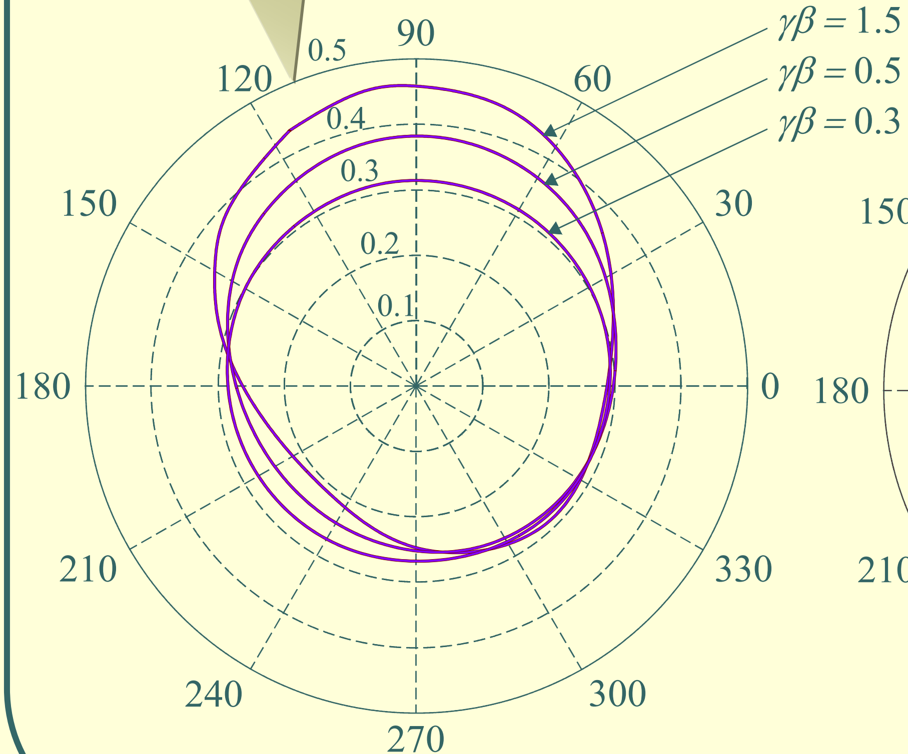
$2 < \gamma < 250$	$\gamma > 1000$
$7 < a_1 < 10$	$1.5 < a_1 < 2$
$0.05 < b_1 < 0.15$	$4\times10^3 < b_1 < 2\times10^6$

$$\overline{W} \approx a_2 \frac{R}{h} e^{-2b_2\left(\frac{h}{R}\right)}$$

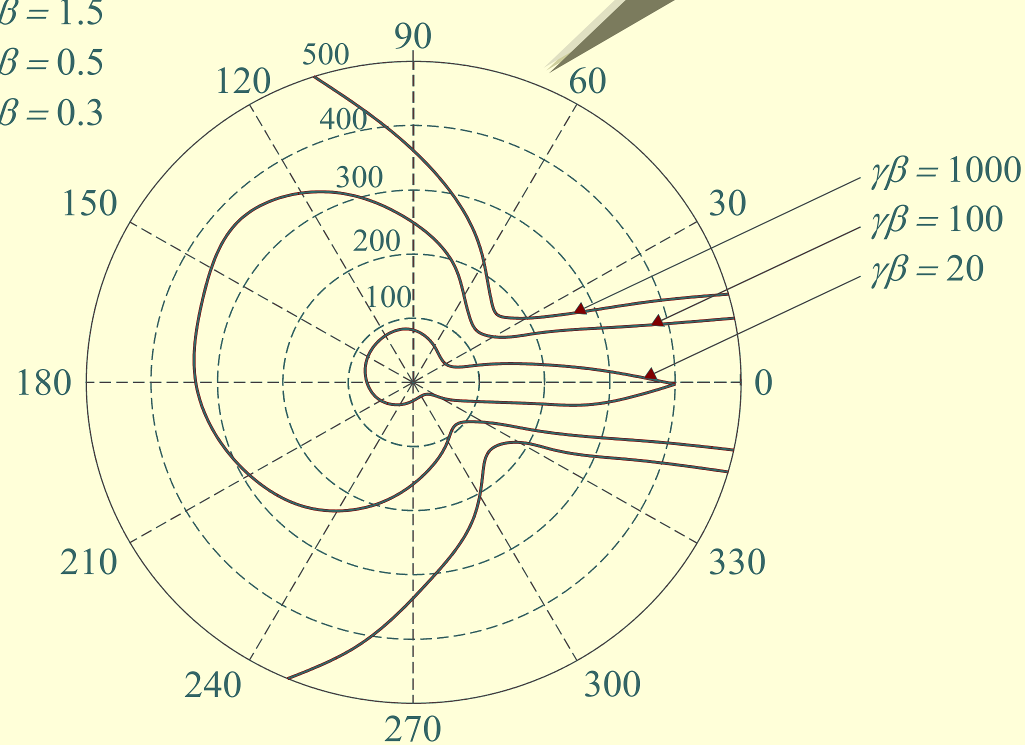
$5 < \gamma < 100;\ \varepsilon_r = 4$
$5 < a_2 < 15$
$3\times10^{-4} < b_2 < 0.3$

Angular Distribution of Radiation

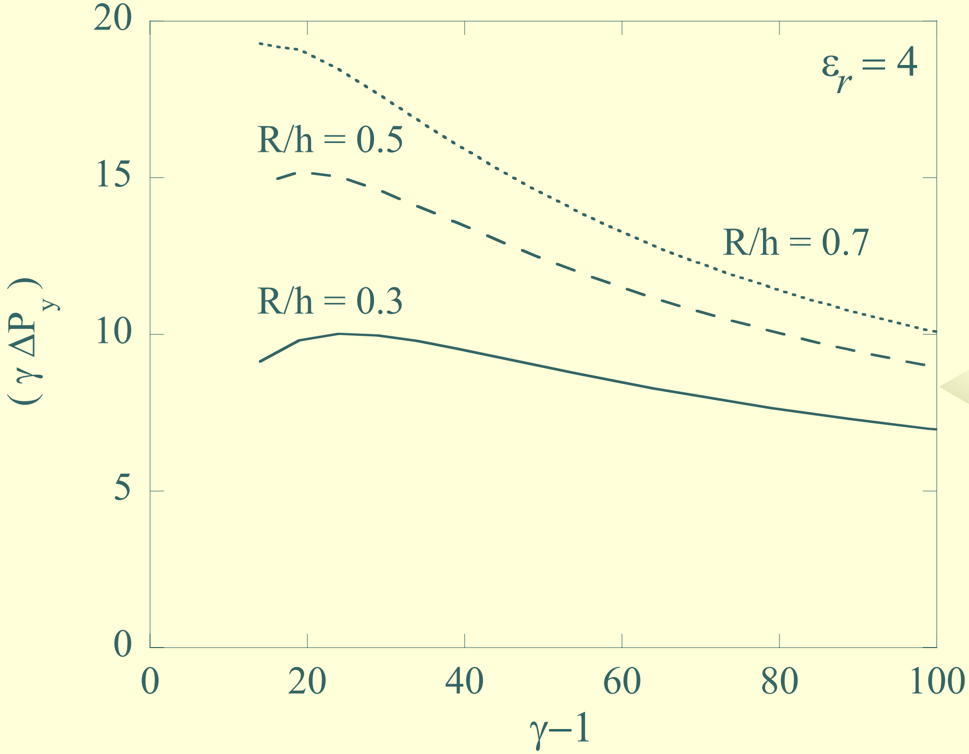
Low Energies



High Energies



Transverse Kick



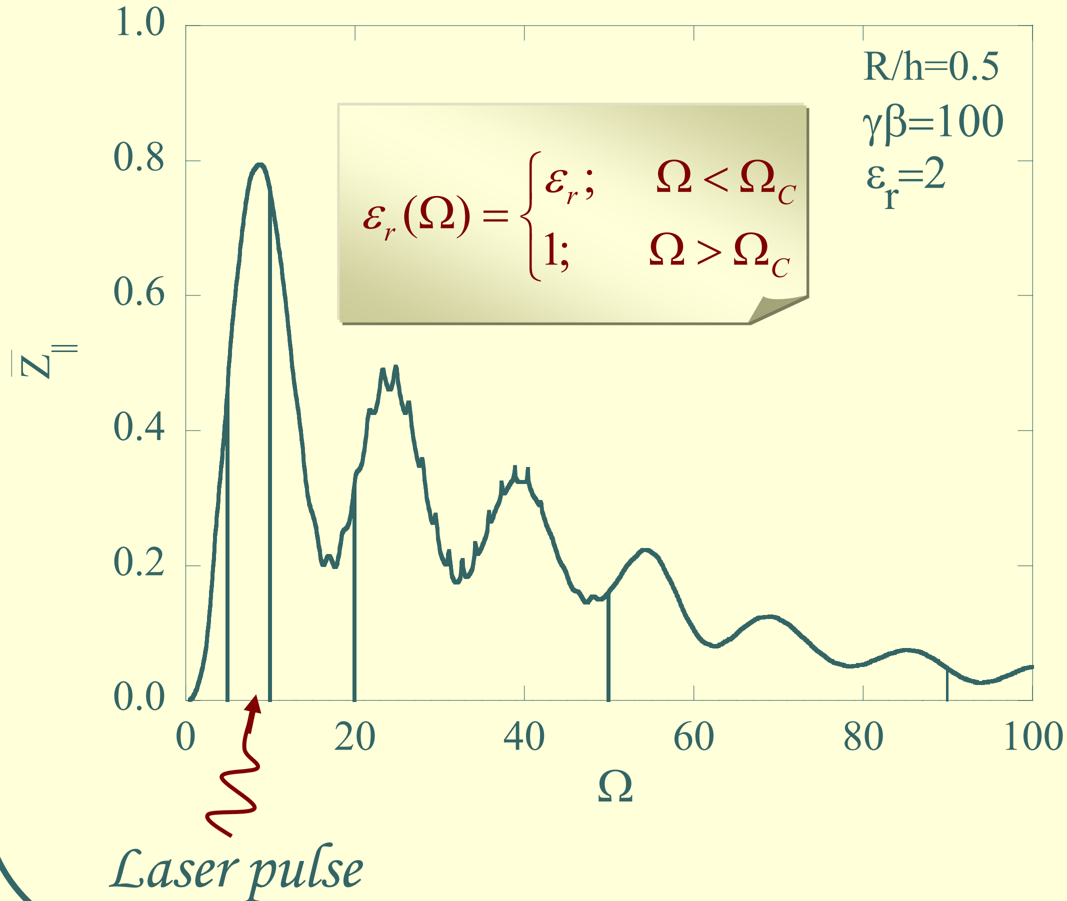
$$\Delta P_y = P_y(t=\infty) - P_y(t=-\infty) = \int_{-\infty}^{\infty} dt F_y$$

$$\Delta \overline{P}_y \equiv \frac{2\pi \Delta P_y}{\lambda^2 \Delta_z \eta_0}$$

$\gamma >> 1$

$$\Delta P_y \approx \frac{\lambda^2 \Delta_z \eta_0}{2\pi} \times \frac{3}{\gamma}$$

Frequency Dependence of ε_r



1.06 μm laser pulse

$\lambda_{peak} \approx 100nm$

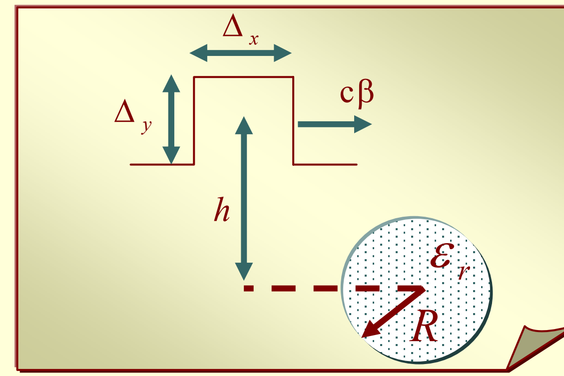
$W(\lambda_c = 0) \approx 2W(\lambda_c = 200nm)$

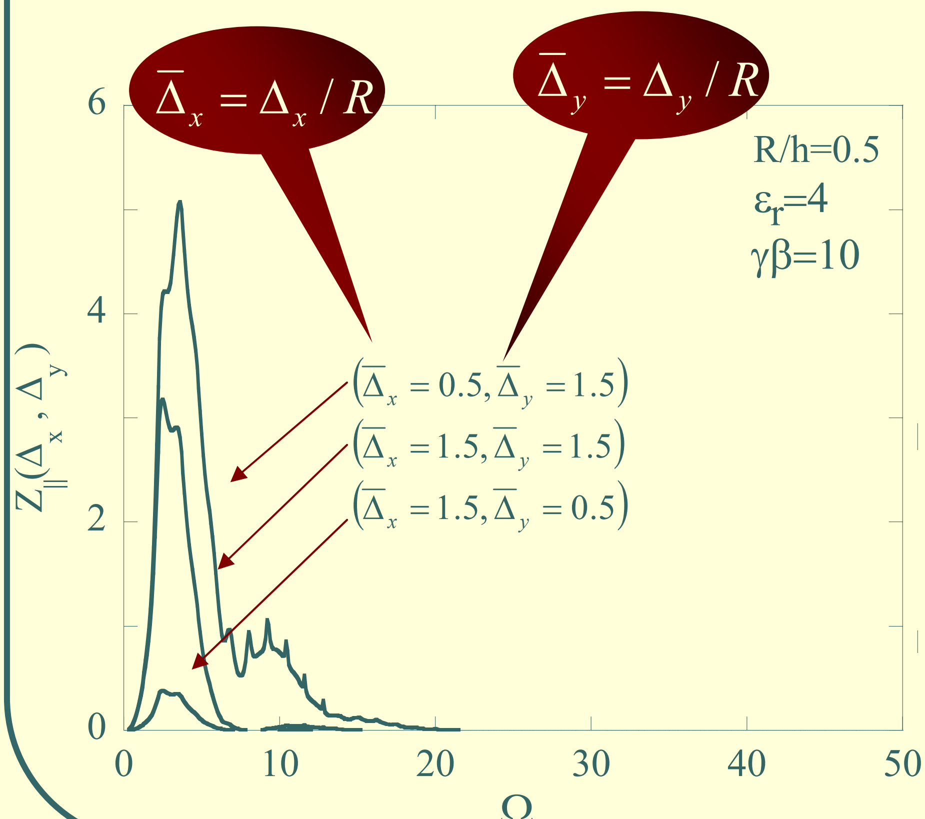
$\overline{W} \approx 0.65\sqrt{\Omega_c}$

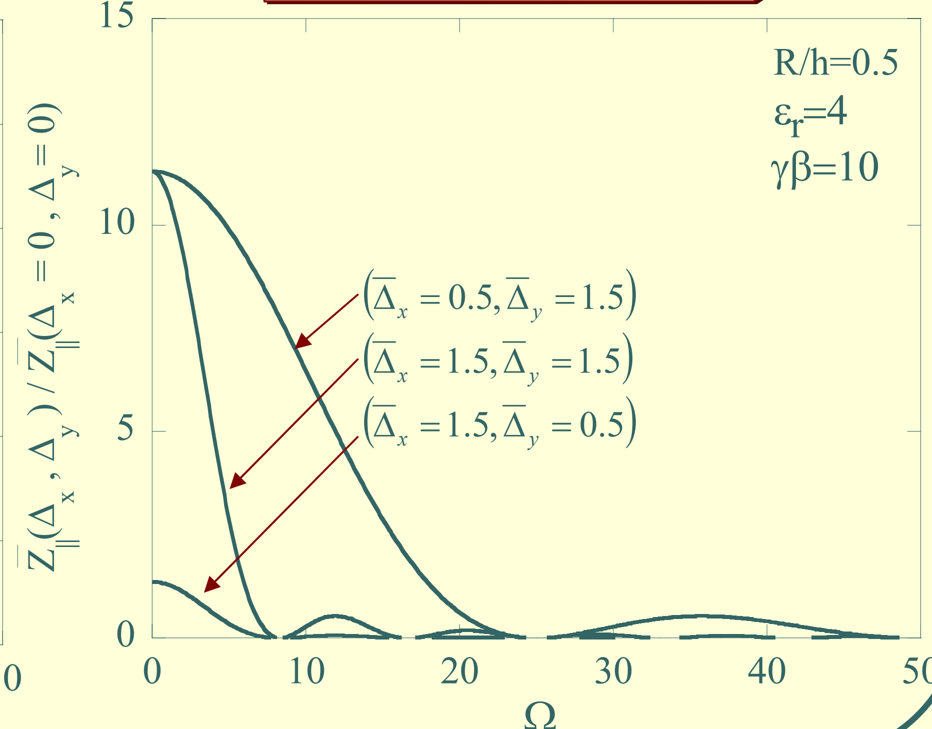
$\varepsilon_r = 4$

$R/h = 0.5$

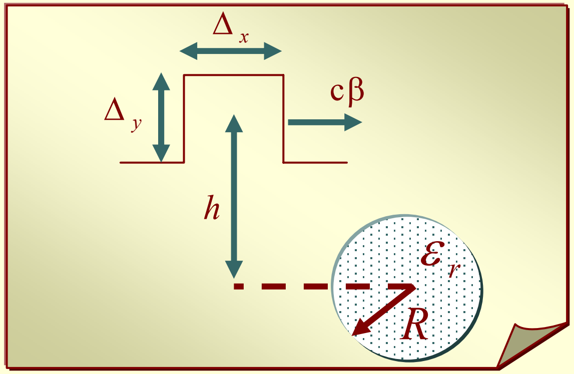
Finite Size Bunch

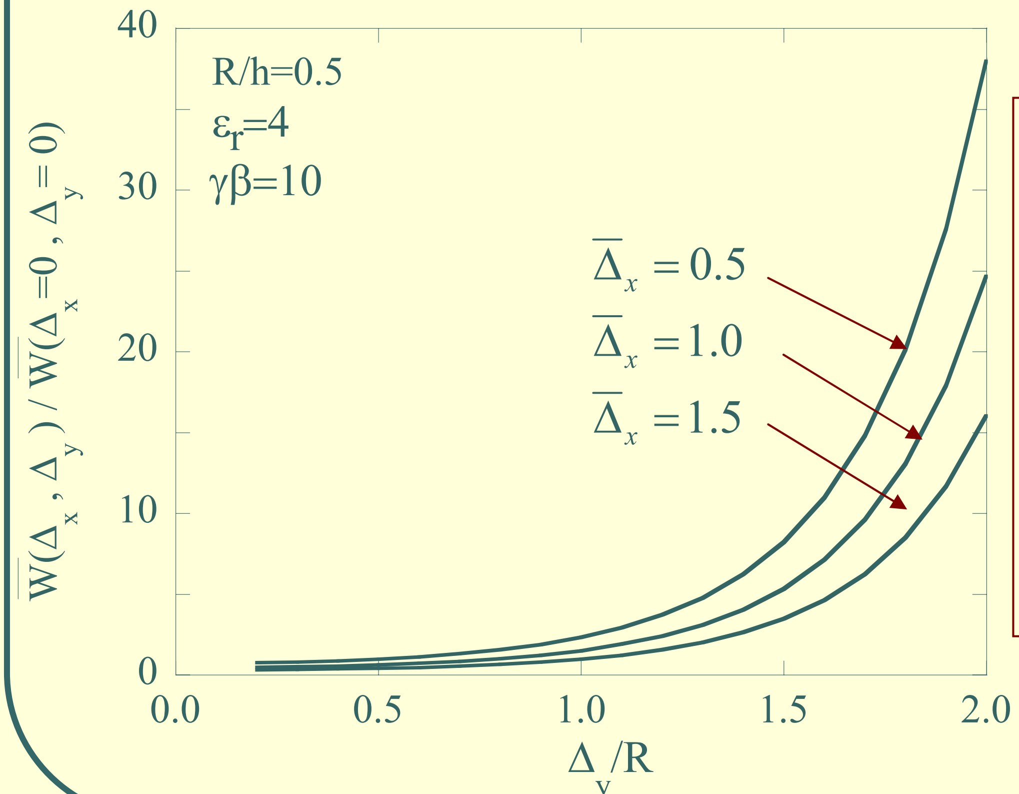






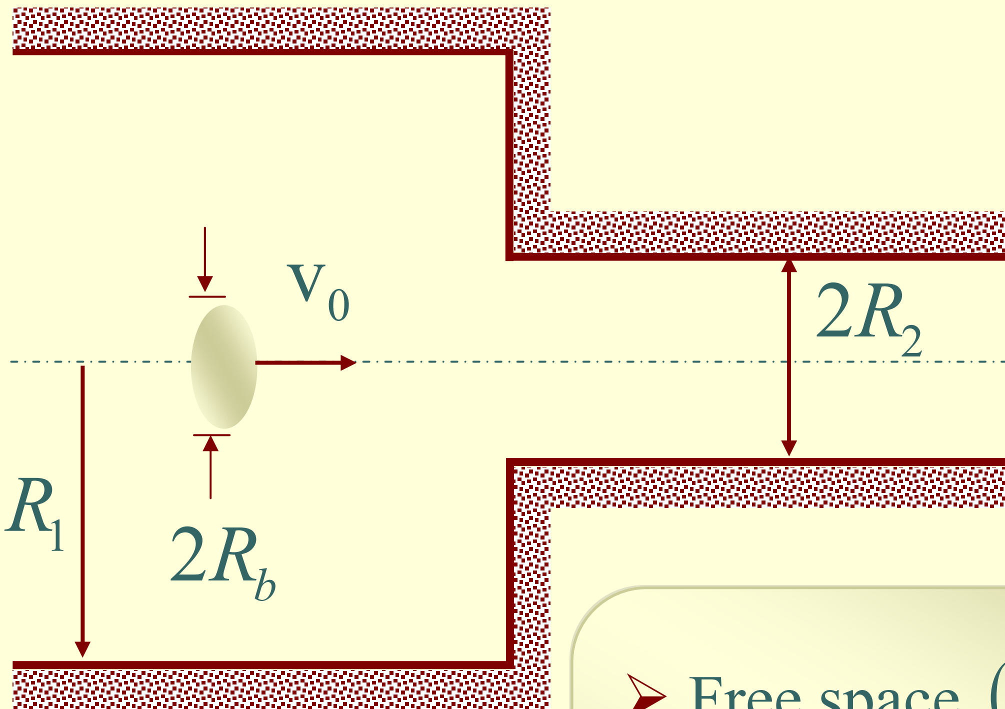
Finite Size Bunch





- Radiated energy increases with the transverse width and decreases with the longitudinal length.
- Radiated energy is Δ_y independent for $\gamma\beta$>>1 and decreases dramatically with the longitudinal length for $\gamma\beta$>>1.

Electron Bunch Traversing a Geometric Discontinuity



- Free space (ε_0, μ_0)
- Symmetric Discontinuity
- Constant velocity

WG Discontinuity: Frequency-Domain Solution

Green's Function

$$J_z(r,z;\omega) = -\frac{q}{(2\pi)^2 r}\delta(r)e^{-j(\omega/\mathrm{v}_0)z}$$

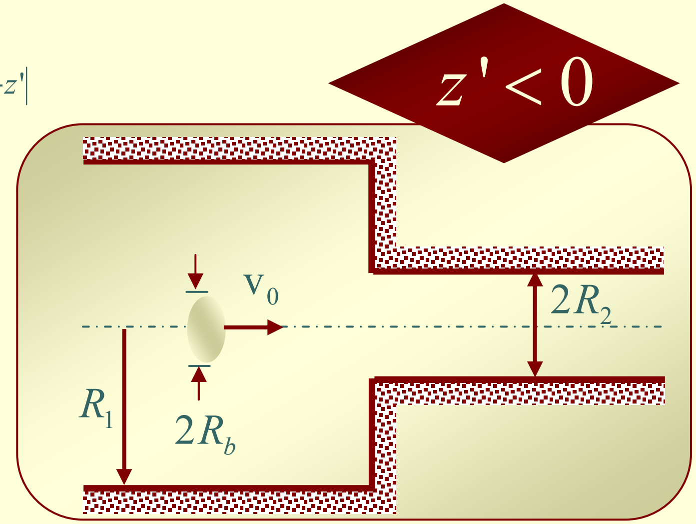
Non-Homogeneous Solution

$$G(z<0,r\,|\,z'<0,r') = \sum_{s=1}^{\infty} \frac{J_0(p_s r/R_1)J_0(p_s r'/R_1)}{\left[\frac{1}{2}R_1^2 J_1^2(p_s)\right]\left[4\pi\Gamma_s^{(1)}\right]} e^{-\Gamma_s^{(1)}|z-z'|}$$

Homogeneous Solution

$$+\sum_{s=1}^{\infty} \rho_s(r',z'<0)J_0(p_s r/R_1)e^{\Gamma_s^{(1)}z}$$

$$G(z>0,r\,|\,z'<0,r') = \sum_{\nu=1}^{\infty} \tau_\nu(r',z'<0)J_0(p_\nu r/R_2)e^{-\Gamma_\nu^{(2)}z}$$



$$\Gamma_s^{(1)} = \sqrt{(p_s/R_1)^2 - (\omega/c)^2}\,; \quad \Gamma_\nu^{(2)} = \sqrt{(p_s/R_2)^2 - (\omega/c)^2}$$

WG Discontinuity: Frequency-Domain Solution

Boundary Conditions

$z = 0$

- ***E_r: continuity over the cross section***
- ***H_φ: continuity across the aperture***

+

- ***Bessel functions orthogonality***
- ***Matrix Formulation***

$$\rho_s(r',z'<0) \rightarrow \underline{R}^{(-)}$$

$$\tau_\nu(r',z'<0) \rightarrow \underline{T}^{(-)}$$

$$g_s^{(1)}(r',z'<0) \rightarrow g^{(1)}$$

$$\underline{R}^{(-)} = \left(\underline{I} + \underline{\underline{Z}}\,\underline{\underline{Y}}\right)^{-1}\left(\underline{I} - \underline{\underline{Z}}\,\underline{\underline{Y}}\right)g^{(1)}$$

$$\underline{T}^{(-)} = \underline{\underline{Y}}\left[\underline{I} + \left(\underline{I} + \underline{\underline{Z}}\,\underline{\underline{Y}}\right)^{-1}\left(\underline{I} - \underline{\underline{Z}}\,\underline{\underline{Y}}\right)\right]g^{(1)}$$

Infinite size Matrices

$$\left\{\underline{\underline{Z}}\right\}_{s,\nu}(\omega) = \frac{\Gamma_\nu^{(2)}}{\Gamma_s^{(1)}}\frac{p_\nu}{p_s}\frac{R_1}{R_2}\frac{1}{J_1^2(p_s)}\frac{2}{R_1^2}\int_0^{R_2} dr\, r J_1\left(p_s\, r/R_1\right) J_1\left(p_\nu\, r/R_2\right)$$

Single Frequency

$$g_s^{(1)}(r',z'<0) = \frac{J_0\left(p_s r'/R_1\right)}{\frac{1}{2}R_1^2 J_1^2(p_s)}\frac{1}{4\pi\Gamma_s^{(1)}}e^{\Gamma_s^{(1)}z'}$$

$z' < 0$

WG Discontinuity: Frequency-Domain Solution

Green's Function

$$J_z(r,z;\omega) = -\frac{q}{(2\pi)^2 r}\delta(r)e^{-j(\omega/\mathrm{v}_0)z}$$

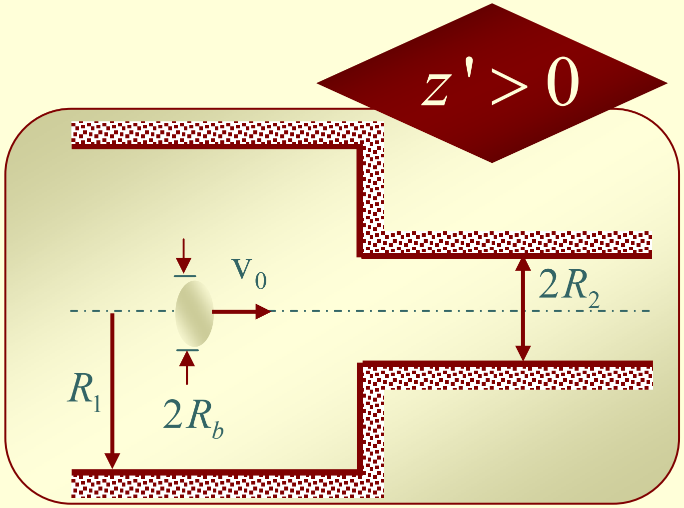
$$G(z>0,r\,|\,z'>0,r') = \sum_{s=1}^{\infty}\rho_s(r',z'>0)J_0\left(p_s\, r/R_2\right)e^{\Gamma_s^{(1)}z}$$

Non-Homogeneous Solution

$$G(z<0,r\,|\,z'>0,r') = \sum_{\nu=1}^{\infty}\frac{J_0\left(p_\nu\, r/R_2\right)J_0\left(p_\nu\, r'/R_2\right)}{\left[\frac{1}{2}R_2^2J_1^2\left(p_\nu\right)\right]\left[4\pi\Gamma_\nu^{(2)}\right]}e^{-\Gamma_\nu^{(2)}|z-z'|}$$

$$+\sum_{\nu=1}^{\infty}\tau_\nu(r',z'>0)J_0\left(p_\nu\, r/R_2\right)e^{-\Gamma_\nu^{(2)}z}$$

Homogeneous Solution



$$\Gamma_s^{(1)} = \sqrt{\left(p_s/R_1\right)^2-\left(\omega/c\right)^2}\,;\quad \Gamma_\nu^{(2)} = \sqrt{\left(p_s/R_2\right)^2-\left(\omega/c\right)^2}$$

WG Discontinuity: Frequency-Domain Solution

Boundary Conditions

$z = 0$

- *E_r: continuity over the cross section*
- *H_φ: continuity across the aperture*

$+$

- *Bessel functions orthogonality*
- *Matrix Formulation*

$$\rho_s(r', z'>0) \rightarrow \underline{R}^{(+)}$$

$$\tau_\nu(r', z'>0) \rightarrow \underline{T}^{(+)}$$

$$g_s^{(2)}(r', z'>0) \rightarrow g^{(2)}$$

$$\underline{R}^{(+)} = \left(\underline{\underline{I}} + \underline{\underline{ZY}}\right)^{-1}\left(\underline{\underline{I}} - \underline{\underline{ZY}}\right) g^{(2)}$$

$$\underline{T}^{(+)} = \underline{\underline{Y}}\left[\underline{\underline{I}} + \left(\underline{\underline{I}} + \underline{\underline{ZY}}\right)^{-1}\left(\underline{\underline{I}} - \underline{\underline{ZY}}\right)\right] g^{(2)}$$

Infinite size Matrices

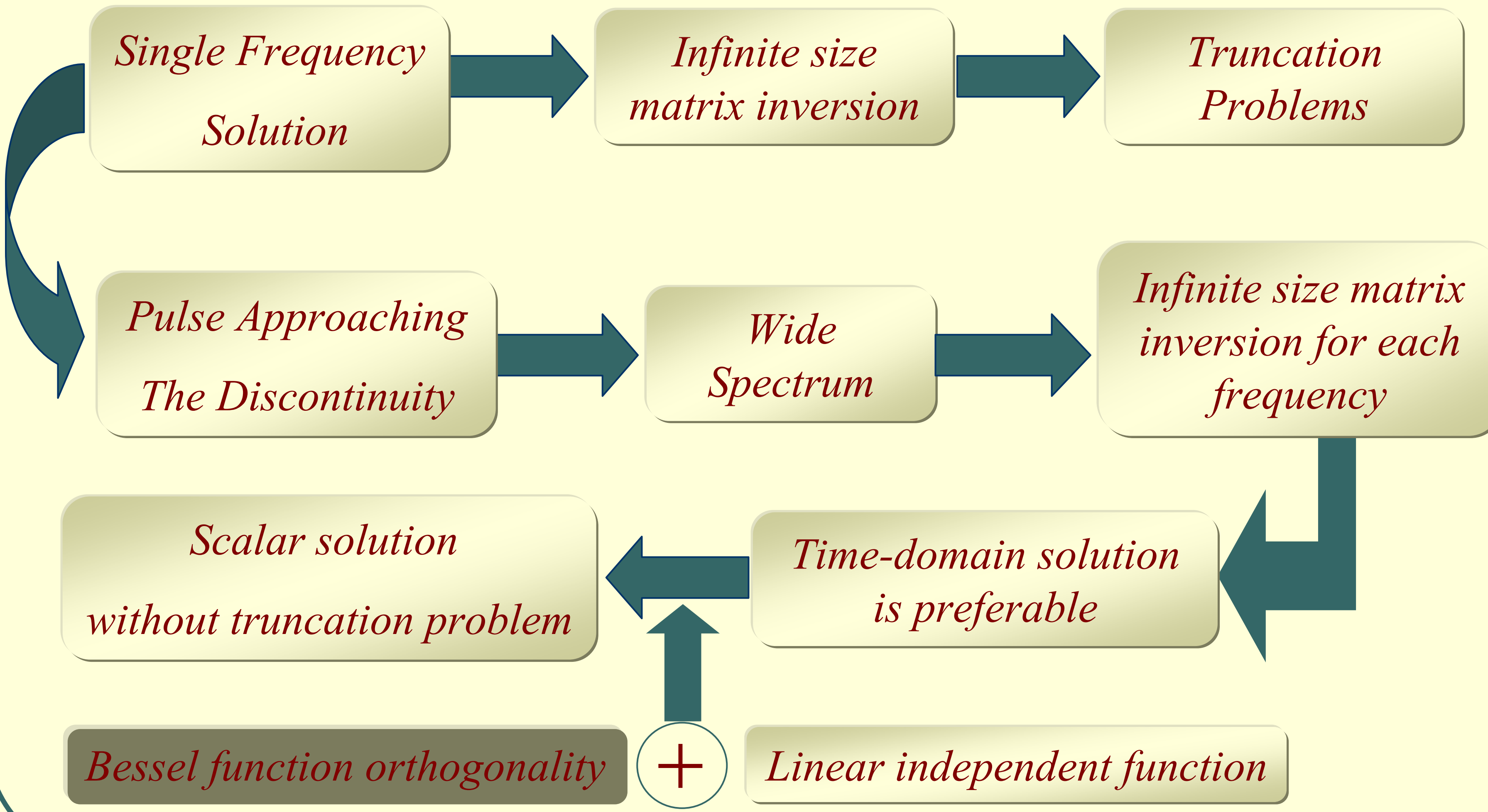
$$\left\{\underline{\underline{Z}}\right\}_{s,\nu}(\omega) = \frac{\Gamma_\nu^{(2)}}{\Gamma_s^{(1)}} \frac{p_\nu}{p_s} \frac{R_1}{R_2} \frac{1}{J_1^2(p_s)} \frac{2}{R_1^2} \int_0^{R_2} dr\, r J_1(p_s r/R_1) J_1(p_\nu r/R_2)$$

Single Frequency

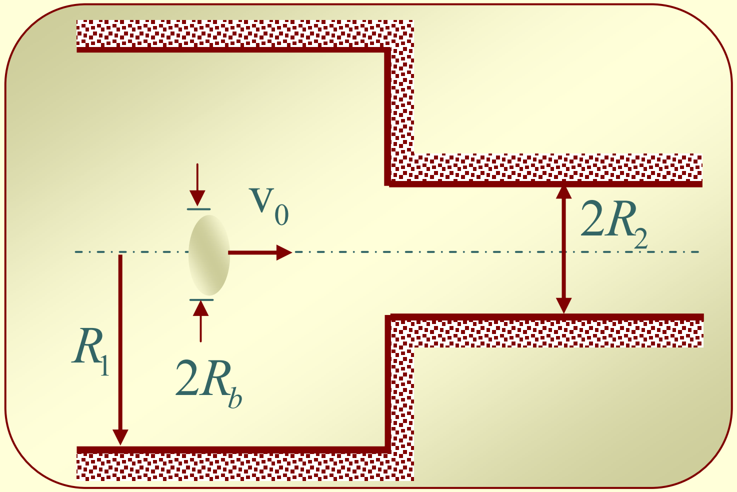
$$g_\nu^{(2)}(r', z'>0) = \frac{J_0(p_\nu r'/R_2)}{\frac{1}{2} R_2^2 J_1^2(p_\nu)} \frac{1}{4\pi \Gamma_\nu^{(2)}} e^{-\Gamma_\nu^{(2)} z'}$$

$z' > 0$

WG Discontinuity: Frequency-Domain Solution



WG Discontinuity: Time-Domain Solution



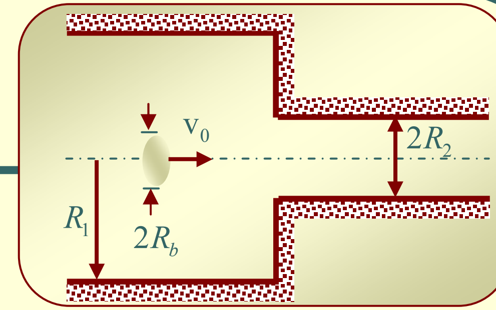
Primary Field

$$A_z^{(p)}(r,z;t) = \begin{cases} \sum_{s=1}^{\infty} \alpha_{s,1} J_0\left(p_s \frac{r}{R_1}\right) e^{-\Omega_{s,1}\left|t-\frac{z}{v_0}\right|}; & t<0, \quad z<0 \\ \sum_{\nu=1}^{\infty} \alpha_{\nu,2} J_0\left(p_\nu \frac{r}{R_2}\right) e^{-\Omega_{\nu,2}\left|t-\frac{z}{v_0}\right|}; & t>0, \quad z>0 \end{cases}$$

$$\Omega_{i,j} = \frac{p_i \gamma \beta c}{R_j}; \quad \alpha_{i,j} = \frac{-q}{2\pi\varepsilon_0 R_j^2}(\gamma\beta)^2 \frac{1}{\Omega_{i,j} J_1^2(p_i)}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}; \quad \beta = \frac{v_0}{c}; \quad i = s,\nu; \; j = 1,2$$

Formulation: Secondary Field t<0



Metallic Boundaries → ***Radial-dependence*** →

$$\begin{cases} J_0\left(\dfrac{p_s r}{R_1}\right); & z<0 \\ J_0\left(\dfrac{p_v r}{R_2}\right); & z>0 \end{cases}$$

Primary Field

$$e^{-\Omega_{s,1}\left|t-\frac{z}{v_0}\right|}$$

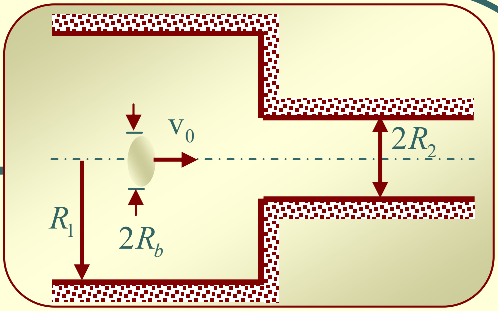
→ ***Time-dependence*** →

$$e^{\Omega_{s,1} t}$$

Long.- dependence ↓

$$\begin{cases} e^{\Omega_{s,1}\frac{z}{v_0}}; & z<0 \\ e^{-z\sqrt{\left(\frac{p_v}{R_2}\right)^2+\left(\frac{p_s\gamma\beta}{R_1}\right)^2}}; & z>0 \end{cases}$$

Formulation: Secondary Field t>0



Metallic Boundaries → **Radial-dependence** →

$$\begin{cases} J_0\left(\dfrac{p_s r}{R_1}\right); & z<0 \\ J_0\left(\dfrac{p_\nu r}{R_2}\right); & z>0 \end{cases}$$

Primary Field

$$e^{-\Omega_{\nu,2}\left|t-\frac{z}{v_0}\right|}$$

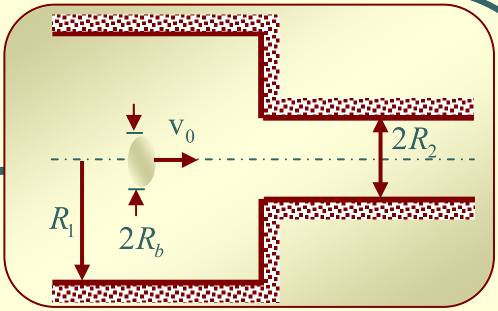
→ **Time-dependence** →

$$e^{-\Omega_{\nu,2}t}$$

Long.- dependence

$$\begin{cases} e^{-\Omega_{\nu,2}\frac{z}{v_0}}; & z>0 \\ e^{z\sqrt{\left(\frac{p_s}{R_1}\right)^2+\left(\frac{p_\nu \gamma\beta}{R_2}\right)^2}}; & z<0 \end{cases}$$

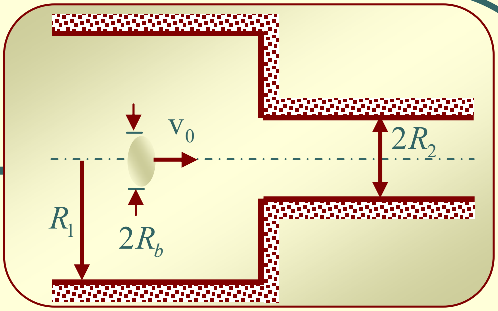
Secondary Field



$$A_z^{(s)}(r,z;t<0)=\begin{cases}\sum_{s=1}^{\infty}\rho_s J_0\left(p_s\frac{r}{R_1}\right)e^{\Omega_{s,1}t}e^{\Omega_{s,1}\frac{z}{v_0}}; & z<0\\ \sum_{s,\nu=1}^{\infty}\tau_{s,\nu}J_0\left(p_\nu\frac{r}{R_2}\right)e^{\Omega_{s,1}t}e^{-z\sqrt{\left(\frac{p_\nu}{R_2}\right)^2+\left(\frac{p_s\gamma\beta}{R_1}\right)^2}}; & z>0\end{cases}$$

$$A_z^{(s)}(r,z;t>0)=\begin{cases}\sum_{s,\nu=1}^{\infty}\bar{\rho}_{s,\nu}J_0\left(p_s\frac{r}{R_1}\right)e^{-\Omega_{\nu,2}t}e^{z\sqrt{\left(\frac{p_s}{R_1}\right)^2+\left(\frac{p_\nu\gamma\beta}{R_2}\right)^2}}; & z<0\\ \sum_{\nu=1}^{\infty}\bar{\tau}_s J_0\left(p_\nu\frac{r}{R_2}\right)e^{-\Omega_{\nu,2}t}e^{-\Omega_{\nu,2}\frac{z}{v_0}}; & z>0\end{cases}$$

Boundary Conditions



$$z = 0; \quad -\infty < t < 0$$

- E_r **: continuity over the cross section**
- H_φ **: continuity across the aperture**

$$z = 0; \quad 0 < t < \infty$$

- E_r **: continuity over the cross section**
- H_φ **: continuity across the aperture**

$$e^{\Omega_{s,1} t}$$

Linearly independent functions

$$J_0\left(p_s r / R\right)$$

Orthogonal set of functions

$$e^{-\Omega_{\nu,2} t}$$

Linearly independent functions

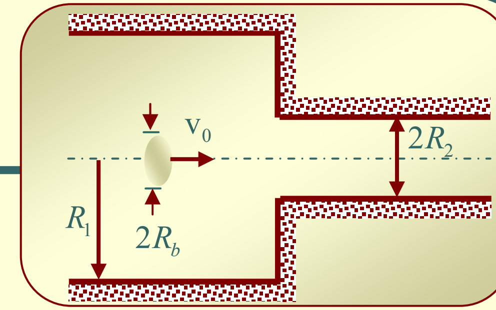
$$\tau_{s,\nu} \;\&\; \rho_s$$

are determined

$$\overline{\rho}_{s,\nu} \;\&\; \overline{\tau}_\nu$$

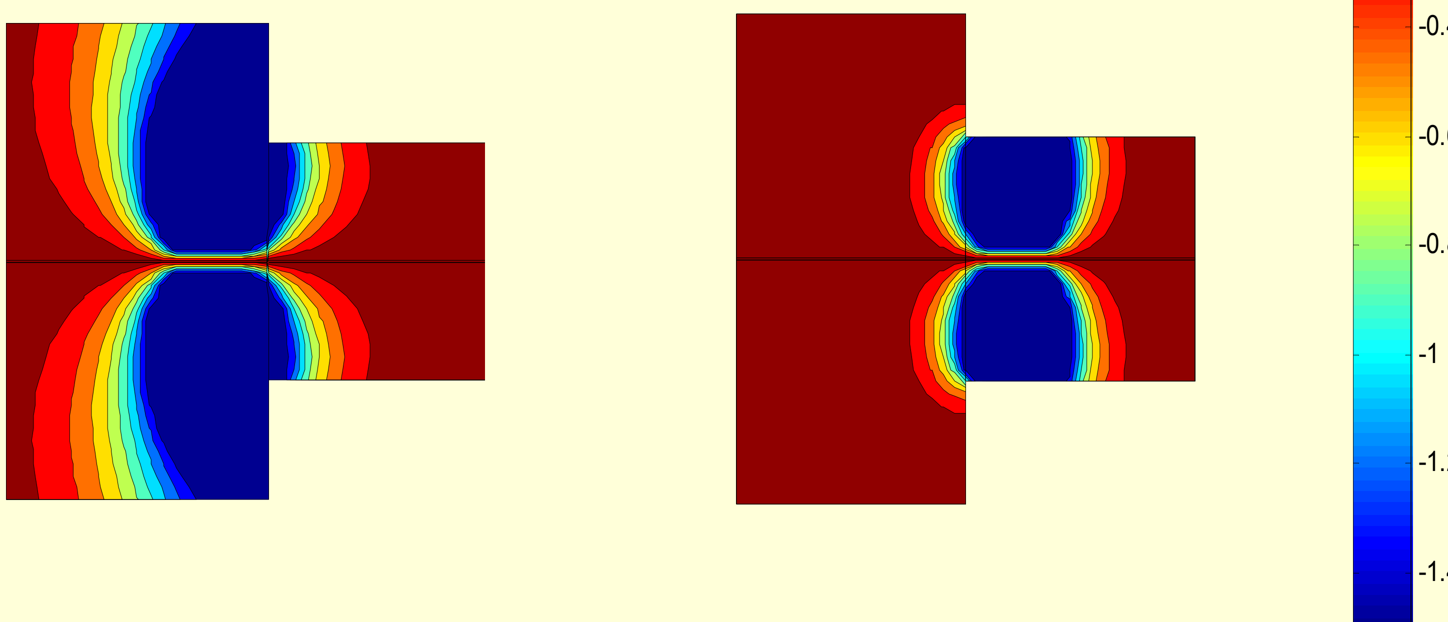
are determined

Magnetic Field



$$t = -0.25\frac{R_1}{c}$$

$$t = 0.25\frac{R_1}{c}$$

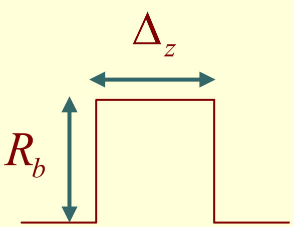


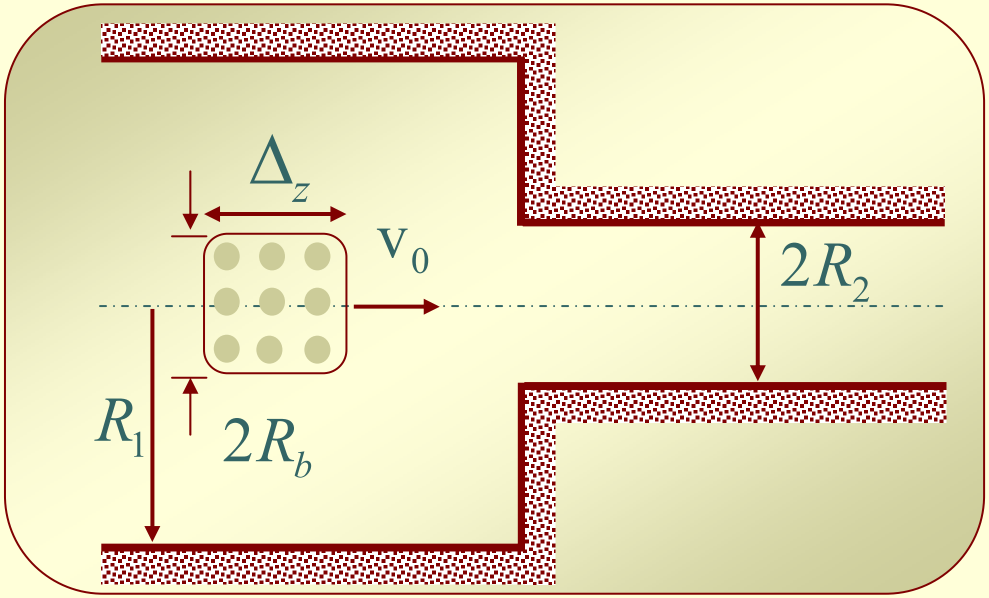
$$\gamma = 2,\ R_2 = 0.5R_1, R_b = 0$$

Finite Size Effect

$$J_z(r,z;t) = -\frac{q\mathrm{v}_0}{2\pi}\frac{2}{R_b^2}U(R_b - r)U\left(\Delta_z/2 - |z - \mathrm{v}_0 t|\right)$$

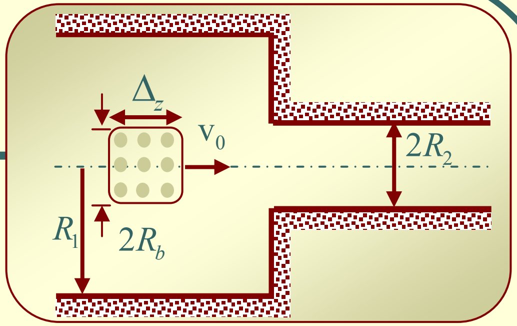
$$U(\xi) = \begin{cases} 0 & \xi \le 0 \\ 1 & \xi > 0 \end{cases}$$



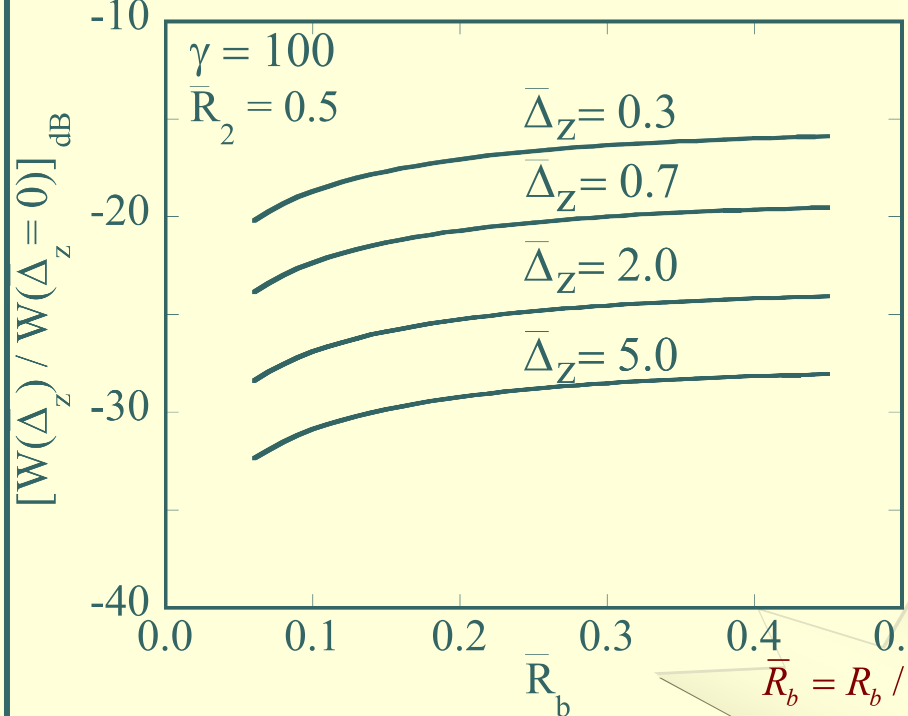


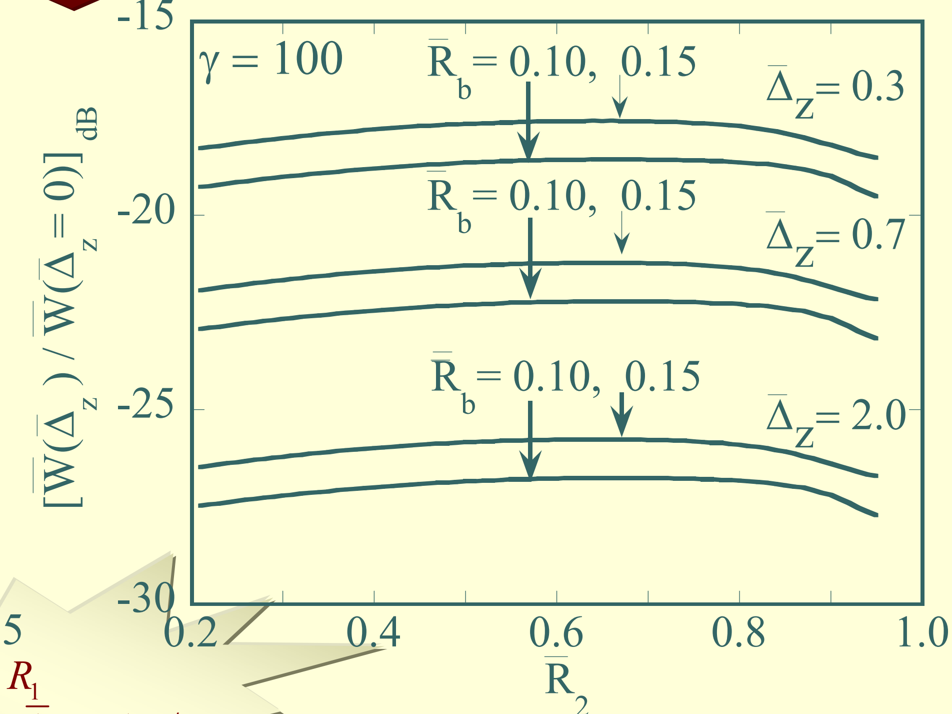
1. $z < 0; \quad t < -\frac{\Delta_z}{2\mathrm{v}_0}$
2. $z < 0; \quad t < \left|\frac{\Delta_z}{2\mathrm{v}_0}\right|$
3. $z > 0; \quad t > \frac{\Delta_z}{2\mathrm{v}_0}$
4. $z > 0; \quad t < \left|\frac{\Delta_z}{2\mathrm{v}_0}\right|$

Radiated Energy



$$\overline{W} \equiv \left(\frac{q^2}{2\pi\varepsilon_0 R_1} \right)^{-1} W$$



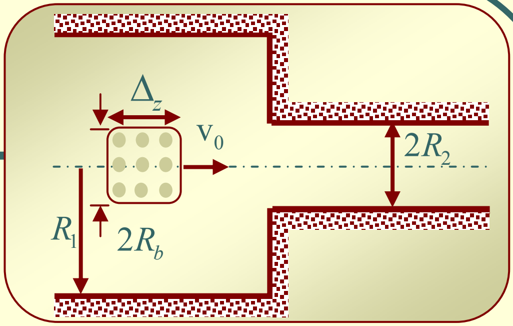


$\overline{R}_b = R_b / R_1$

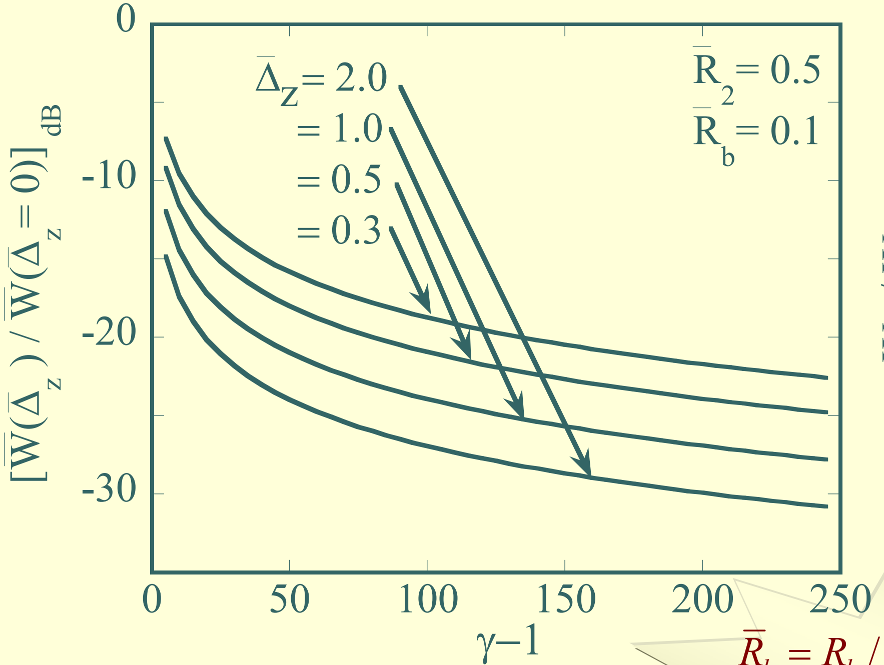
$\overline{\Delta}_z = \Delta_z / R_1$

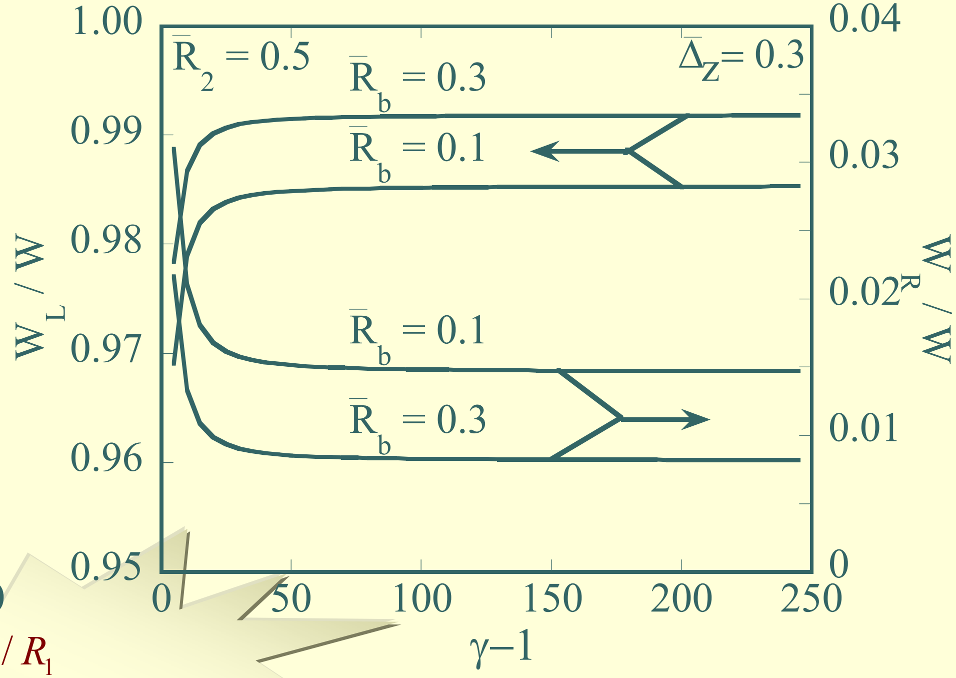
$\overline{R}_2 = R_2 / R_1$

Radiated Energy



$$\overline{W} \equiv \left(\frac{q^2}{2\pi\varepsilon_0 R_1} \right)^{-1} W$$



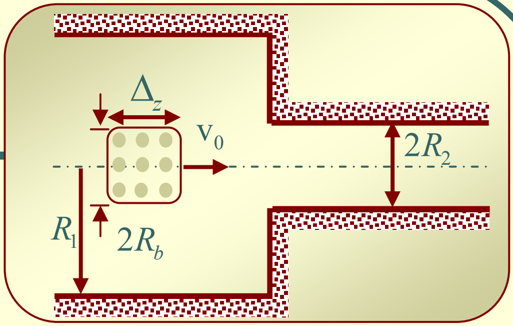


$\overline{R}_b = R_b / R_1$

$\overline{\Delta}_z = \Delta_z / R_1$

$\overline{R}_2 = R_2 / R_1$

Radiated Energy



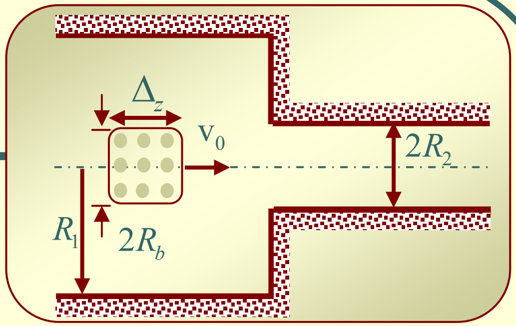
$$\overline{W} \approx \frac{1}{\gamma} \frac{1}{\left(R_b / R_1\right)^{0.38}} \frac{1 - R_2 / R_1}{\cosh^2\left[\left(\gamma \Delta_z / R_1\right)^{0.25}\right]}$$

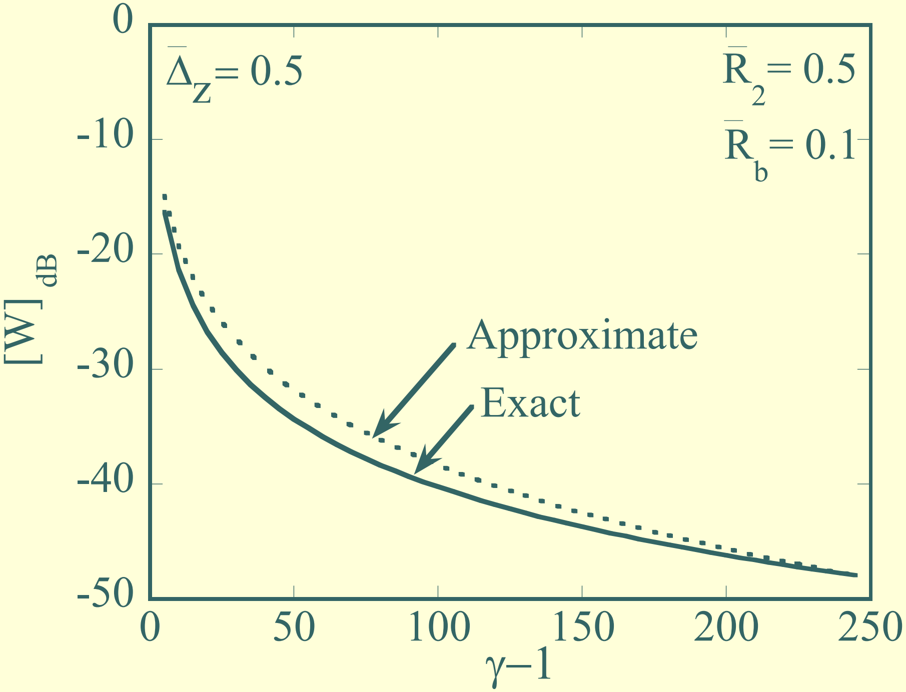
$$\overline{W} \equiv \left(\frac{q^2}{2\pi\varepsilon_0 R_1} \right)^{-1} W$$

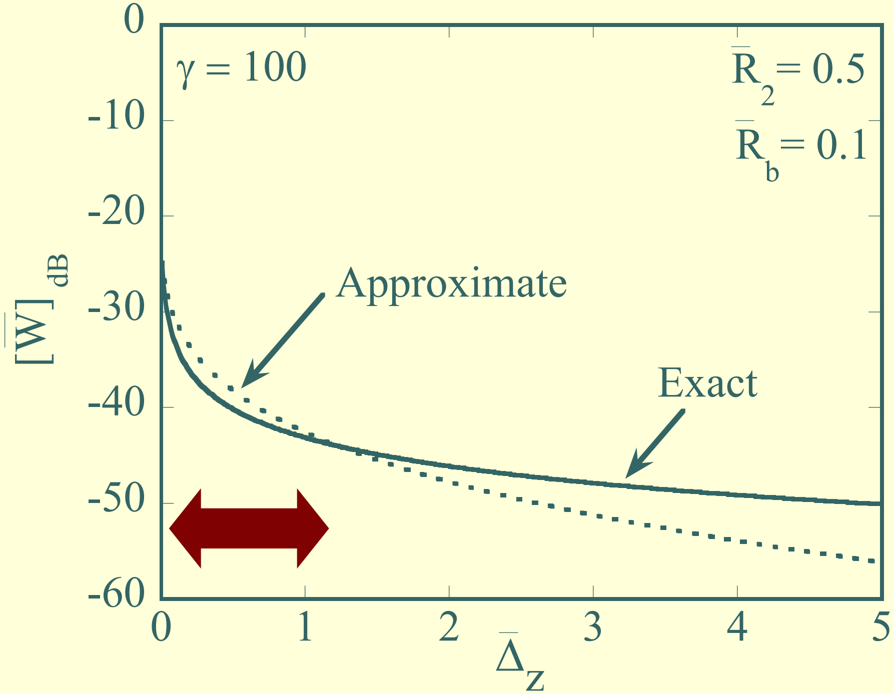
1. $0 \le \Delta_z < R_1$
2. $0 < R_b < R_2 < R_1$
3. $\gamma > 10$

1. $R_b \to 0$	$\Rightarrow \quad \overline{W} \to \infty$	Point Charge
2. $R_2 \to R_1$	$\Rightarrow \quad \overline{W} \to 0$	Uniform Waveguide
3. $\gamma \to \infty$	$\Rightarrow \quad \overline{W} \to 0$	Ultra-Relativistic

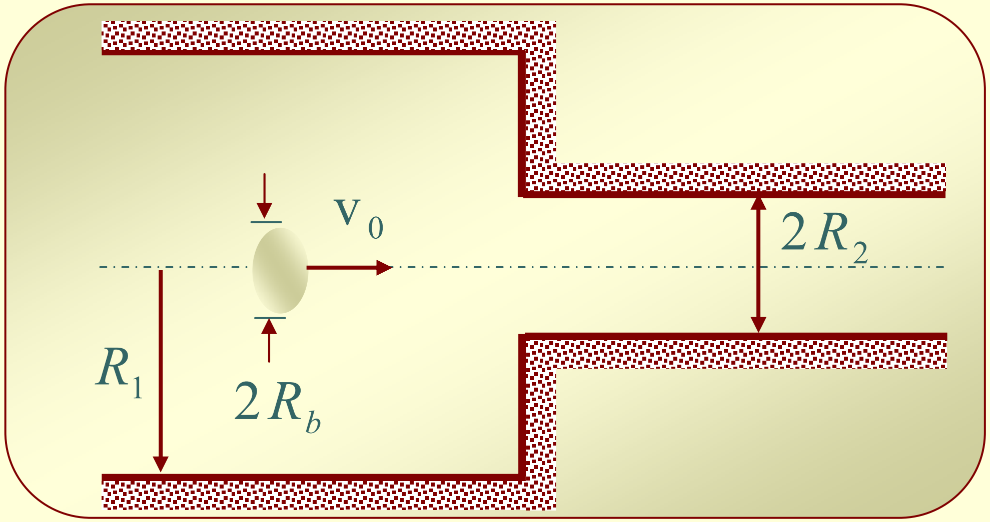
Exact vs. Approximate







WG Discontinuity: Time-Domain Solution



- **Time-domain** analytic solution of the evanescent field in the region of the discontinuity.
- The solution uses the **orthogonality** of the Bessel functions and the fact that exponential functions describing the temporal behavior are **linearly independent.**
- Simulations based on this formulation enabled the development of an **analytic** expression for the energy required to maintain the velocity of the bunch constant.

Surface Roughness: Motivation

X-band

$$\lambda \approx 1cm$$

$$accuracy \approx 1\mu m$$

Optical

$$\lambda \approx 1\mu m$$

$$accuracy \overset{?}{\approx} 1\overset{\circ}{A}$$

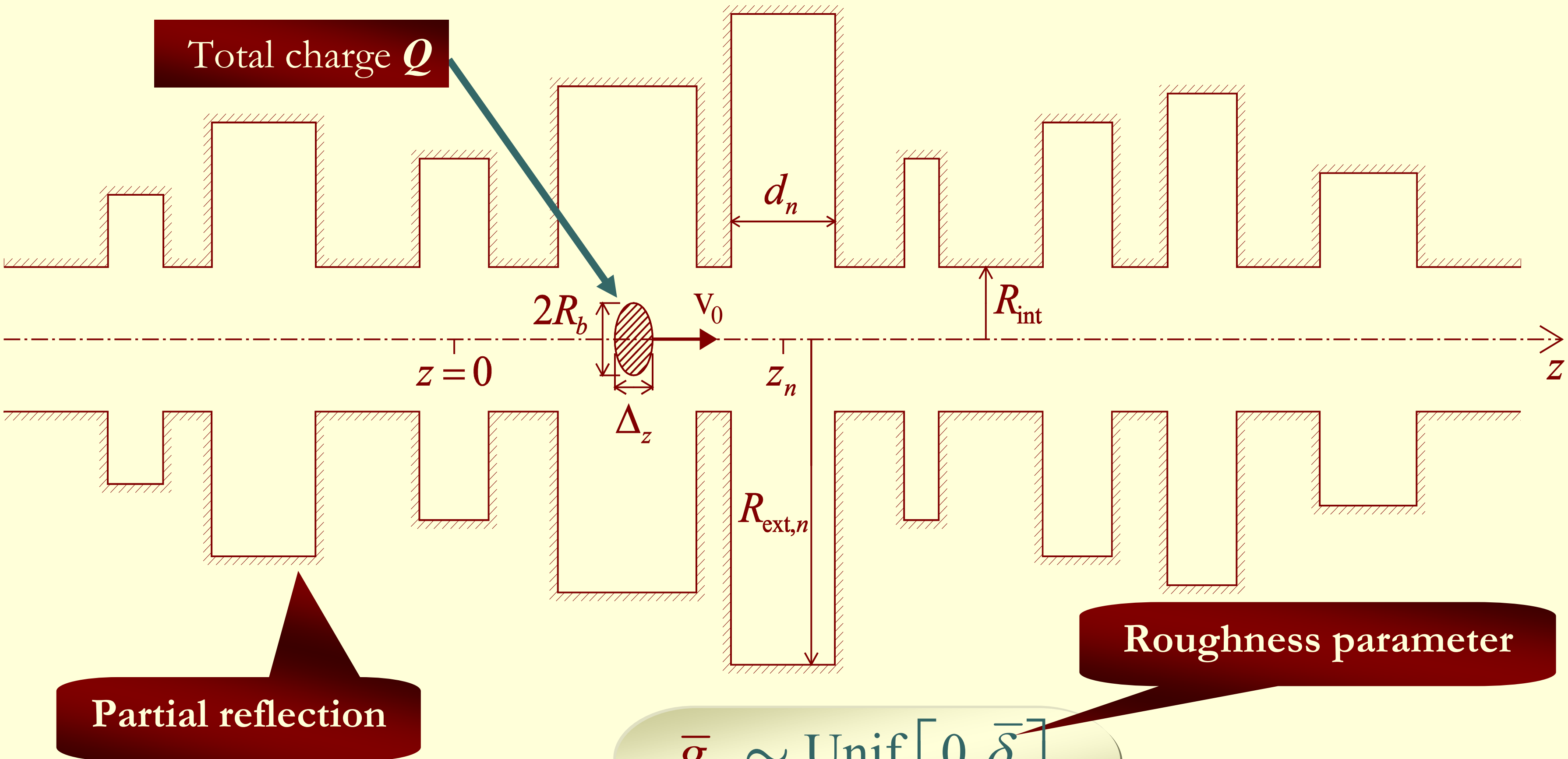
Dimensions Constrains

- Atomic level engineering.
- The size of the bunch as well as the roughness of the structure are anticipated to be of the same order of magnitude.

Source Properties

- Superposition of plane waves.
- Broad frequency spectrum.
- An assumption of small perturbation is not necessarily relevant.

Configuration



$$\bar{g}_n \sim \mathrm{Unif}\left[0, \bar{\delta}\right]$$

$$\bar{R}_{\text{ext},n} \equiv \frac{R_{\text{ext},n}}{R_{\text{int}}} = 1 + \bar{g}_n\,; \quad \bar{d}_n \equiv \frac{d_n}{R_{\text{int}}} = \bar{g}_n\,; \quad \bar{z}_1 \equiv \frac{z_1}{R_{\text{int}}} = 0; \quad \bar{z}_{n+1} = \bar{z}_n + \frac{\bar{d}_n}{2} + \frac{\bar{d}_{n+1}}{2} + \bar{g}_n$$

Wake-Field

Current Density

$$J_z(r,z;\omega) = -\frac{Q}{(2\pi)^2}\frac{2}{R_b^2}U(R_b - r)\,\mathrm{sinc}\left(\frac{1}{2}\frac{\omega}{v_0}\Delta_z\right)e^{-j\omega\frac{z}{v_0}}$$

$$\mathrm{sinc}(\xi) \equiv \frac{\sin(\xi)}{\xi}$$

$$U(\zeta) \equiv \begin{cases} 1; & \zeta > 0 \\ 0; & \zeta \le 0 \end{cases}$$

Magnetic Vector Potential

$$\left[\nabla^2 + \left(\frac{\omega}{c}\right)^2\right]A_z(r,z;\omega) = -\mu_0 J_z(r,z;\omega)$$

Wake-Field

$r < R_{\text{int}}$

$$A_z(r,z;\omega) = \underbrace{2\pi\mu_0 \int_0^{R_b} dr'r' \int_{-\infty}^{\infty} dz' G(r,z\,|\,r',z') J_z(r',z';\omega)}_{\text{Non-homogeneous solution}} + \underbrace{\int_{-\infty}^{\infty} dk A(k) e^{-jkz} \mathrm{I}_0(\Gamma r)}_{\text{homogeneous solution}}$$

$$G(r,z\,|\,r',z') = \int_{-\infty}^{\infty} dk e^{-jk(z-z')} g_k(r\,|\,r')$$

Green's function in a boundless space

$$g_k(r\,|\,r') = \frac{1}{(2\pi)^2} \begin{cases} \mathrm{I}_0(\Gamma r)\mathrm{K}_0(\Gamma r'); & 0 \le r \le r' \\ \mathrm{K}_0(\Gamma r)\mathrm{I}_0(\Gamma r'); & r' \le r < \infty \end{cases}$$

$$\Gamma^2 \equiv k^2 - \left(\frac{\omega}{c}\right)^2$$

Wake-Field

$R_b < r < R_{\text{int}}$

$$A_z\left(r > R_b, z;\omega\right) = \int_{-\infty}^{\infty} dk\left[A(k)\mathrm{I}_0(\Gamma r) + B(k)\mathrm{K}_0(\Gamma r)\right]e^{-jkz}$$

$$B(k) = -Q\mu_0\frac{1}{(2\pi)^2}\mathrm{sinc}\left(\frac{1}{2}\frac{\omega}{\mathrm{v}_0}\Delta_z\right)\left[\frac{2\mathrm{I}_1(u)}{u}\right]\delta\left(k - \frac{\omega}{\mathrm{v}_0}\right)$$

$u \equiv \frac{\omega}{\mathrm{v}_0\gamma}R_b$

n^th groove

$$A_z^n\left(r, z;\omega\right) = D_n(\omega)\mathrm{T}_{0,n}\left(\frac{\omega}{c}r\right)$$

Single mode approximation

$$\mathrm{T}_{0,n}\left(\frac{\omega}{c}r\right) \equiv \mathrm{J}_0\left(\frac{\omega}{c}r\right)\mathrm{Y}_0\left(\frac{\omega}{c}R_{\text{ext},n}\right) - \mathrm{Y}_0\left(\frac{\omega}{c}r\right)\mathrm{I}_0\left(\frac{\omega}{c}R_{\text{ext},n}\right)$$

Wake-Field : Continuity of E_z & H_φ

$$\Omega \equiv \frac{\omega}{c} R_{\text{int}}$$

$$A_z(k) = -\frac{1}{2\pi}\frac{\Omega^2}{\Delta^2 \mathrm{I}_0(\Delta)}\sum_{n=1}^{N} D_n(\omega)\mathrm{T}_{0.n}(\Omega)\Im_n(k) - B(k)\frac{\mathrm{K}_0(\Delta)}{\mathrm{I}_0(\Delta)}$$

$$\Delta \equiv \Gamma R_{\text{int}}$$

$$\sum_{m=1}^{N} \tau_{n,m}(\omega) D_m(\omega) = S_n(\omega)$$

$$\tau_{n,m}(\omega) \equiv \mathrm{T}_{1,n}(\Omega)\delta_{n,m} - \mathrm{T}_{0,m}(\Omega)\chi_{n,m}$$

$$\mathrm{T}_{1,n}(\Omega) \equiv \mathrm{J}_1(\Omega)\mathrm{Y}_0\left(\frac{\omega}{c}R_{\text{ext},n}\right) - \mathrm{Y}_1(\Omega)\mathrm{I}_0\left(\frac{\omega}{c}R_{\text{ext},n}\right)$$

$$\Im_n(k) = e^{jkz_n}\operatorname{sinc}\left(\frac{1}{2}kd_n\right)$$

$$S_n = \frac{1}{\Omega}\int_{-\infty}^{\infty} dk \frac{1}{\mathrm{I}_0(\Delta)} B(k)\Im_n^*(k)$$

Power & Energy

Emitted Power

Secondary field due to metallic surface

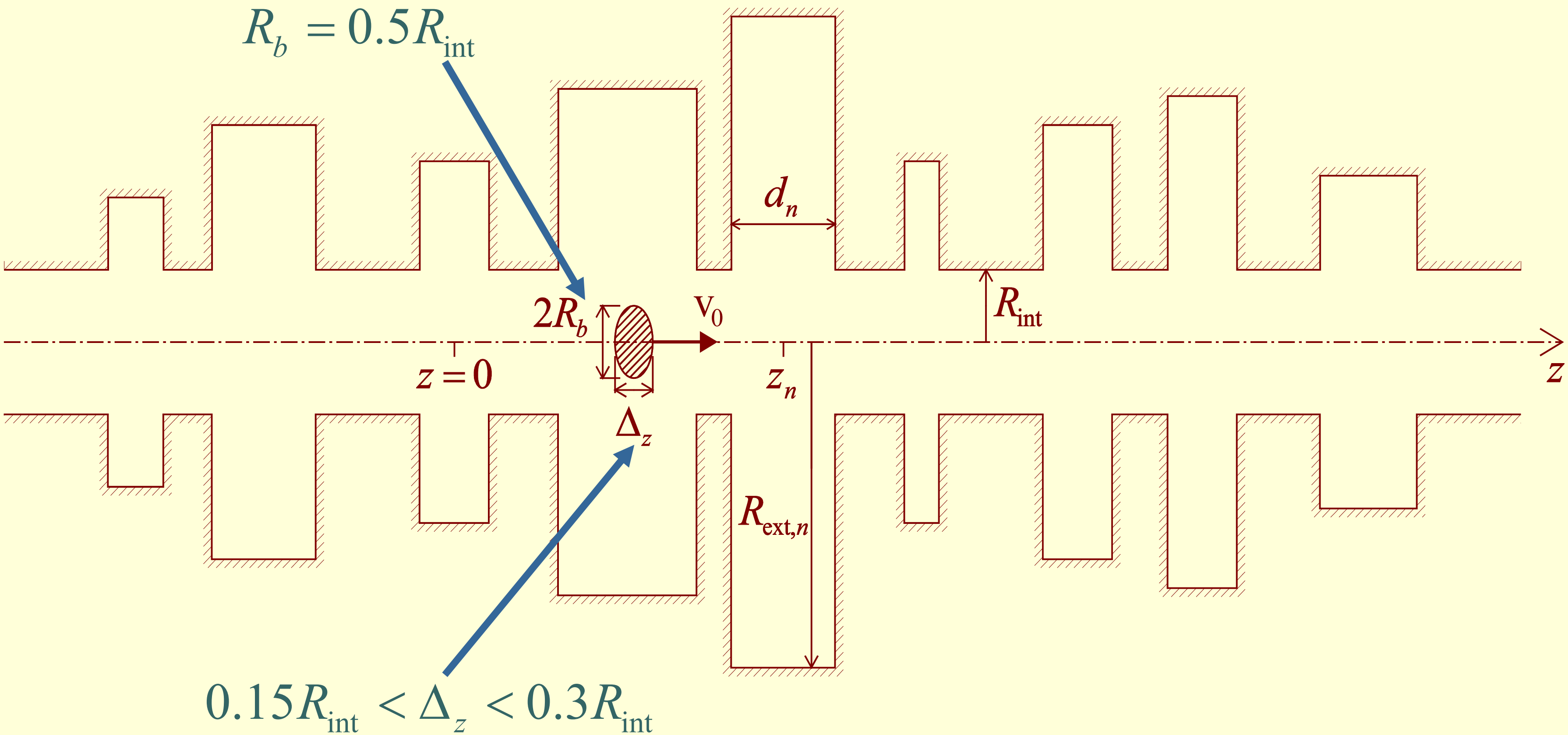
$$P(t) = 2\pi \int_0^{R_b} r\,dr \int_{-\infty}^{\infty} dz\, J_z(r,z;t)\, E_z^{(s)}(r,z;t)$$

Emitted Energy

$$W \equiv \int_{-\infty}^{\infty} dt\; P(t) = \frac{Q^2}{4\pi\varepsilon_0 R_{\text{int}}} \operatorname{Re}\left[\int_0^{\infty} d\Omega\; S(\Omega) \right] \equiv \frac{Q^2}{4\pi\varepsilon_0 R_{\text{int}}} \bar{W}$$

Normalized spectrum

Typical Dimensions

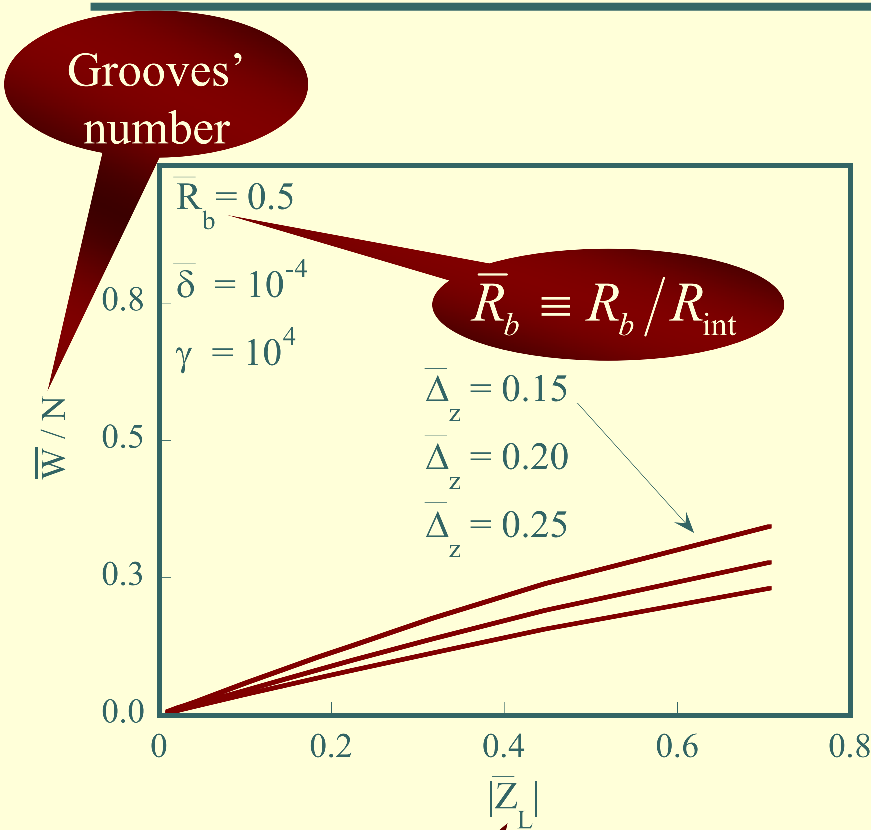


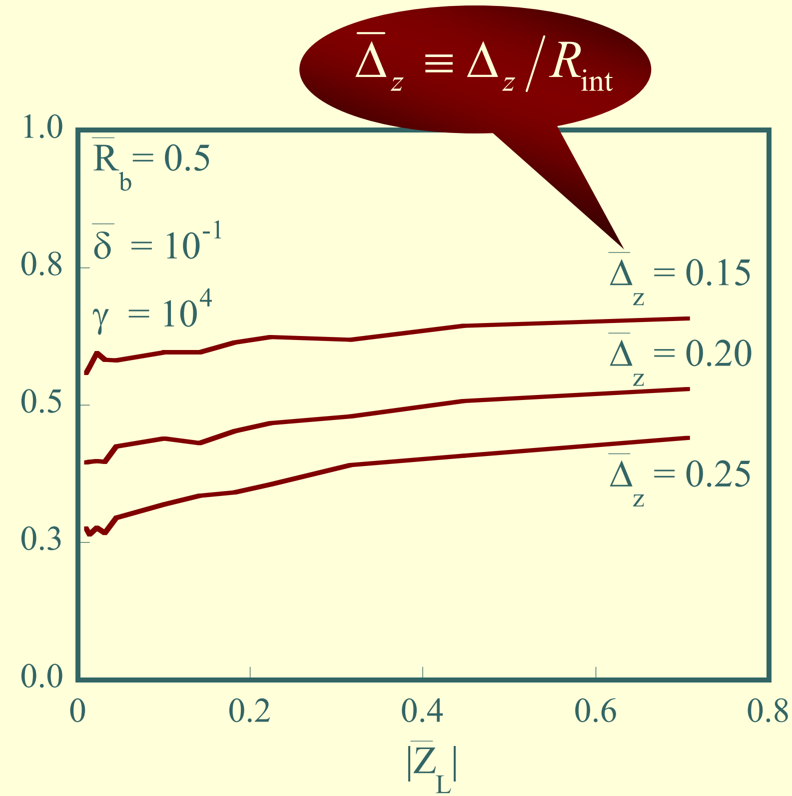
$R_{int} \approx 0.5\mu m \Rightarrow 0 \le \overline{\delta} \le 0.2$

Bunch: $30^\circ \div 45^\circ \approx 1\mu m$

Each data point is a result of averaging over 80 different distributions for a given value of $\overline{\delta}$

Partial Reflection in Grooves

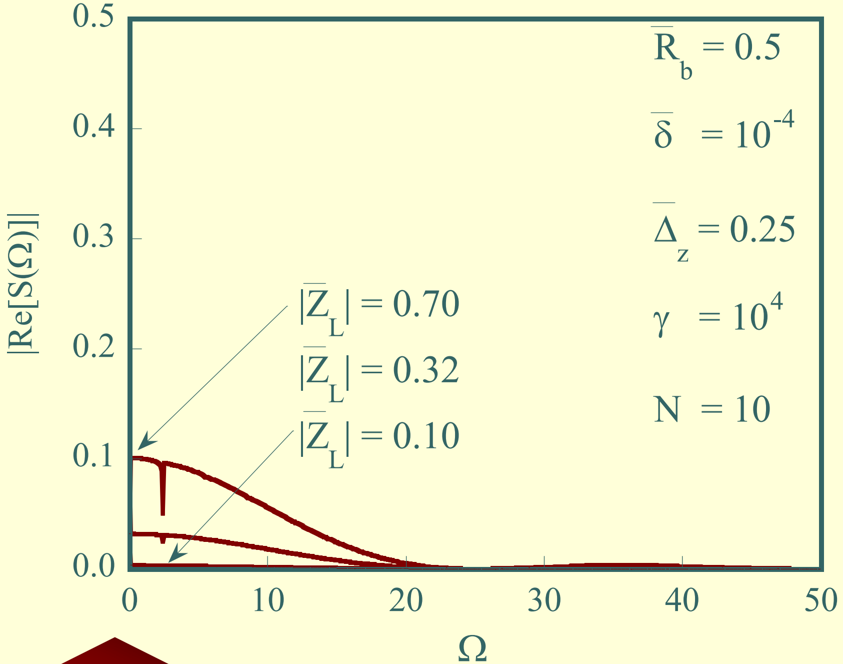


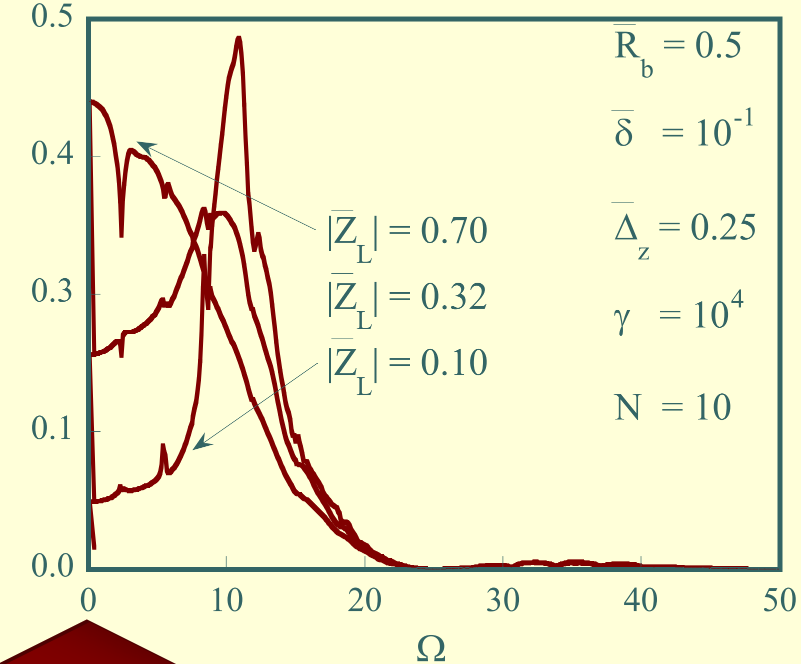


Normalized impedance @ grooves' end

- Energy almost linear with the load impedance @ $\overline{\delta} = 10^{-4}$.
- Energy weakly dependent on the load impedance @ $\overline{\delta} = 10^{-1}$.

Partial Reflection in Grooves





$\bar{\delta} = 10^{-4}$

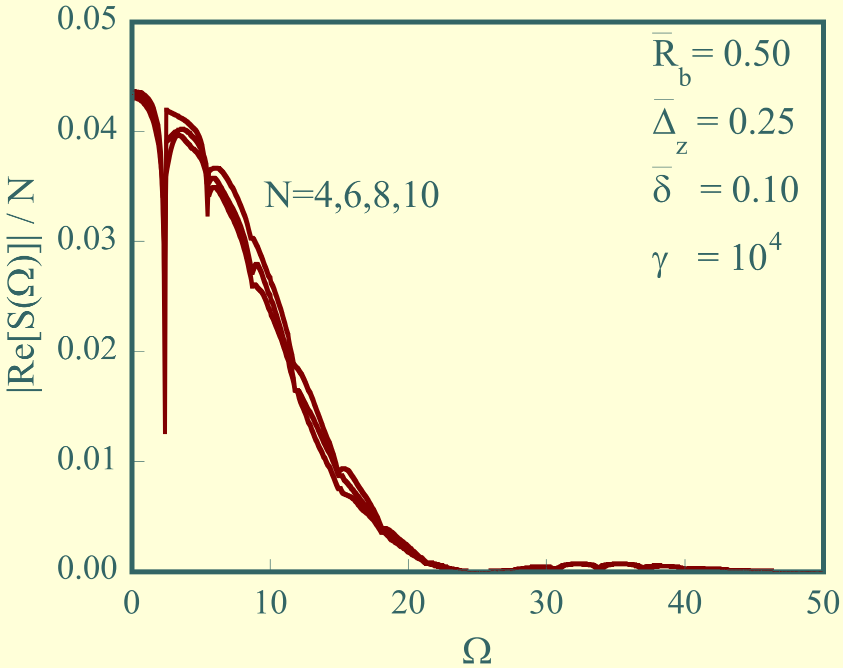
- Spectrum's peak close to $\Omega = 0$.
- Spectrum's width determined by the bunch's spectrum: **sinc²** shape.

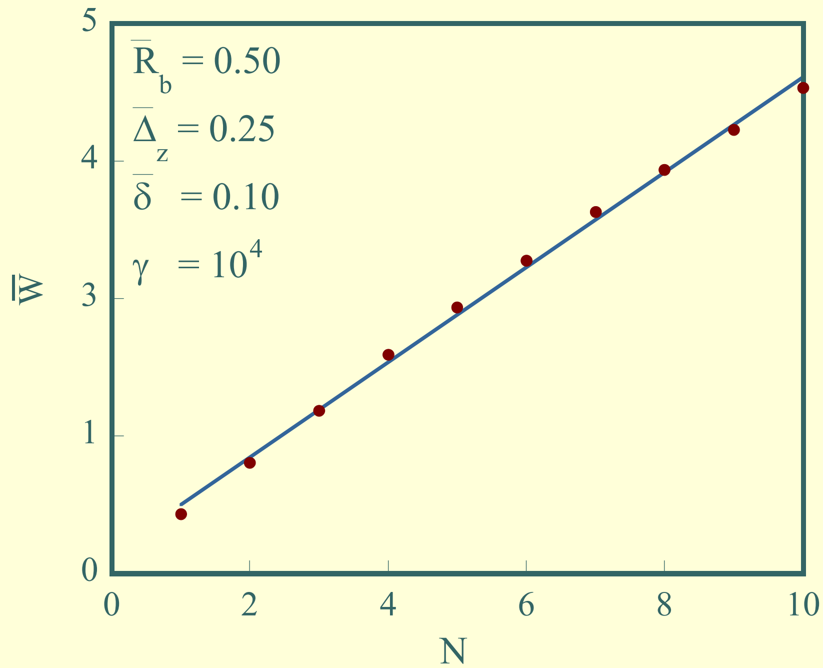
$\bar{\delta} = 10^{-1}$

- Increasing the load impedance shifts the main peak of the spectrum towards $\Omega = 0$.

Number of Grooves

$\left|\bar{Z}_L\right| = 1/\sqrt{2}$



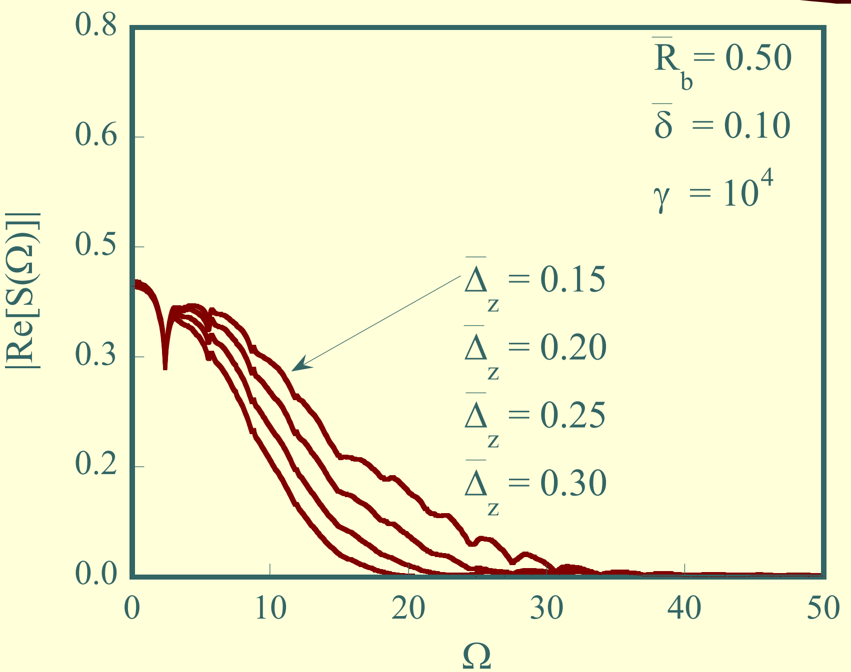


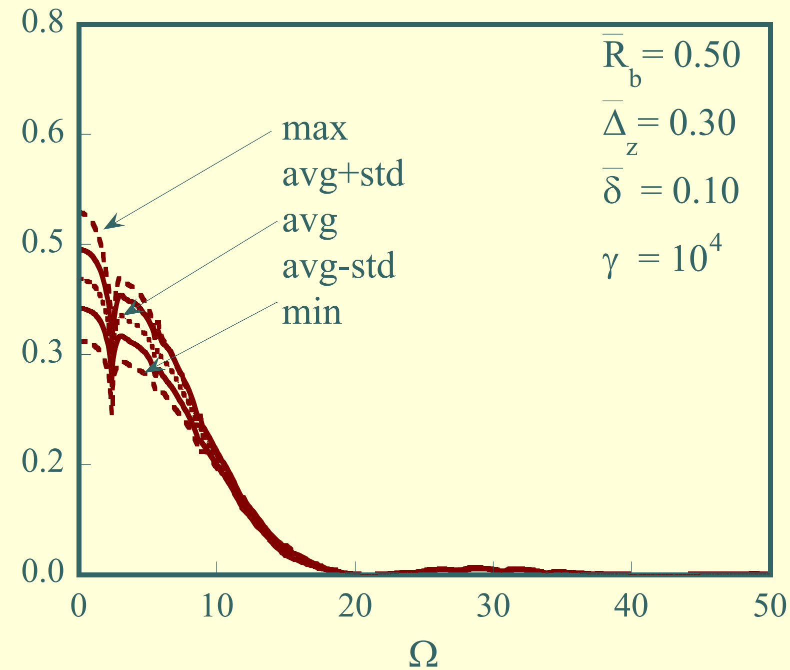
- The spectrum per number of grooves is virtually the same.
- The total emitted energy increases linearly with the number of grooves.

Spectrum

$$\bar{W} \equiv \mathrm{Re}\left[\int_0^\infty d\Omega S(\Omega)\right]$$

$$\left|\bar{Z}_L\right| = 1/\sqrt{2}$$



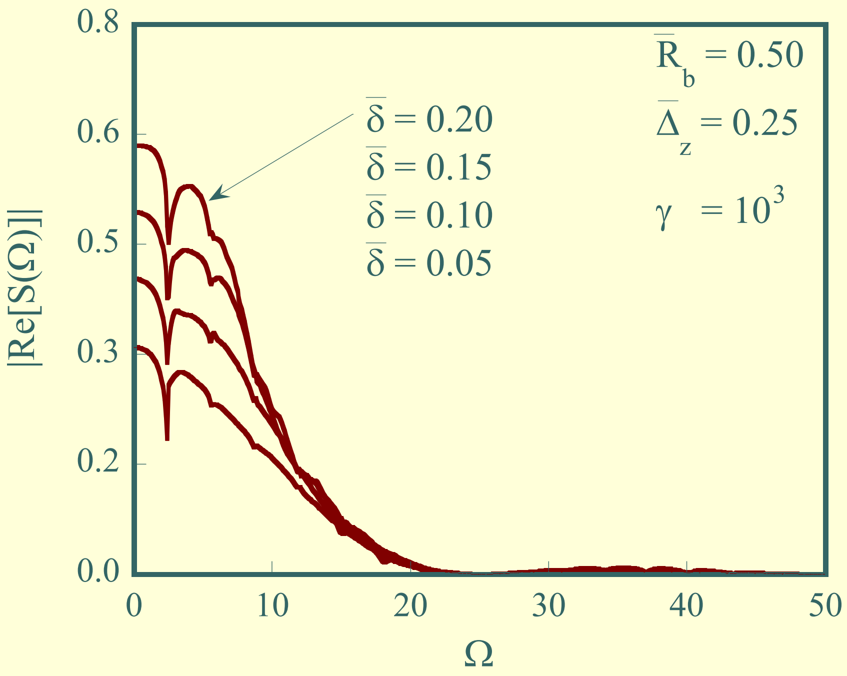


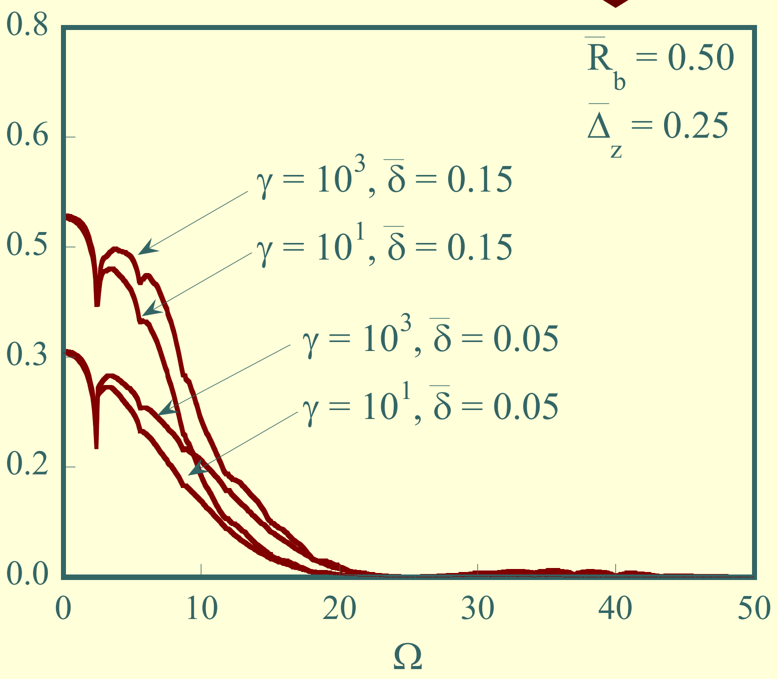
- The main peak of the spectrum is almost $\bar{\Delta}_z$ independent.
- The spectrum width decreases with the increase of $\bar{\Delta}_z$.

- At low freq. Significant difference between average and min or max spectrum
- At high freq. all curves coincide.

Spectrum

$\left|\bar{Z}_L\right| = 1/\sqrt{2}$





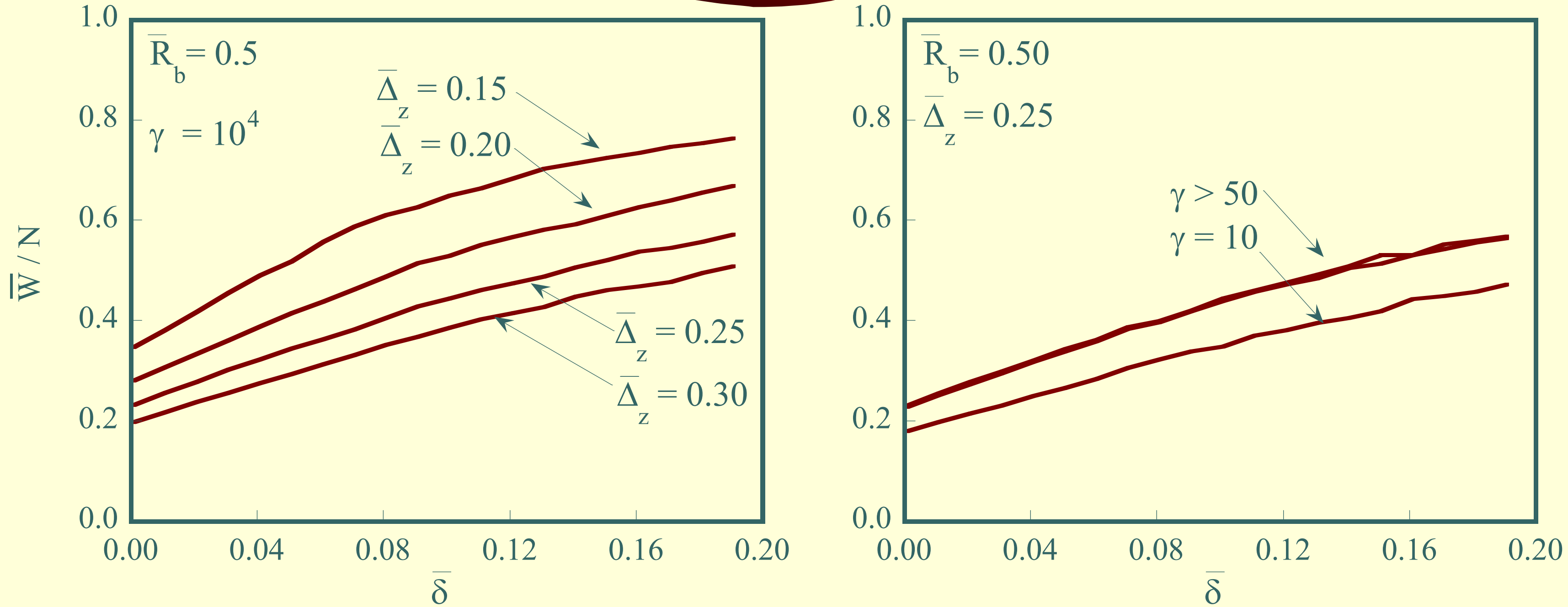
- Average spectrum increases with $\bar{\delta}$.
- Main contribution from $\Omega < 20$.

- The spectrum is weakly dependent on γ.

Emitted Energy

$$\left|\bar{Z}_L\right| = 1/\sqrt{2}$$

$$W \equiv \frac{Q^2}{4\pi\varepsilon_0 R_{\text{int}}}\bar{W}$$

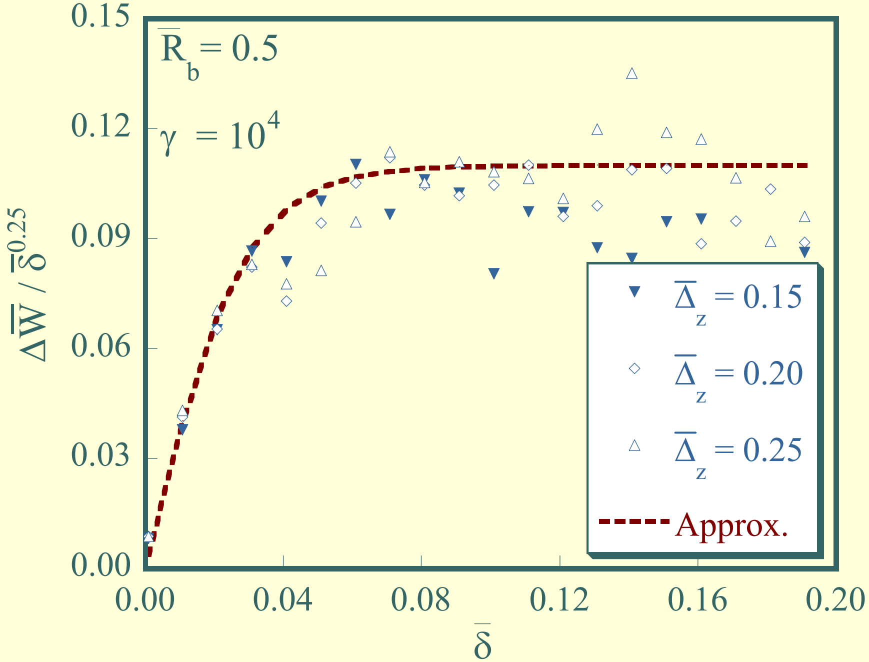


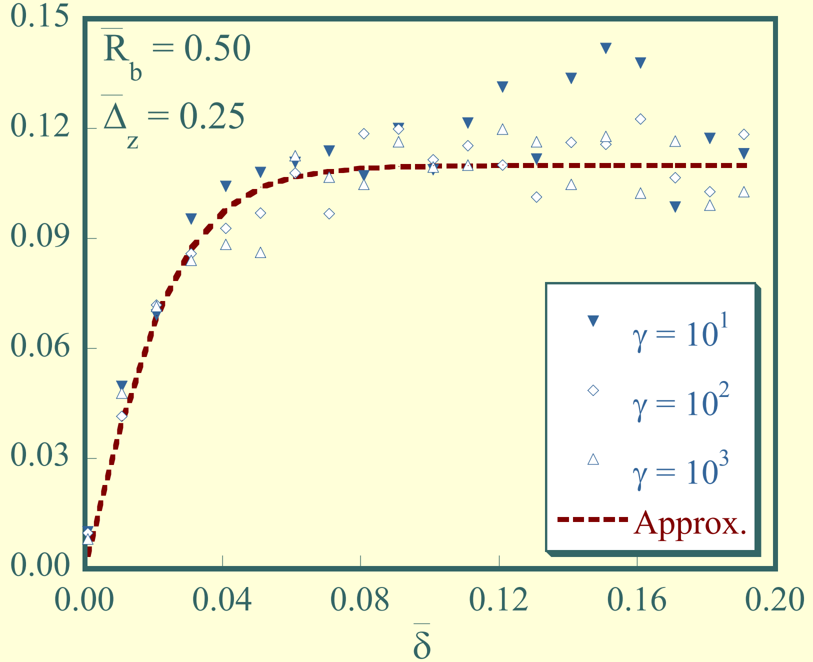
- Average energy per groove increases with $\bar{\delta}$.
- More energy generated by shorter bunch.
- The impact of γ is almost negligible.

Emitted Energy: Standard-Deviation

$|\bar{Z}_L| = 1/\sqrt{2}$

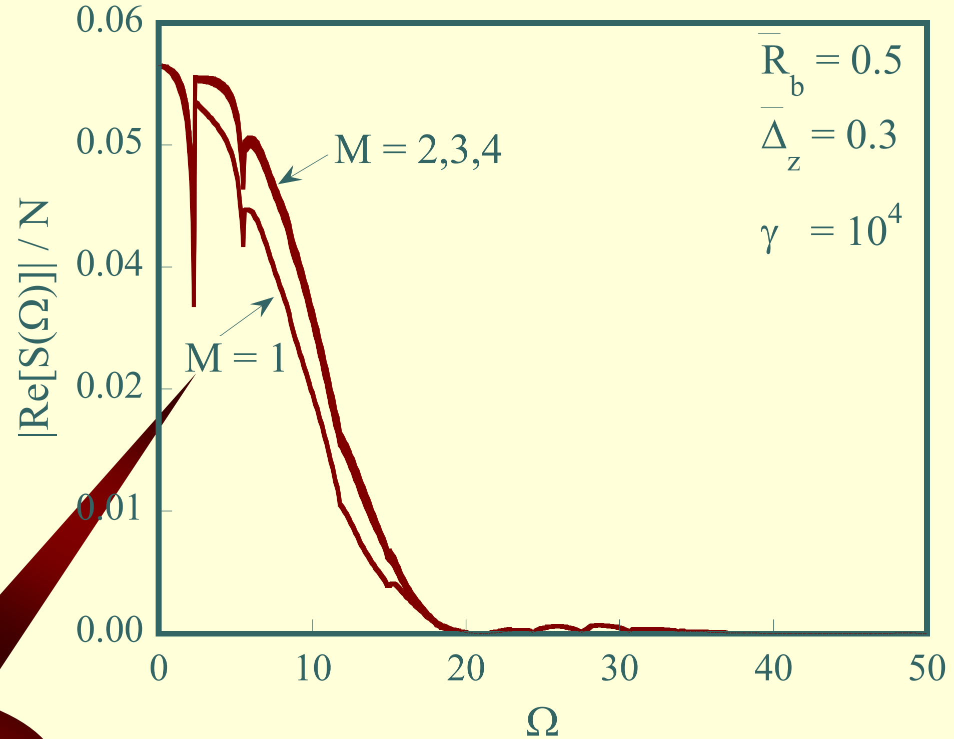
$$\Delta\bar{W} \equiv \frac{\sqrt{\langle \bar{W}_i^2 \rangle - \langle \bar{W}_i \rangle^2}}{\langle \bar{W}_i \rangle}$$





- Energy spread per groove increases weakly with average $\bar{\delta}^{0.25}$.

Single Mode Approximation



Number of modes

- Using a single mode is sufficient for all practical purposes.
- Use of many modes introduces numerical noise.

Summary: Scaling-Laws

average roughness

Average Energy

$$\frac{\langle W \rangle}{\dfrac{Q^2}{4\pi\varepsilon_0 R_{\text{int}}} \times N} \simeq 0.57 \tanh\left(\frac{45 \dfrac{\langle g \rangle}{R_{\text{int}}}}{1 + 20.72 \dfrac{\Delta_z}{R_{\text{int}}}} \right) + \frac{1.429}{1 + 20.72 \dfrac{\Delta_z}{R_{\text{int}}}}$$

roughness std.

Standard Deviation

$$\frac{\sqrt{\langle W^2 \rangle - \langle W \rangle^2}}{\dfrac{Q^2}{4\pi\varepsilon_0 R_{\text{int}}} \times N} \simeq 0.15 \left(\frac{\Delta g}{R_{\text{int}}} \right)^{0.25} \tanh\left(121.2 \frac{\Delta g}{R_{\text{int}}} \right)$$

$$\times \left[0.57 \tanh\left(\frac{45 \dfrac{\langle g \rangle}{R_{\text{int}}}}{1 + 20.72 \dfrac{\Delta_z}{R_{\text{int}}}} \right) + \frac{1.429}{1 + 20.72 \dfrac{\Delta_z}{R_{\text{int}}}} \right]$$

Summary: Scaling-Laws

Point-Charge

$\bar{\Delta}_z = 0$

Average Energy

$$\frac{\langle W \rangle}{N} \simeq \frac{Q^2}{4\pi\varepsilon_0 R_{\text{int}}} \times 2$$

Standard Deviation

$$\frac{\sqrt{\langle W^2 \rangle - \langle W \rangle^2}}{N} \simeq \frac{Q^2}{4\pi\varepsilon_0 R_{\text{int}}} \times 0.3 \left(\frac{\Delta g}{R_{\text{int}}} \right)^{0.25}$$

- Point-charge moving in a cylinder of radius R_{int} bored in a dielectric or metallic medium
- Point-charge moving in a cylindrical wave-guide with periodic wall of arbitrary but azimuthally symmetric geometry.