Department of Electrical Engineering
Technion—Israel Institute of Technology

ELECTROMAGNETIC FIELDS – #44140

based upon lectures delivered by

Prof. D. Schieber

and by

Prof. L. Schächter

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Preface

Electromagnetic Theory provides accurate explanations to most of the phenomena surrounding us. Within the framework of our course "Electromagnetic Fields" we lay down the foundations of this theory with special emphasis on its engineering aspects. As a trivial example of such an aspect, let us consider here a simple capacitor; its capacitance may be calculated in a straightforward manner by resorting to static theory; however, in practice, this capacitance is occasionally used up to frequencies in the range of a few gigahertz?! Does this result break down at a certain frequency? And if so, what is that frequency? Due to the wide variety of topics electrical engineers may be exposed to, we focus in these lecture-notes mainly on general concepts in an attempt to present systematic ways to the analysis of practical electromagnetic problems. In order to elucidate the various concepts involved, simplified models are being developed highlighting on the one hand the basics while detailing on the other hand most of the intermediary mathematical steps. Special emphasis is put on energy conversion and conservation, coupling phenomena, uniqueness of electromagnetic solutions and superposition. Finally, our lecture-notes give a very brief glimpse on fields and matter interaction as well as a very short introduction to wave phenomena.

From the didactic perspective our lecture-notes rely heavily on concepts studied within the framework of the first years courses such as "Calculus", "Partial Differential Equations", "Fourier Series and Integral Transforms" and "Functions of Complex Variables" consequently, a thorough recapitulation of these courses is recommended to the student taking the course.

David Schieber
Levi Schächter
November 1st, 2005
Haifa

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1. MAXWELL'S EQUATIONS

1.1. Introduction

Maxwell's equations constitute the theoretical basis of most phenomena encountered by electrical engineers in their practice; these equations elucidate the underlying physical processes occurring in electronic circuits, in microwave, optical and X-ray systems, in power transmission, computer tomography, geophysical remote sensing, etc. In many practical cases of interest various tools or "laws" have been developed in order to simplify the analysis e.g. – Kirchoff's laws, which are the basis of circuit theory. Although Kirchoff's laws are perfectly valid starting at dc and up to the GHz frequency range, their validity must be carefully considered for frequencies in the ten GHz range or higher. In order to envision the difficulties associated with the higher frequency range, we have to bear in mind that in lumped circuits operating at low frequencies, the electric energy is assumed to be stored in capacitors only, whereas magnetic energy is assumed to be stored exclusively in inductors. However, at high frequencies (wave phenomena), electric energy is also stored in inductors and vice versa, i.e. magnetic energy is stored in capacitors. In fact, this becomes evident if we consider a plane harmonic electromagnetic wave in which the electric and magnetic energies co-exist equally in the same volume of space.

These days many young students focus their interest on computer and communication related topics; although it may not seem evident at first sight, both these areas rely heavily on electromagnetic theory.

At the micro-chip level, for example, electromagnetic theory is resorted to in order to design circuits without "cross-talk" between elements; on the other hand in modern communication systems, the entire broadcasting and receiving process is either microwave or optically based. This list may be extended to include generation and control of electrons as in TV sets or in computer monitors, where a single electron beam covers each point on the screen at high repetition rates; these devices are being replaced by flat displays where arrays of electron emitters generate micro-electron beams hitting a fluorescent panel, each beamlet, in turn, being decoupled from the others. In other cases controlling electron emission entails avoiding it in spite of presence of intense electric fields – for example one must know how to design micro-circuits capable of withstanding high electric fields. Similarly, one must know how to shield circuits from external signals, and how to reduce noise levels. Electromagnetic field theory is obviously the very basic topic that a student must study if interested in areas of microwave systems, optics, electro-optics and opto-electronics.

1.2. Maxwell Equations — Differential Form

For an analysis of physical phenomena it is necessary to introduce a frame of reference facilitating the description of a phenomenon in space-time. Such a frame of reference may consist of a coordinate system comprising three orthogonal "rulers" enabling the measurement of the distance between any two points, with each point of interest "carrying" a clock providing information about the local running time; all the clocks are assumed to be initially synchronized. At this stage we restrict ourselves to a Cartesian system of coordinates, schematically illustrated in Figure 1.1.

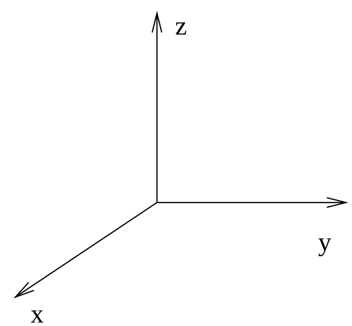

Figure 1.1: Cartesian system of coordinates.

As an integral part of this coordinate system we define three unit vectors

$$\vec{1}_x, \qquad \vec{1}_y, \qquad \vec{1}_z, \tag{1.2.1}$$

forming a right-handed system - see Figure 1.2.

In addition, two differential operators playing an important role in the description of electromagnetic

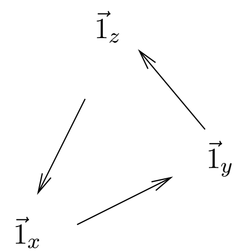

Figure 1.2: Right-handed system of coordinates.

phenomena, are introduced; thus given a vector $\vec{A} \equiv A_x \vec{1_x} + A_y \vec{1_y} + A_z \vec{1_z}$, we define its divergence by

$$\vec{\nabla} \cdot \vec{A} \equiv \partial_x A_x + \partial_y A_y + \partial_z A_z \tag{1.2.2}$$

as well as its curl i.e.,

$$\vec{\nabla} \times \vec{A} \equiv \begin{vmatrix} \vec{1_x} & \vec{1_y} & \vec{1_z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{matrix} +\vec{1}_x & (\partial_y A_z - \partial_z A_y) \\ -\vec{1}_y & (\partial_x A_z - \partial_z A_x) \\ +\vec{1}_z & (\partial_x A_y - \partial_y A_x) \,. \end{matrix} \tag{1.2.3}$$

Based on these two differential operators and assuming the electromagnetic field to be adequately described by three dimensional vectors, it is possible to write a set of equations, called Maxwell's Equations (ME) describing all known electromagnetic phenomena. In their *differential* form these equations read

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \qquad \text{Faraday's law} \tag{1.2.4}$$

$$\vec{\nabla} \times \vec{H} - \partial_t \vec{D} = \vec{J} \qquad \text{Ampere's law} \tag{1.2.5}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{Magnetic induction conservation} \tag{1.2.6}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \text{Gauss' law} \tag{1.2.7}$$

This is a set of 8 equations which indicate that:

$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$ $\Rightarrow$ time variations of the magnetic induction are associated with spatial variations of the electric field and vice versa;

$\vec{\nabla} \times \vec{H} = \vec{J} + \partial_t \vec{D}$ $\Rightarrow$ spatial variation of the magnetic field implies time variation of the electric induction and/or the presence of an electric current density.

The interpretation of the two scalar equations (1.2.6) and (1.2.7) will be discussed within the context of their integral reformulation.

Although nomenclature may vary, we resort to the following description:

$\vec{E}$ –	electric field	[V/m]	$\vec{D}$ –	electric induction	[C/m^2]
$\vec{H}$ –	magnetic field	[A/m]	$\vec{B}$ –	magnetic induction	[V sec/m^2] or [T]
$\vec{J}$ –	current density	[A/m^2]	ρ –	charge density	[C/m^3].

In order to examine the inter-dependence of this set of equations we resort to the well known differential equation satisfied by a vector $\vec{A}$, i.e.,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \equiv 0 \,. \tag{1.2.8}$$

Examining now Faraday's law and applying the divergence operator $\vec{\nabla}$, we conclude that

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E} + \partial_t \vec{B}) = 0 \,. \tag{1.2.9}$$

According to (1.2.8) the first term is identically zero

$$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E})}_{=0} + \partial_t (\vec{\nabla} \cdot \vec{B}) = 0 \,, \tag{1.2.10}$$

and, therefore, upon interchanging the order of the operators $\vec{\nabla}$ and ∂_t, we conclude that $\vec{\nabla} \cdot \vec{B}$ equals a time-independent scalar function $f(\vec{r})$. Assuming this function to have been zero in the remote past, we obtain

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.2.11}$$

which is identical to (1.2.6). In other words, based on a mathematical identity as well as on experiment [i.e. $f(\vec{r}) = 0$] we have found that Faraday's law comprises information associated with the continuity of magnetic induction.

1.3. Equation of Electric Charge Continuity

Applying the same procedure to Ampere's law we obtain

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H} - \partial_t \vec{D}) = \vec{\nabla} \cdot \vec{J}\,, \tag{1.3.1}$$

where the first term on the left once more vanishes, and according to Gauss' law [(1.2.7)]

$$\underbrace{\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{H}\right)}_{=0} - \partial_t \underbrace{\left(\vec{\nabla} \cdot \vec{D}\right)}_{\rho} = \vec{\nabla} \cdot \vec{J}\,. \tag{1.3.2}$$

We are thus left with the continuity equation of the electric charge

$$\boxed{\vec{\nabla} \cdot \vec{J} + \partial_t \rho = 0\,.} \tag{1.3.3}$$

As we shall subsequently see, this relation implies *charge conservation*; therefore if we postulate (or assume) the latter, Ampere's law includes physical information associated with Gauss' law. It is therefore possible to conclude that the vector equations (Faraday's and Ampere's law) are "stronger", since in conjunction with the identity in Eq.(1.2.8), they include most of the information associated with the two scalar equations; we shall later on return to this point.

1.4. Maxwell Equations — Integral Form:

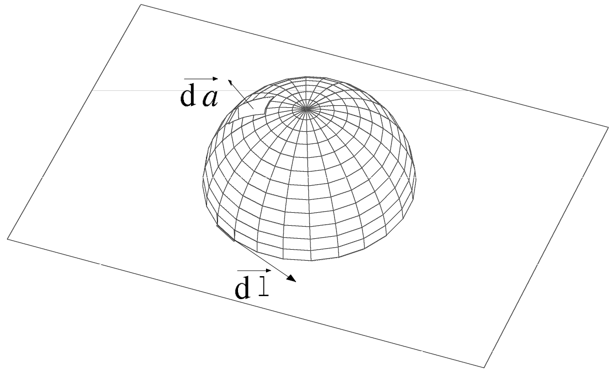

Figure 1.3: Surface and contour of integration.

In a variety of cases it is more convenient to use Maxwell's equations (ME) in their integral form rather than in their differential form (1.2.4-7). For this purpose it is necessary to examine these equations in conjunction with two integral theorems of calculus i.e. Stokes' and Gauss' theorems. Stokes' theorem states that the surface integral over the curl of any continuous vector field ($\vec{A}$) equals the contour integral over the same vector field; the integration contour has to follow the circumference of the initial surface, see Figure 1.3 i.e.,

$$\iint_{S} \mathrm{d}\vec{a} \cdot (\vec{\nabla} \times \vec{A}) = \oint_{\Gamma} \vec{A} \cdot \mathrm{d}\vec{l}; \tag{1.4.1}$$

$\mathrm{d}\vec{l}$ is an infinitesimal vector tangential to the circumference of the surrounding contour (counter clockwise), Γ is the contour encompassing the surface S and $\mathrm{d}\vec{a}$ is an infinitesimal surface element pointing outwards.

On the other hand, Gauss' mathematical theorem states that the volume integral of the divergence of a vector field ($\vec{A}$) equals the surface integral of the same vector field i.e.,

$$\int_{\mathrm{v}} \mathrm{d}v\,(\vec{\nabla}\cdot\vec{A}) \equiv \oint_{\mathrm{S}} \mathrm{d}\vec{a}\cdot\vec{A}\,; \tag{1.4.2}$$

the surface enclosing the whole volume of integration, v ; dv is an infinitesimal volume element in v.

1.4.1. Faraday's Law in Integral Form

Based upon Faraday's law,

$$\iint \mathrm{d}\vec{a}\cdot[\vec{\nabla}\times\vec{E}+\partial_t\vec{B}] = 0\,, \tag{1.4.3}$$

Stokes' theorem implies

$$\oint \mathrm{d}\vec{l}\cdot\vec{E} + \iint \mathrm{d}\vec{a}\cdot\partial_t\vec{B} = 0 \tag{1.4.4}$$

and assuming that the surface of integration does not vary in time,

$$\oint \mathrm{d}\vec{l}\cdot\vec{E} + \frac{\mathrm{d}}{\mathrm{d}t}\iint \mathrm{d}\vec{a}\cdot\vec{B} = 0 \tag{1.4.5}$$

This expression may be interpreted in the following way: the negative temporal change of the magnetic

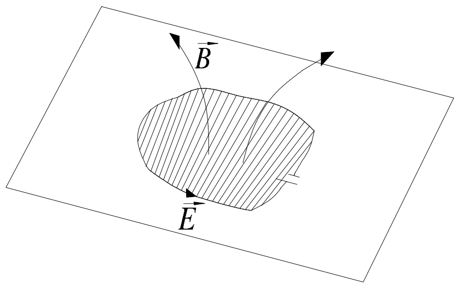

Figure 1.4: Flux through a metallic loop.

flux,

$$\Phi = \int \mathrm{d}\vec{a} \cdot \vec{B}\,, \tag{1.4.6}$$

across a given surface (see Figure 1.4) equals the line integral of the electric field along the contour of the same surface

$$\oint \mathrm{d}\vec{l} \cdot \vec{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}\,. \tag{1.4.7}$$

Consequently, Kirchoff voltage law

$$\oint \mathrm{d}\vec{l} \cdot \vec{E} = 0 \tag{1.4.8}$$

may be used in one of three cases:

1. Vanishing magnetic induction ($B = 0$).

2. Time-independent magnetic induction ($\partial_t = 0$).

3. Time-dependent magnetic induction, yet the integration surface and the magnetic induction are perpendicular to each other locally ($\vec{B} \perp \mathrm{d}\vec{a}$) - see Figure 1.4.

In any case, for an electric circuit located in a varying (external) magnetic field, Kirchoff's voltage law must, in principle, be extended to include the effect of the external magnetic flux on the circuit.

1.4.2. Ampere's law in Integral Form

Ampere's law,

$$\iint \mathrm{d}\vec{a} \cdot (\vec{\nabla} \times \vec{H} - \partial_t \vec{D}) = \iint \mathrm{d}\vec{a} \cdot \vec{J}\,, \tag{1.4.9}$$

in conjunction with Stokes theorem reads

$$\oint \mathrm{d}\vec{l} \cdot \vec{H} - \frac{\mathrm{d}}{\mathrm{d}t} \iint \mathrm{d}\vec{a} \cdot \vec{D} = \iint \mathrm{d}\vec{a} \cdot \vec{J} = \sum I\,. \tag{1.4.10}$$

wherein $\sum I$ denotes the total current crossing the (stationary) surface of interest – see Figure 1.5. Disregarding for the sake of simplicity at this stage temporal variations, the last equation implies that the current crossing a given surface $\vec{a}$ is linked to the magnetic field so that the latter's line integral along the contour of the surface equals exactly that current.

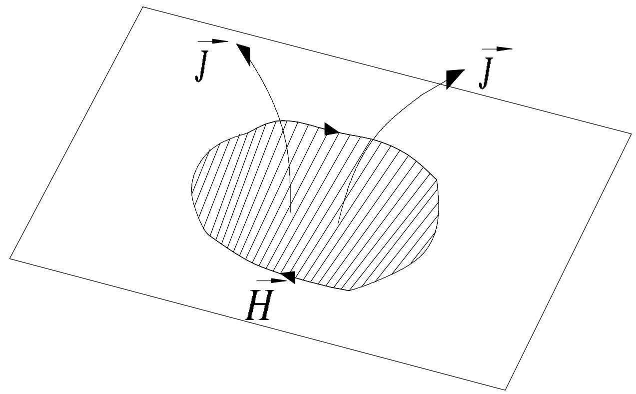

Figure 1.5: Surface and contour of integration.

1.4.3. Gauss' Law

Based on Gauss' mathematical theorem it is possible to write Gauss' law

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{1.4.11}$$

as

$$\underbrace{\int \mathrm{d}v(\vec{\nabla} \cdot \vec{D})}_{\oiint_S \mathrm{d}\vec{a} \cdot \vec{D}} = \underbrace{\int \mathrm{d}v\, \rho}_{Q} \tag{1.4.12}$$

hence

$$\boxed{\oiint_S \mathrm{d}\vec{a} \cdot \vec{D} = Q} \tag{1.4.13}$$

This result indicates that the electric charge inside a given volume is directly linked to the electric induction ($\vec{D}$) - see Figure 1.6. Obviously, the relation $Q = 0$ does not imply $\vec{D}$ to be of zero value, but rather that no net sources exist in the specific region considered.

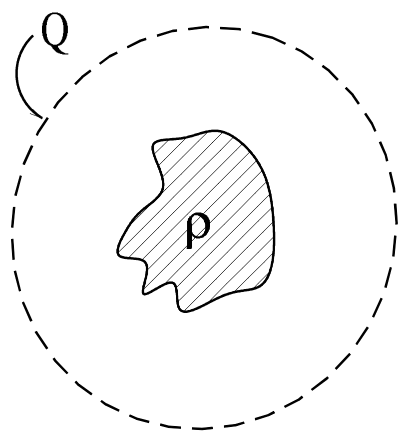

Figure 1.6: The total change enclosed in a surface is directly linked to the electric induction.

In a similar way one may examine the magnetic induction

$$\int_{\mathrm{v}} \mathrm{d}v(\vec{\nabla} \cdot \vec{B}) = 0 \Rightarrow \boxed{\oiint_{\mathrm{A}} d\vec{a} \cdot \vec{B} = 0\,.} \tag{1.4.14}$$

This relation indicates that the magnetic induction has no sources resembling those of the electric field.

At the fundamental (microscopic) level this entails that no magnetic monopole has ever been observed in nature (so far)!!

1.4.4. Charge Conservation

The approach followed in Section 1.4.3 may also be applied to the law of charge conservation,

$$\vec{\nabla} \cdot \vec{J} + \partial_t \rho = 0\,. \tag{1.4.15}$$

Integration of this relation over a given volume, v, implies

$$\int_{\mathrm{v}} \mathrm{d}v \, (\vec{\nabla} \cdot \vec{J} + \partial_t \rho) = 0 \tag{1.4.16}$$

and with the aid of Gauss' theorem

$$\oiint \mathrm{d}\vec{a} \cdot \vec{J} + \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int \mathrm{d}v \, \rho}_{\equiv Q} = 0 \tag{1.4.17}$$

furthermore, the surfaces considered in both Faraday's and Ampere's laws are neither moving nor deforming. One finally obtains

$$\boxed{\oiint \mathrm{d}\vec{a} \cdot \vec{J} + \frac{\mathrm{d}}{\mathrm{d}t} Q = 0\,.} \tag{1.4.18}$$

Explicitly, this equation states that the negative time-change of the total charge "stored" in a given volume, is balanced by the current crossing the envelope encompassing the volume considered.

Kirchoff's current law (KCL) may be now derived from this relation coupled with the assumption that the amount of charge stored in any junction is zero (all charge being stored in capacitors), and consequently,

$$\oiint \mathrm{d}\vec{a} \cdot \vec{J} = \sum I = 0\,. \tag{1.4.19}$$

1.4.5. Summary of Maxwell's Equations — Integral Form

It is convenient at this stage to summarize the various relation developed in this subsection:

Faraday's Law:

$$\boxed{\oint \mathrm{d}\vec{l} \cdot \vec{E} + \frac{\mathrm{d}}{\mathrm{d}t} \iint \mathrm{d}\vec{a} \cdot \vec{B} \quad = 0}$$

Ampere's Law:

$$\boxed{\oint \mathrm{d}\vec{l} \cdot \vec{H} - \frac{\mathrm{d}}{\mathrm{d}t} \iint \mathrm{d}\vec{a} \cdot \vec{D} \quad = I}$$

Gauss' Law:

$$\boxed{\oiint \mathrm{d}\vec{a} \cdot \vec{D} \qquad = Q}$$

Magnetic Induction Conservation:

$$\boxed{\oiint \mathrm{d}\vec{a} \cdot \vec{B} \qquad = 0}$$

Obviously all the equations are formulated in an un-accelerated frame of reference.

1.5. Maxwell's Equations in the Presence of Magnetic Sources

As indicated in Section 1.4.3, at the *microscopic* level, no magnetic monopole has as yet been detected; however, at the *macroscopic* level, it is sometimes convenient to artificially *postulate* the existence of such sources, and the question now is what is the form of Maxwell's equations in such a case. In order to address this question, let us consider at first a fictitious magnetic charge density denoted by ρ_m. Similar to Gauss' law, this magnetic charge density is linked to the magnetic induction by means of the relation

$$\vec{\nabla} \cdot \vec{B} = \rho_m \, . \tag{1.5.1}$$

Moreover, associated with this magnetic charge density, we postulate the existence of a so called magnetic current density denoted by $\vec{J}_m$. It is further assumed that similar to the electric charge, the magnetic charge also satisfies the continuity equation namely,

$$\vec{\nabla} \cdot \vec{J}_m + \partial_t \rho_m = 0 \, , \tag{1.5.2}$$

and, consequently, Faraday's law is augmented to read

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = -\vec{J}_m \, . \tag{1.5.3}$$

Hence, in the presence of electric as well as of magnetic sources Maxwell's equations read

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = -\vec{J}_m \,, \tag{1.5.4}$$

$$\vec{\nabla} \times \vec{H} - \partial_t \vec{D} = \vec{J} \,, \tag{1.5.5}$$

$$\vec{\nabla} \cdot \vec{B} = \rho_m \,, \tag{1.5.6}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \,. \tag{1.5.7}$$

Exercise: Formulate Maxwell's equations in the presence of magnetic sources in integral form.

1.6. Constitutive Relations

1.6.1. Lorentz Force Approach

As they stand, Maxwell's equations [Eqs. (1.2.4-7) or Eqs. (1.5.4-7)] cannot be solved since they represent in effect 6 independent equations for 12 variables, assuming that the source terms are known. To proceed towards a set of equations that may be solved, we have to determine 6 other equations determining the relations among $\vec{E}$, $\vec{D}$, $\vec{H}$ and $\vec{B}$. At this stage we shall consider only free space ("vacuum") conditions and, specifically, we shall point out that from the point of view of electrical measurement $\vec{E}$ and $\vec{B}$ are well defined since they determine the force acting on an electrically charged particle (q)

$$\vec{F}_{\mathrm{e}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) , \tag{1.6.1}$$

or the force density acting on an electric charge distribution ρ

$$\vec{f}_{\mathrm{e}} = \rho\left(\vec{E} + \vec{v} \times \vec{B}\right) ; \tag{1.6.2}$$

here $\vec{v}$ represents the velocity of the particle – relative to an inertial frame of reference. These are two equivalent representations of Lorentz's law; the subscript "e" emphasizes the fact that this is the force acting on an *electric* charge. It is evident from (1.6.2) that assuming the charge-density of the body to be known, it is possible to deduct, in principle, the intensity of the electric field $\vec{E}$ or the value of the magnetic induction $\vec{B}$ just from the motion of the charged body.

Basically, in analogy to the electric charge q, the Lorentz' force definition may be generalized to include a magnetic charge q_{m} namely,

$$\vec{F}_{\mathrm{m}} = q_{\mathrm{m}} \left(\vec{H} - \vec{v} \times \vec{D} \right) , \tag{1.6.3}$$

or in case of a magnetic charge density ρ_{m},

$$\vec{f}_{\mathrm{m}} = \rho_{\mathrm{m}} \left(\vec{H} - \vec{v} \times \vec{D} \right) . \tag{1.6.4}$$

Once more, in complete analogy to the electric charge, the force density determines the two other field components ($\vec{H}$ and $\vec{D}$), or more explicitly, the way each one of them could, in principle, be measured. Measurement of the various field components *in vacuum* leads to the conclusion that $\vec{E}$ and $\vec{D}$ are proportional and so are $\vec{H}$ and $\vec{B}$. Consequently, in SI units we may write

$$\vec{D} = \varepsilon_0 \vec{E} , \tag{1.6.5}$$

wherein $\varepsilon_0 \equiv 8.85 \times 10^{-12}$[V][m]/[C] stands for the *permittivity coefficient of vacuum* and

$$\vec{H} = \frac{1}{\mu_0} \vec{B} , \tag{1.6.6}$$

$\mu_0 \equiv 4\pi \times 10^{-7}$[T][m]/[A] stands for the *permeability coefficient of vacuum* (by agreement).

Example # 1. For the sake of simplicity, we shall assume a "world" consisting of a motionless point-charge Q located at the origin of a spherical coordinate system (r, ϕ, θ). Gauss' law and symmetry considerations imply that the electric induction is radial and given by

$$\vec{D} = D_r \vec{1_r} = \frac{Q}{4\pi r^2} \vec{1_r} \, . \tag{1.6.7}$$

Again, due to symmetry ($\partial_\phi = 0, \partial_\theta = 0$) and assuming an isotropic "universe", Faraday's law for the static case ($\vec{\nabla} \times \vec{E} = 0$) implies that the only non-zero electric field component is $E_r(r)$. Based upon Faraday's law it is not, however, possible to determine this field without some further information. In order to establish this additional information, it is necessary to go one step further and introduce a test-charge q located at a distance R from the origin. According to Lorentz' law, the force of Q on a motionless test particle is

$$F_r = q E_r(r = R) \, . \tag{1.6.8}$$

Moreover, it is *experimentally* known that the force between two charged particles is proportional to each one of the charges and inversely proportional to the square of the distance between the particles,

$$F_r \propto \frac{qQ}{R^2} \, . \tag{1.6.9}$$

From the last two relations we may conclude that the electric field is proportional to Q and inversely proportional to R^2,

$$E_r(r = R) \propto \frac{Q}{R^2} \, . \tag{1.6.10}$$

Comparing equation (1.6.10) and (1.6.7) at $r = R$, it is readily seen that the ratio between the E_r and D_r is constant, i.e.

$$\frac{D_r}{E_r} = \text{const.} \tag{1.6.11}$$

hence the generalization $\vec{D} = \varepsilon_0 \vec{E}$ as described by (1.6.5).

1.6.2. Maxwell's Approach

In the previous section the measurement of $\vec{D}$ and $\vec{H}$ was in principle based upon the postulate of the existence of a macroscopic magnetic charge and, further, upon the generalization of Lorentz' force law. Alternatively, it is possible to take advantage of well known experimental evidence regarding the force between two charges or two solenoids (wires) in order to measure $\vec{D}$ and $\vec{H}$ - in conjunction with Ampere's law and Gauss' law. Let us start with the measurement of $\vec{D}$. Imagine a region in space where a uniform electric field ($\vec{E}$) exists comprising two thin metal foils touching each other; for the sake of simplicity, both are assumed to be perpendicular to the field. The joining foils now exhibit a positive charge (Q) on one side, and negative charge on the other; this charge is proportional to the external electric field, $|\vec{E}|$. At this stage, we separate the two foils pulling them out of the field domain. Between the two foils the electric induction is proportional to the electric charge and inversely proportional to the area (A) of the foils – $|\vec{D}| \sim Q/A$ – as will become clearer in the next section; the charge Q may now readily be measured. Consequently, since Q is proportional to the electric field, the latter and the electric induction are proportional. This reasoning first developed by Maxwell, will now be extended to the magnetic induction and to the magnetic field.

Consider a region in space where the magnetic induction ($\vec{B}$) is uniform. Into this domain we insert an N winding solenoid of length l, fed by a direct current source I. According to Ampere's law, this solenoid generates a magnetic field H that in its center is proportional to NI/l. Moreover, inserting a small compass into the solenoid we find that for a certain current I there is zero force on the arm of the compass implying that the magnetic field generated by the solenoid is associated to a magnetic induction that exactly cancels the initial magnetic induction. Since on the one hand based upon experiment the compensating induction field is proportional to I and, on the other hand, according to Ampere's law $\vec{H}$ is proportional to the current, we conclude that $\vec{B}$ and $\vec{H}$ are proportional.

Example # 2. We shall follow now an approach similar to that in *Example #1* in order to determine the ratio between $|\vec{H}|$ and $|\vec{B}|$ in vacuum. Consider this time a thin wire "infinitely" long in the z-direction and carrying a current I. Due to symmetry, outside the wire circular magnetic field is excited

$$H_\phi(r) = \frac{I}{2\pi r}\,; \tag{1.6.12}$$

[we have used here the cylindrical coordinate system (r, ϕ, z)]. Since the system extends to infinity along the z-direction ($\partial_z = 0$) and azimuthally symmetric i.e., $\partial_\phi = 0$ there is only one non-zero component of the magnetic induction – namely, B_ϕ. However, without some additional information there is no way we can determine it. As in the case of the electric field (and induction) we employ the *second term of the Lorentz force* $(\vec{v} \times \vec{B})$ that acts on a unit length (Δ_z) of test wire carrying a current I_t parallel to the original wire but located at a distance R from the latter,

$$F_r \propto \Delta_z I_t B_\phi(r = R)\,. \tag{1.6.13}$$

It is well known *experimentally* that the force per unit length is proportional to both currents and inversely proportional to the distance between the wires i.e.,

$$F_r \propto \frac{\Delta_z\, I_t\, I}{R}\,. \tag{1.6.14}$$

Therefore, the magnetic induction is proportional to I and inversely proportional to R and consequently, the ratio of the magnetic induction and the magnetic field is a constant

$$\frac{B_\phi}{H_\phi} = \text{const.} \tag{1.6.15}$$

hence the generalization $\vec{B} = \mu_0 \vec{H}$ as described by (1.6.6).

1.6.3. Some Other Simple Constitutive Relations

For a significant group of materials the relation between $\vec{D}$ and $\vec{E}$ is linear as in vacuum but the coefficient is different i.e.

$$\vec{D} = \varepsilon_0 \varepsilon_{\rm r} \vec{E} \tag{1.6.16}$$

wherein $\varepsilon_{\rm r}$ stands for the *relative permittivity* or for the relative dielectric coefficient. Typical values of ε_r range from unity to ten, few material exhibiting a dielectric coefficient of a few thousands and very few showing even larger values.

Similarly for many materials, the relation between the magnetic induction and the magnetic field is linear, the slope differing from the free space slope, i.e.

$$\vec{B} = \mu_0 \mu_{\rm r} \vec{H} \tag{1.6.17}$$

wherein $\mu_{\rm r}$ represents the *relative permeability*; its value may exceed several hundreds of thousands in some metallic alloys.

Another constitutive relation that will be used in what follows is the experimental relation between the current density and the electric field in metals. For sufficiently low electric field ($|\vec{E}| \ll E_0 \equiv mc^2/ec\tau$ where τ is the mean free time between two collisions of the electron in the metal) the relation between the current density and the electric field is given by Ohm's law

$$\vec{J} = \sigma \vec{E} \tag{1.6.18}$$

where σ is the conductivity of the material – for typical values see next Table.

Metals and Alloys in Solid State	
	$\sigma - S/m$ **at 20^{o}C**
Aluminium	3.54×10^7
Copper, annealed	5.80×10^7
Copper, hard drawn	5.65×10^7
Gold, pure drawn	4.10×10^7
Iron, 99.98%	1.0×10^7
Steel	$0.5 - 1.0 \times 10^7$
Lead	0.48×10^7
Magnesium	2.17×10^7
Nichrome	0.10×10^7
Nickel	1.28×10^7
Silver, 99.98%	6.14×10^7
Tungsten	1.81×10^7

1.7. Boundary Conditions

In this section we examine the behavior of the electromagnetic field in the presence of a discontinuity. A difficulty occurs due to the fact that the physical situation is in principle described by *differential* equations and the derivatives are obviously not defined in the vicinity of a sharp discontinuity. To obliterate this obstacle it is necessary to resort to Maxwell's equations in their integral form. As a first case let us consider the discontinuity in the magnetic field due to a current-density flowing along a very thin layer. Specifically consider the following setup: the current flows along the layer as illustrated in Figure 1.7 parallel to the unit vector $\vec{1}_a$. We consider a test area $w\ell$ which this current density crosses. The basic assumption is that at limit $w \to 0$ the current density is sufficiently high so that

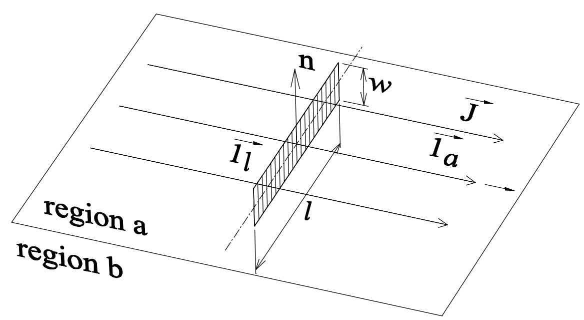

Figure 1.7: Discontinuity of the tangential magnetic field is linked with a surface current density.

$$\lim_{\substack{w\to 0 \\ J\to\infty}} \left(\vec{J}w\right) \equiv \vec{J}_s \ . \tag{1.7.1}$$

The limiting value $\vec{J}_s$ denotes the so-called *surface current-density* its units being Ampere per meter. The next step is to examine Ampere's law in view of the configuration described above namely,

$$\oint d\vec{\ell} \cdot \vec{H} = \int d\vec{a} \cdot \vec{J} + \frac{d}{dt} \int d\vec{a} \cdot \vec{D} \, . \tag{1.7.2}$$

At the limit of an extremely narrow range ($w \to 0$) this integral equation reads, for finite values of the field

$$\vec{\ell} \cdot [\vec{H}_a - \vec{H}_b] + \underbrace{w \cdot [\vec{H}_a - \vec{H}_b]}_{\to 0} = \vec{1}_a \cdot \ell \underbrace{w\vec{J}}_{\vec{J}_s} + \frac{d}{dt} \vec{1}_a \cdot \underbrace{\ell w \vec{D}}_{=0} \tag{1.7.3}$$

however, the second terms on both the left as well as on the right hand side vanish, so that one obtains the relation

$$\ell \vec{1}_\ell \cdot [\vec{H}_a - \vec{H}_b] = \vec{1}_a \cdot \vec{J}_s \ell \, . \tag{1.7.4}$$

Resorting to the definition of the normal to the surface $\vec{1}_n \equiv \vec{1}_\ell \times \vec{1}_a \Rightarrow \vec{1}_\ell = \vec{1}_a \times \vec{1}_n$ (right handed coordinates) there results the relation

$$(\vec{1}_a \times \vec{1}_n) \cdot [\vec{H}_a - \vec{H}_b] = \vec{1}_a \cdot \vec{J}_s \tag{1.7.5}$$

or $\vec{1}_a \cdot [\vec{1}_n \times (\vec{H}_a - \vec{H}_b)] = \vec{1}_a \cdot \vec{J}_s$. Since this is valid for an arbitrary unit vector $\vec{1}_a$ we conclude that

$$\boxed{\vec{1}_n \times (\vec{H}_a - \vec{H}_b) = \vec{J}_s \, .} \tag{1.7.6}$$

Clearly, this boundary condition is related to Ampere's law

$$\vec{\nabla} \times \vec{H} - \partial_t \vec{D} = \vec{J} \qquad \Rightarrow \qquad \vec{1}_n \times (\vec{H}_a - \vec{H}_b) = \vec{J}_s \,. \tag{1.7.7}$$

Hence, by analogy it may be readily concluded that Faraday's law implies

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = -\vec{J}_m \Rightarrow \vec{1}_n \times (\vec{E}_a - \vec{E}_b) = -\vec{J}_{m,s} \,. \tag{1.7.8}$$

Here $\vec{J}_{m,s}$ is defined similar to the definition of $\vec{J}_s$ in (1.7.1) i.e. $\lim_{\substack{w \to 0 \\ J_m \to \infty}} \left(\vec{J}_m w \right) \equiv \vec{J}_{m,s}$. Note that if the magnetic surface current density is zero, then

$$\boxed{\vec{1}_n \times (\vec{E}_a - \vec{E}_b) = 0 \,.} \tag{1.7.9}$$

The normal, $\vec{1}_n$, points from b to a. The last relation therefore links the *tangential* components of the electric field in two different domains imposing their continuity – in the absence of any magnetic surface current density. Similarly, Eq.(1.7.6) indicates that the discontinuity of the tangential components of the magnetic field is determined by the surface current density.

We now turn our attention to the boundary conditions associated with the divergence equations. Similar to the definition of $\vec{J}_s$ we define the surface charge density – see Figure 1.8 – as

$$\lim_{\substack{w \to 0 \\ \rho \to \infty}} (\rho w) \equiv \rho_s \,. \tag{1.7.10}$$

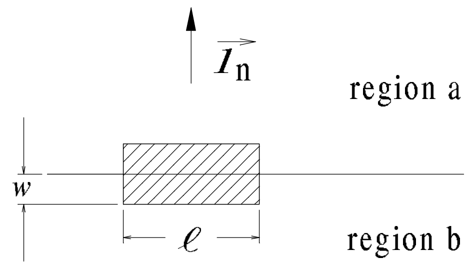

Figure 1.8: Surface charge density is linked to the discontinuity in perpendicular electric induction.

With this definition we now examine Gauss' law in its integral form

$$\vec{1}_n \ell \Delta_z \cdot (\vec{D}_a - \vec{D}_b) = w\ell\Delta_z \rho = \ell\Delta_z \rho_s \tag{1.7.11}$$

hence

$$\boxed{\vec{1}_n \cdot (\vec{D}_a - \vec{D}_b) = \rho_s} \tag{1.7.12}$$

implying that any discontinuity in the *perpendicular* electric induction is compensated by a surface charge density. In the absence of the latter, this induction component is continuous.

An identical approach may be used to formulate the discontinuity of the magnetic induction. Since Gauss' law leads to (1.7.12), i.e.

$$\vec{\nabla} \cdot \vec{D} = \rho \Rightarrow \vec{1}_n \cdot (\vec{D}_a - \vec{D}_b) = \rho_s \tag{1.7.13}$$

we conclude that the conservation of the magnetic induction, $\vec{\nabla} \cdot \vec{B} = \rho_m$, implies

$$\boxed{\vec{1}_n \cdot (\vec{B}_a - \vec{B}_b) = \rho_{m,s}} \tag{1.7.14}$$

where

$$\rho_{m,s} \equiv \lim_{\substack{w \to 0 \\ \rho_m \to \infty}} [\rho_{\rm m} w] \ . \tag{1.7.15}$$

Furthermore, in case of zero magnetic charge density, the perpendicular components of the magnetic induction are continuous.

Each boundary condition so far was associated with one of Maxwell's equations. At this point it is possible to proceed one step further and develop the boundary conditions associated with the charge conservation. In its integral form, the latter reads

$$\int \mathrm{d}v \vec{\nabla} \cdot \vec{J} + \int \mathrm{d}v \frac{\partial \rho}{\partial t} = 0 \tag{1.7.16}$$

Without loss of generality let us consider a volume element of width w and area A. The discontinuity separates this volume in two halves. This element is infinitesimally small - see Figure 1.9. Without loss of generality, we shall assume that the z-axis is perpendicular to the discontinuity hence

$$\int \mathrm{d}v \vec{\nabla} \cdot \vec{J} = \int \mathrm{d}v \left(\frac{\partial J_z}{\partial z} + \vec{\nabla}_\parallel \cdot \vec{J}_\parallel \right) \ ; \tag{1.7.17}$$

$\vec{J}_\parallel$ denoting the two components in the plane of the discontinuity whereas $\vec{\nabla}_\parallel$ is the divergence operator in the same plane. The first term in the right hand side may be simplified to read

$$\int \mathrm{d}v \frac{\partial J_z}{\partial z} = A \int \mathrm{d}z \frac{\partial J_z}{\partial z} = A \left(J_{z,1} - J_{z,2} \right) \tag{1.7.18}$$

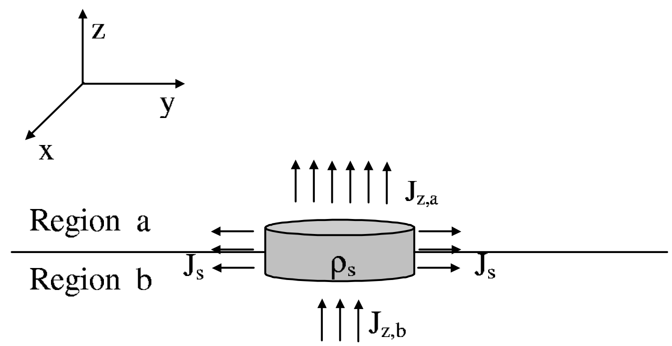

Figure 1.9: Boundary condition linked to charge conservation.

and the second term

$$\int \mathrm{d}v(\vec{\nabla}_{\parallel} \cdot \vec{J}_{\parallel}) = Aw(\vec{\nabla}_{\parallel} \cdot \vec{J}_{\parallel}) = A\vec{\nabla}_{\parallel} \cdot \vec{J}_s \tag{1.7.19}$$

Following a similar approach used earlier, the second term of equation (1.7.16) reads

$$\int \mathrm{d}v\frac{\partial \rho}{\partial t} = Aw\frac{\partial \rho}{\partial t} = A\frac{\partial \rho_s}{\partial t} \tag{1.7.20}$$

hence

$$\vec{1}_n \cdot \left(\vec{J}_1 - \vec{J}_2\right) + \vec{\nabla}_{\parallel} \cdot \vec{J}_s + \frac{\partial \rho_s}{\partial t} = 0 \tag{1.7.21}$$

Based on this expression we may conclude that the change (in time) of the surface charge-density may be linked to a discontinuity of the normal current density or the 2D divergence of the surface current density - or vice versa.

1.8. Operation Regimes

After establishing the constitutive relations and the boundary conditions it is possible, in principle, to determine the complete solution of the electromagnetic field. However, it is not always necessary to solve the full set of Maxwell's equations in order to determine the solution required. The Table below shows the equations that describe the electromagnetic phenomena in three main regimes: on the left, appear the equations that describe static phenomena. In the middle of the table the equations describing the quasi-statics regime are listed, i.e. the regimes for which time variations may occur but as will be discussed extensively subsequently, they occur on time scales much longer than the time it takes light to traverse a distance in vacuum similar to the dimension of the investigated device. On the third, the "complete" set of dynamic equations are presented.

Statics		**Quasi-Statics**		**Dynamics**
Electro	**Magneto**	**Electro**	**Magneto**	**Electromagnetics**
$\vec{\nabla}\times\vec{E}=0$		$\vec{\nabla}\times\vec{E}=0$	$\vec{\nabla}\times\vec{E}+\partial_t\vec{B}=0$	$\vec{\nabla}\times\vec{E}+\partial_t\vec{B}=0$
	$\vec{\nabla}\times\vec{H}=\vec{J}$	$\vec{\nabla}\times\vec{H}-\partial_t\vec{D}=\vec{J}$	$\vec{\nabla}\times\vec{H}=\vec{J}$	$\vec{\nabla}\times\vec{H}-\partial_t\vec{D}=\vec{J}$
$\vec{\nabla}\cdot\vec{D}=\rho$		$\vec{\nabla}\cdot\vec{D}=\rho$		$\vec{\nabla}\cdot\vec{D}=\rho$
	$\vec{\nabla}\cdot\vec{B}=0$		$\vec{\nabla}\cdot\vec{B}=0$	$\vec{\nabla}\cdot\vec{B}=0$

2. POTENTIALS AND SUPERPOTENTIALS

Maxwell's equations are a set of first order differential equations which, in vacuum, are also linear. In many cases it is convenient to solve *one* second order equation rather than *two* first order equations. With this purpose in mind it is quite convenient to introduce the so-called potential functions. It is important to emphasize that within the context of the present discussion, these functions are in effect mathematical tools where the quantities determined by practical measurements are always the fields. Although widely used for solving electromagnetic problems, only in some specific cases these potentials have also a useful physical interpretation.

2.1. Magnetic Vector Potential and Electric Scalar Potential

Consider for simplicity Maxwell equations in vacuum,

$$\vec{D} = \varepsilon_0 \vec{E}\,, \tag{2.1.1}$$

$$\vec{B} = \mu_0 \vec{H}\,, \tag{2.1.2}$$

in the presence of electric charge sources $(\vec{J}, \rho)$ yet in the absence of magnetic sources $(\vec{J}_m = 0\,,\ \rho_m = 0)$ i.e.,

$$\vec{\nabla} \times \vec{E} = -\partial_t \mu_0 \vec{H} \tag{2.1.3}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \partial_t \varepsilon_0 \vec{E} \tag{2.1.4}$$

$$\vec{\nabla} \cdot \varepsilon_0 \vec{E} = \rho \tag{2.1.5}$$

$$\nabla \cdot \mu_0 \vec{H} = 0\,. \tag{2.1.6}$$

The last equation [(2.1.6)] may be reduced to an *identity* if we bear in mind that any vector field $\vec{V}$ satisfies the relation $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0$. It is therefore natural to define

$$\vec{B} = \mu_0 \vec{H} \equiv \vec{\nabla} \times \vec{A}\,, \tag{2.1.7}$$

where $\vec{A}(\vec{r}, t)$ is referred to as the *magnetic vector potential*. Substitution of this definition into Faraday's law implies the relations

$$\nabla \times \vec{E} + \mu_0 \partial_t \vec{H} = 0 \quad \Rightarrow \quad \nabla \times \vec{E} + \partial_t (\nabla \times \vec{A}) = 0 \quad \Rightarrow \quad \nabla \times [\vec{E} + \partial_t \vec{A}] = 0\,. \tag{2.1.8}$$

At this point it is natural to invoke a second identity of the vector calculus namely: any scalar function, ϕ, satisfies

$$\vec{\nabla} \times (\nabla \phi) \equiv 0\,. \tag{2.1.9}$$

Consequently, the electric field is given by

$$\vec{E} = -\partial_t \vec{A} - \nabla \phi \tag{2.1.10}$$

and in this context ϕ is called the *electric scalar potential.*

So far we have determined the relation between the electromagnetic field and the two potential functions. Bearing in mind that $\vec{E}$ and $\vec{H}$ satisfy Maxwell equations, the question we shall address next is what differential equations do $\vec{A}$ and ϕ satisfy? In order to reply to this question we substitute $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\partial_t \vec{A} - \nabla\phi$ in Ampere's law

$$\frac{1}{\mu_0} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{J} + \varepsilon_0 \partial_t \left[-\partial_t \vec{A} - \nabla\phi \right] . \tag{2.1.11}$$

In Cartesian coordinates the double curl operator reads

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \equiv \nabla(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \tag{2.1.12}$$

and thus

$$\nabla(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \left[\vec{J} - \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{A} - \varepsilon_0 \partial_t \nabla\phi \right] \tag{2.1.13}$$

or

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \right] \vec{A} = -\mu_0 \vec{J} + \nabla \left[\vec{\nabla} \cdot \vec{A} + \varepsilon_0 \mu_0 \partial_t \phi \right] . \tag{2.1.14}$$

Similarly we substitute (2.1.9) in Gauss' law

$$\begin{aligned} &\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \quad \Rightarrow \quad \vec{\nabla} \cdot [-\partial_t \vec{A} - \nabla\phi] = \rho/\varepsilon_0 \\ &\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} - \partial_t [\vec{\nabla} \cdot \vec{A}] . \end{aligned} \tag{2.1.15}$$

In order to continue from this point one has to bear in mind that in general a time-independent vector field is determined by its curl ($\vec{\nabla} \times \vec{V}$) and by the divergence ($\vec{\nabla} \cdot \vec{V}$). The latter has not been defined so far for our vector potential function. According to Eqs. (2.1.15) we observe that there are two "natural" possibilities:

2.1.1. Coulomb Gauge

The first choice is motivated by (2.1.15) namely

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{2.1.16}$$

in which case ϕ satisfies the ordinary Poisson equation

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \tag{2.1.17}$$

whereas the magnetic vector potential obeys the differential equation

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \vec{A} = -\mu_0 \vec{J} + \nabla(\varepsilon_0 \mu_0 \partial_t \phi)\,. \tag{2.1.18}$$

These are the two differential equations satisfied by $\vec{A}$ and ϕ.

2.1.2. Lorentz Gauge

The second choice relies on the fact that by imposing the condition

$$\vec{\nabla} \cdot \vec{A} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \phi = 0 \tag{2.1.19}$$

$\vec{A}$ and ϕ satisfy each the decoupled wave equation

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \right] \vec{A} = -\mu_0 \vec{J} \,, \tag{2.1.20}$$

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \right] \phi = -\frac{\rho}{\varepsilon_0} \,. \tag{2.1.21}$$

Note that

$$\varepsilon_o \mu_o = \frac{1}{c^2} \,,$$

where c is the speed of light in vacuum. Furthermore, if in Cartesian coordinates we apply $\vec{\nabla}$ on (2.1.20), apply ∂_t on (2.1.21), multiply the latter by $\mu_0 \varepsilon_0$ and add the two, we obtain

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \right] \left(\vec{\nabla} \cdot \vec{A} + \varepsilon_0 \mu_0 \partial_t \phi \right) = -\mu_0 \left(\vec{\nabla} \cdot \vec{J} + \partial_t \rho \right)$$

i.e. by virtue of Lorentz gauge as well as by the law of charge conservation a zero identity $0 = 0$ results!

In case the magnetic vector potential is time-independent or zero, the scalar potential ϕ is closely related to the voltage. In order to realize the relation between the potential and the voltage recall that $\vec{E} = -\nabla\phi$ and as indicated in Section 1.4.1 Kirchoff's voltage law can be formulated in terms of $\oint \vec{E} \cdot d\vec{l} = 0$ or in other words in a closed loop the sum of all voltages vanishes. Let us examine instead of the closed integral, the integral between two points l_1 and l_2. Now, the voltage between these two points is denoted by $V_{1,2}$ i.e.,

$$V_{1,2} = -\int_{l_1}^{l_2} d\vec{l} \cdot \vec{E}\,.$$

However, bearing in mind that $\vec{E} = -\nabla\phi$, then

$$V_{1,2} = \int_{l_1}^{l_2} d\vec{l} \cdot \nabla\phi = \phi(l_2) - \phi(l_1)\,.$$

This is to say that for such cases the voltage represents the difference between the potentials at the two different locations.

2.2. Electric Vector Potential and Magnetic Scalar Potential

As in Section 2.1 we shall assume a free-space region [i.e. Eqs. (2.1.1–2)]; however, now the electric charge sources vanish ($\vec{J} = 0$ and $\rho = 0$) yet the magnetic charge sources are non zero i.e. $J_{\rm m} \neq 0$ and $\rho_{\rm m} \neq 0$. Maxwell's equations in this case read

$$\vec{\nabla} \times \vec{E} + \partial_t \mu_0 \vec{H} = -\vec{J}_{\rm m} \tag{2.2.1}$$

$$\vec{\nabla} \times \vec{H} - \partial_t \varepsilon_0 \vec{E} = 0 \tag{2.2.2}$$

$$\vec{\nabla} \cdot \varepsilon_0 \vec{E} = 0 \tag{2.2.3}$$

$$\vec{\nabla} \cdot \mu_0 \vec{H} = \rho_{\rm m}\,. \tag{2.2.4}$$

We now take advantage of Eq. (2.2.3) i.e. $\vec{\nabla} \cdot \vec{D} = 0$, and define

$$\varepsilon_0 \vec{E} = -\vec{\nabla} \times \vec{C}\,, \tag{2.2.5}$$

which renders (2.2.3) into an identity [since $\nabla \cdot (\nabla \times \vec{C}) \equiv 0$]. In Eq. (2.2.5) $\vec{C}$ stands for the so-called *electric vector potential*. Substituting this definition in Ampere's law Eq. (2.2.2) we obtain

$$\begin{aligned} &\vec{\nabla} \times \vec{H} - \partial_t[\varepsilon_0 \vec{E}] = 0 \quad \Rightarrow \quad \vec{\nabla} \times \vec{H} - \partial_t[-\vec{\nabla} \times \vec{C}] = 0 \\ &\vec{\nabla} \times [\vec{H} + \partial_t \vec{C}] = 0 \end{aligned} \tag{2.2.6}$$

implying that the magnetic field is determined by the relation

$$\vec{H} = -\partial_t \vec{C} - \nabla \psi\,, \tag{2.2.7}$$

wherein ψ stands for the *magnetic scalar potential*; and here advantage has been taken of the identity

$$\vec{\nabla} \times (\nabla \psi) = 0\,. \tag{2.2.8}$$

The next step is to determine the equations satisfied by $\vec{C}$ and ψ. For this purpose we substitute (2.2.5) in Faraday's law; the result is

$$\begin{aligned}
&\vec{\nabla} \times \vec{E} = -\vec{J}_{\mathrm{m}} - \partial_t \mu_0 \vec{H} \\
&\vec{\nabla} \times \frac{1}{\varepsilon_0}(-\vec{\nabla} \times \vec{C}) = -\vec{J}_{\mathrm{m}} - \mu_0 \partial_t [-\partial_t \vec{C} - \nabla \psi] \\
&\vec{\nabla} \times (\vec{\nabla} \times \vec{C}) = \varepsilon_0 \vec{J}_{\mathrm{m}} - \varepsilon_0 \mu_0 \left[\frac{\partial^2}{\partial t^2} \vec{C} + \partial_t \nabla \psi \right] \\
&\nabla(\vec{\nabla} \cdot \vec{C}) - \nabla^2 \vec{C} = \varepsilon_0 \vec{J}_{\mathrm{m}} - \varepsilon_0 \mu_0 \left[\frac{\partial^2}{\partial t^2} \vec{C} + \partial_t \nabla \psi \right] \\
&\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \right] \vec{C} = -\varepsilon_0 \vec{J}_{\mathrm{m}} + \nabla \left[\vec{\nabla} \cdot \vec{C} + \varepsilon_0 \mu_0 \partial_t \psi \right].
\end{aligned} \tag{2.2.9}$$

Similarly, substituting (2.2.7) in (2.2.4) we have

$$\begin{aligned}
\vec{\nabla} \cdot \vec{H} &= \rho_{\mathrm{m}} / \mu_0 \\
- \quad \vec{\nabla} \cdot (\vec{\nabla} \psi + \partial_t \vec{C}) &= \rho_{\mathrm{m}} / \mu_0 \\
\nabla^2 \psi &= -\rho_{\mathrm{m}} / \mu_0 - \partial_t \vec{\nabla} \cdot \vec{C}\,.
\end{aligned} \tag{2.2.10}$$

In this case the equivalent of the Coulomb gauge reads

$$\vec{\nabla} \cdot \vec{C} = 0 \tag{2.2.11}$$

which implies

$$\nabla^2 \psi = -\frac{\rho_{\mathrm{m}}}{\mu_0}\,, \tag{2.2.12}$$

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \vec{C} = -\varepsilon_0 \vec{J}_{\mathrm{m}} + \nabla(\varepsilon_0 \mu_0 \partial_t \psi)\,. \tag{2.2.13}$$

In a similar way if the equivalent of Lorentz gauge

$$\vec{\nabla} \cdot \vec{C} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \psi = 0 \tag{2.2.14}$$

is imposed, we obtain

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \vec{C} \;\; = - \;\; \varepsilon_0 \vec{J}_{\mathrm{m}}\,, \tag{2.2.15}$$

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \psi \;\; = - \;\; \frac{\rho_{\mathrm{m}}}{\mu_0}\,. \tag{2.2.16}$$

2.3. Superpotentials

The electromagnetic field has 6 components that satisfy Maxwell's equation. In the previous section it was shown that rather than solving a set of first order differential equations it is possible to define *four* functions ($\vec{A}$ and ϕ) that satisfying a second order differential equation i.e the wave equation. In fact if we take into consideration the constraint imposed by the Lorentz gauge there are actually *three* independent functions which have to be determined. These three functions may be the three components of a vector field called the vector superpotential. Let us recapitulate the case of electric sources assuming the Lorentz gauge:

$$\vec{B} = \vec{\nabla} \times \vec{A}\,, \tag{2.3.1}$$

$$\vec{E} = -\partial_t \vec{A} - \nabla\phi\,, \tag{2.3.2}$$

$$\left[\nabla^2 - \varepsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right] \vec{A} = -\mu_0 \vec{J}\,, \tag{2.3.3}$$

$$\left[\nabla^2 - \varepsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right] \phi = -\rho/\varepsilon_0\,, \tag{2.3.4}$$

$$\vec{\nabla} \cdot \vec{A} + \varepsilon_0\mu_0 \partial_t \phi = 0\,, \tag{2.3.5}$$

$$\vec{\nabla} \cdot \vec{J} + \partial_t \rho = 0\,. \tag{2.3.6}$$

If we now define the Hertz superpotential $\vec{\Pi}$ and the generalized source$\vec{\mathcal{P}}$ such that

$$\vec{A} = \varepsilon_0 \mu_0 \partial_t \vec{\Pi}\,, \tag{2.3.7}$$

$$\phi = -\vec{\nabla} \cdot \vec{\Pi}\,, \tag{2.3.8}$$

$$\vec{J} = \partial_t \vec{\mathcal{P}}\,, \tag{2.3.9}$$

$$\rho = -\vec{\nabla} \cdot \vec{\mathcal{P}}\,, \tag{2.3.10}$$

then the first two definitions render the Lorentz gauge [(2.3.5)] into an identity, whereas the second two imply that the continuity equation (2.3.6) reduces to an identity. Substituting (2.3.7) and (2.3.9) in (2.3.3) we have

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \varepsilon_0 \mu_0 \partial_t \vec{\Pi} = -\mu_0 \partial_t \vec{\mathcal{P}} \tag{2.3.11}$$

hence

$$\left[\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right] \vec{\Pi} = -\vec{\mathcal{P}}/\varepsilon_0\,. \tag{2.3.12}$$

One may repeat the same procedure for the case of "magnetic" charge sources.

$$\varepsilon_0\vec{E} = -\vec{\nabla} \times \vec{C}\,, \tag{2.3.13}$$

$$\vec{H} = -\nabla\psi - \partial_t\vec{C}\,, \tag{2.3.14}$$

$$\left[\nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right]\vec{C} = -\varepsilon_0\vec{J}_{\rm m}\,, \tag{2.3.15}$$

$$\left[\nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right]\psi = -\rho_{\rm m}/\mu_0\,, \tag{2.3.16}$$

$$\vec{\nabla}\cdot\vec{C} + \varepsilon_0\mu_0\frac{\partial\psi}{\partial t} = 0\,, \tag{2.3.17}$$

$$\vec{\nabla}\cdot\vec{J}_{\rm m} + \frac{\partial\rho_{\rm m}}{\partial t} = 0\,. \tag{2.3.18}$$

If we now define

$$\vec{C} = \varepsilon_0\mu_0\partial_t\vec{F}\,, \tag{2.3.19}$$

$$\psi = -\vec{\nabla}\cdot\vec{F}\,, \tag{2.3.20}$$

$$\vec{J}_{\rm m} = \partial_t\vec{\mathcal{M}}\,, \tag{2.3.21}$$

$$\rho_{\rm m} = -\vec{\nabla}\cdot\vec{\mathcal{M}}\,, \tag{2.3.22}$$

we obtain

$$\left[\nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right]\vec{F} = -\vec{\mathcal{M}}/\mu_0\,, \tag{2.3.23}$$

$\vec{F}$ is often called the Fitzgerald superpotential and $\vec{\mathcal{M}}$ stands for the generalized magnetic source.

2.4. Equivalence of the Two Gauges

It was previously emphasized that the potentials are just mathematical tools. However, according to (2.1.17-18) and (2.1.20-21)) the differential equations they satisfy are quite different. In this section we show that for a *given source* the electromagnetic field is the same though the equations for the potentials differ! For simplicity we shall translate the problem into $\omega - \vec{k}$ space, namely every quantity being written in the form:

$$U(x, y, z, t) = \int d\omega d^3k\, \overline{U}(k_x, k_y, k_z, \omega) e^{j\omega t - j\vec{k}\cdot\vec{r}}\,.$$

The advantage of this notation is that the differential operators become algebraic operators i.e.

$$\partial_t \vec{U} \;\; \rightarrow \;\; j\omega\vec{U} \tag{2.4.1}$$

$$\vec{\nabla}\cdot\vec{U} \;\; \rightarrow \;\; -j\vec{k}\cdot\vec{U}\,. \tag{2.4.2}$$

Within the framework of Coulomb gauge we have

$$\nabla^2\phi = -\rho/\varepsilon_0 \qquad \Rightarrow k^2\overline{\phi} = \overline{\rho}/\varepsilon_0 \tag{2.4.3}$$

and thus

$$\overline{\phi} = \frac{\overline{\rho}}{\varepsilon_0}\frac{1}{k^2} \tag{2.4.4}$$

along with

$$\left[\nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0\vec{J} + \nabla[\varepsilon_0\mu_0\partial_t\phi] \qquad \Rightarrow$$

$$\begin{aligned}\overline{\vec{A}} &= \quad \frac{-\mu_0}{-k^2+\omega^2\varepsilon_0\mu_0}\overline{\vec{J}} + \frac{-j\vec{k}(\varepsilon_0\mu_0)j\omega}{-k^2+\omega^2\varepsilon_0\mu_0}\overline{\phi} \\ \overline{\vec{A}} &= - \quad \frac{-\mu_0}{-k^2+\omega^2\varepsilon_0\mu_0}\overline{\vec{J}} + \frac{\omega(\varepsilon_0\mu_0)\vec{k}}{-k^2+\omega^2\varepsilon_0\mu_0}\frac{\overline{\rho}}{\varepsilon_0 k^2}\,.\end{aligned} \tag{2.4.5}$$

For comparison, in the framework of the Lorentz gauge

$$\overline{\phi} = \frac{\overline{\rho}}{\varepsilon_0}\frac{1}{k^2-\omega^2\varepsilon_0\mu_0}\,, \tag{2.4.6}$$

$$\overline{\vec{A}} = \frac{\overline{\vec{J}}\mu_0}{k^2-\omega^2\varepsilon_0\mu_0}\,. \tag{2.4.7}$$

Clearly the two expressions differ; however, the the electromagnetic fields are identical as next shown. Thus, using the Coulomb gauge

$$
\begin{aligned}
\vec{E} &= -\; j\omega\vec{\overline{A}} + j\vec{k}\overline{\phi} \\
&= -\; j\omega\left[\frac{-\mu_0\vec{\overline{J}}}{-k^2+\omega^2\varepsilon_0\mu_0} + \frac{\omega(\varepsilon_0\mu_0)\vec{k}}{-k^2+\omega^2\varepsilon_0\mu_0}\,\frac{\overline{\rho}}{\varepsilon_0}\,\frac{1}{k^2}\right] + j\vec{k}\,\frac{\overline{\rho}}{\varepsilon_0}\,\frac{1}{k^2} \\
&= \frac{j\omega\mu_0}{-k^2+\omega^2\varepsilon_0\mu_0}\vec{\overline{J}} + \overline{\rho}\frac{1}{\varepsilon_0}\,\frac{1}{k^2}\left[j\vec{k} + \frac{-j\omega^2(\varepsilon_0\mu_0)\vec{k}}{-k^2+\omega^2\varepsilon_0\mu_0}\right] \\
&= \frac{j\omega\mu_0}{-k^2+\omega^2\varepsilon_0\mu_0}\vec{\overline{J}} + \overline{\rho}\frac{1}{\varepsilon_0}\,\frac{j\vec{k}}{k^2}\cdot\frac{-k^2}{-k^2+\omega^2\varepsilon_0\mu_0} \\
&= \frac{j\omega\mu_0}{-k^2+\omega^2\varepsilon_0\mu_0}\vec{\overline{J}} - j\vec{k}\frac{\overline{\rho}}{\varepsilon_0}\,\frac{1}{-k^2+\omega^2\varepsilon_0\mu_0}\,.
\end{aligned}
\tag{2.4.8}
$$

On the other hand, reverting now to the Lorentz gauge, the expressions:

$$
\begin{aligned}
\vec{\overline{E}} &= -\; j\omega\vec{\overline{A}} + j\vec{k}\overline{\phi} \\
&= -\; j\omega\left[\frac{-\mu_0}{-k^2+\omega^2\varepsilon_0\mu_0}\vec{\overline{J}}\right] + j\vec{k}\left[\frac{-\overline{\rho}/\varepsilon_0}{-k^2+\omega^2\varepsilon_0\mu_0}\right] \\
&= \frac{j\omega\mu_0}{-k^2+\omega^2\varepsilon_0\mu_0}\vec{\overline{J}} - j\vec{k}\frac{\overline{\rho}}{\varepsilon_0}\,\frac{1}{-k^2+\omega^2\varepsilon_0\mu_0}
\end{aligned}
\tag{2.4.9}
$$

result. As expected Eqs. (2.4.8) and (2.4.9) are identical.

Exercise: Repeat this procedure for the couple $\vec{C}$ and ψ.

3. CONSERVATION LAWS

3.1. Poynting Theorem – Time Domain

Maxwell equations "contain" information about energy conservation, or more precisely, about power balance. In the first part of this section we develop the expression which governs this balance; subsequently we shall investigate some of its aspects and implications.

Consider first Faraday and Ampere laws in matter-free space, i.e.

$$\vec{\nabla} \times \vec{E} + \partial_t(\mu_0 \vec{H}) = 0 \,, \tag{3.1.1}$$

$$\vec{\nabla} \times \vec{H} - \partial_t(\varepsilon_0 \vec{E}) = \vec{J} \,. \tag{3.1.2}$$

Multiplying scalarly the first equation by $\vec{H}$ and the second by $\vec{E}$ and subtracting the resulting expressions one obtains the relation

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) + \vec{H} \cdot \partial_t(\mu_0 \vec{H}) + \vec{E} \cdot \partial_t(\varepsilon_0 \vec{E}) = -\vec{J} \cdot \vec{E} \,. \tag{3.1.3}$$

It is possible to show that (prove it!!)

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = (\nabla \times \vec{E}) \cdot \vec{H} - (\vec{\nabla} \times \vec{H}) \cdot \vec{E} \tag{3.1.4}$$

which in turn implies that (3.1.3) may be rewritten as

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \partial_t \left[\frac{1}{2} \varepsilon_0 \vec{E} \cdot \vec{E} + \frac{1}{2} \mu_0 \vec{H} \cdot \vec{H} \right] = -\vec{J} \cdot \vec{E} \,. \tag{3.1.5}$$

At this point we define the so-called *Poynting vector*

$$\vec{S} \equiv \vec{E} \times \vec{H} \tag{3.1.6}$$

whose units are [Watt/m^2]. To the second term in (3.1.5) we ascribe the term *electromagnetic energy density*

$$w_{\mathrm{EM}} = \frac{1}{2}\,\varepsilon_0\,\vec{E}\cdot\vec{E} + \frac{1}{2}\,\mu_0\,\vec{H}\cdot\vec{H} \tag{3.1.7}$$

whose units are Joules/m^3 whereas the third stands for the power conversed per unit volume or otherwise expressed, generated (or absorbed) by the "sources" or "sinks"

$$\vec{J}\cdot\vec{E}\ ; \tag{3.1.8}$$

the units of this term are Watt/m^3. The first term in (3.1.7) being regarded as the electric energy density, $w_{\mathrm{E}} = \frac{1}{2}\,\varepsilon_0\,\vec{E}\cdot\vec{E}$, whereas the second term stands for the magnetic energy density, $w_{\mathrm{M}} = \frac{1}{2}\,\mu_0\,\vec{H}\cdot\vec{H}$. In conclusion, from Maxwell equations we directly obtain the so-called *Poynting theorem* which, in its differential form, reads

$$\boxed{\vec{\nabla}\cdot\vec{S} + \partial_t w_{\mathrm{EM}} = -\vec{J}\cdot\vec{E}} \tag{3.1.9}$$

or in integral form

$$\boxed{\oint\!\!\!\oint \mathrm{d}\vec{a}\cdot\vec{S} + \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}v\, w_{\mathrm{EM}} = -\int \mathrm{d}v\,(\vec{J}\cdot\vec{E})\ .} \tag{3.1.10}$$

The last expression indicates that the power flow across the envelope in conjunction with the time-variation of the energy stored in the volume, equal the power generated/absorbed in the volume confined by the envelope. In the following examples this theorem will be examined somewhat more closely.

Example #1: Capacitor. Let us now obtain a more intuitive picture of this theorem. Consider a capacitor consisting of two circular plates of radius R separated by an air gap h with $h \ll R$ being assumed that R is much smaller than the wavelength ($\lambda = 2\pi c/\omega$) of the exciting electromagnetic field. A voltage (V) is applied to the upper plate whereas the lower one is grounded. Ignoring any possible edge effects, we may approximate the electric field between the plates by its axial component only, i.e.

$$E_z(t) = -\frac{V(t)}{h} \,. \tag{3.1.11}$$

According to Ampere's law and bearing in mind the circular symmetry of the system, we have

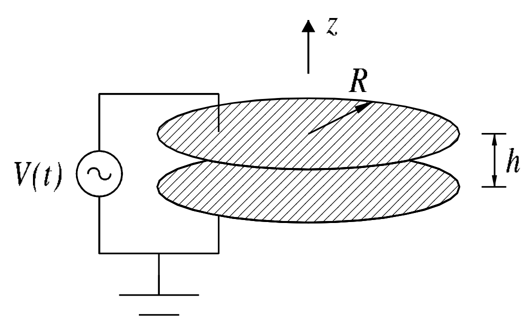

Figure 3.1: Simple model for a capacitor.

$$\varepsilon_0 \frac{\partial E_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) \Rightarrow H_\varphi = \frac{r}{2}\varepsilon_0 \frac{\partial E_z}{\partial t}\,,$$
$$H_\varphi \simeq -\frac{1}{2}\, r\varepsilon_0 \frac{1}{h}\frac{dV}{dt}\,. \tag{3.1.12}$$

Once the field components have been established, the Poynting vector is quite easily calculated:

$$\begin{aligned} \vec{S} = \vec{E}\times\vec{H} &= \left(-\frac{V}{h}\,\vec{1}_z\right)\times\left(-\frac{1}{2}\, r\varepsilon_0 \frac{1}{g}\frac{dV}{dt}\,\vec{1}_\varphi\right) \\ &= -\frac{1}{2}\frac{r}{h^2}\,\varepsilon_0 V \frac{dV}{dt}\,\vec{1}_r = -\frac{1}{2}\frac{r}{h^2}\frac{\varepsilon_0}{2}\frac{dV^2}{dt}\,\vec{1}_r\,. \end{aligned} \tag{3.1.13}$$

The Poynting vector comprises a *radial* component only and therefore

$$\begin{aligned} \vec{\nabla}\cdot\vec{S} &= \frac{1}{r}\frac{\partial}{\partial r}(rS_r) = \frac{1}{r}\frac{\partial}{\partial r} r\left[-\frac{1}{2}\frac{r}{g^2}\frac{\varepsilon_0}{2}\frac{dV^2}{dt}\right] \\ &= -\frac{1}{h^2}\frac{\varepsilon_0}{2}\frac{dV^2}{dt}\,. \end{aligned} \tag{3.1.14}$$

In the integral form:

$$\begin{aligned} \oiint \vec{S}\cdot d\vec{a} &= \oiint S_r\,(Rd\varphi)dz = S_r\Big|_{r=R} h\,(2\pi R) \\ &= h(2\pi R)\left[-\frac{1}{2}\frac{R}{h^2}\,\varepsilon_0\frac{1}{2}\frac{d}{dt}V^2\right] \\ &= -\frac{d}{dt}\left[\frac{1}{2}CV^2\right] = -\frac{d}{dt}W_E \end{aligned} \tag{3.1.15}$$

wherein,

$$C = \varepsilon_0 \, \frac{\pi R^2}{h} \, , \tag{3.1.16}$$

is defined as the low frequency (static) capacitance of the system, and thus

$$\oiint \vec{S} \cdot d\vec{a} + \frac{d}{dt} \, W_E = 0 \, . \tag{3.1.17}$$

As it stands at the moment, the magnetic energy stored in the capacitor does not contribute to the energy balance. Justification of this result will be provided subsequently when discussing electro and magneto-quasi-statics.

Example #2: Resistor.

A second example of interest relates to a resistor. Consider the system illustrated in Fig. 3.3. The inner cylinder consists of a conducting material σ. Between the two plates the electric field is again (see discussion prior to Eq. (3.1.11)) approximated by

$$E_z = -\frac{V}{h} \tag{3.1.18}$$

and consequently, the relevant axial current density is

$$J_z = \sigma \, E_z = -\sigma \, \frac{V}{h} \, . \tag{3.1.19}$$

In the presence of this current a circular magnetic field is established

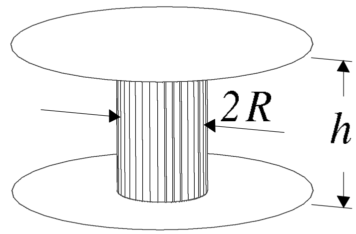

Figure 3.2: Simple model for a resistor.

$$\vec{\nabla} \times \vec{H} = \vec{J} + \partial_t \vec{D} \rightsquigarrow \frac{1}{r}\,\partial_r\,(rH_\varphi) = J_z\,. \tag{3.1.20}$$

Hence, for $r \leq R$

$$H_\varphi \simeq -\frac{1}{2}\,r\,\sigma\,\frac{V}{h} \tag{3.1.21}$$

so that the Poynting vector comprises (in our approximation) only a radial component namely

$$S_r = -E_z H_\varphi = -\left(-\frac{V}{h}\right)\left[-\frac{1}{2}\,r\,\sigma\frac{V}{h}\right] \tag{3.1.22}$$

and, therefore, the power flow across the surface at $r = R$ is

$$
\begin{aligned}
\oiint d\vec{a} \cdot \vec{S} &= -2\pi R\, h \left[-\frac{V}{h}\right] \left[-\frac{1}{2}\, R\, \sigma\, \frac{V}{h}\right] \\
&= - \frac{\pi R^2}{h}\, \sigma\, V^2
\end{aligned}
\tag{3.1.23}
$$

i.e.,

$$
\oiint \vec{S} \cdot d\vec{a} = - \frac{1}{\mathcal{R}}\, V^2 = -P
\tag{3.1.24}
$$

where $\mathcal{R}$ stands for the low frequency (static) resistance

$$
\mathcal{R} = \frac{1}{\sigma}\, \frac{h}{\pi R^2}\,.
\tag{3.1.25}
$$

This result illustrates that the power dissipated in the metallic material “enters” in the system through its boundaries, being propagated by means of the Poynting vector.

3.2. Complex Poynting Theorem

In many cases of practical interest the source of the electromagnetic field varies periodically in time at a constant angular frequency (ω). It is convenient to adopt under this circumstance the well-known phasor notation. Consider a function

$$G(x, y, z, t) = G_o(x, y, z) \cos[\omega t + \theta(x, y, z)] \,, \tag{3.2.1}$$

introducing now a complex notation $e^{j\omega t}$ where $j = \sqrt{-1}$ we have

$$G = \frac{1}{2} \left[G_o \, e^{j\omega t + j\theta} + G_o \, e^{-j\omega t - j\theta} \right] \tag{3.2.2}$$

or

$$G = \text{Re} \left\{ G_o \, e^{j\theta} \, e^{j\omega t} \right\} \,. \tag{3.2.3}$$

We now define the *phasor* of G as

$$\underline{G}(x, y, z) \equiv G_o(x, y, z) \, e^{j\theta\,(x,y,z)} \tag{3.2.4}$$

so that

$$G(x, y, z, t) = \text{Re} \left[\underline{G}(x, y, z) \, e^{j\omega t} \right] \,. \tag{3.2.5}$$

Based upon this notation, and assuming a *linear* medium, Maxwell's equations take in the following form:

$$\vec{\nabla} \times \underline{\vec{E}} + j\omega\mu_0\underline{\vec{H}} = 0 \tag{3.2.6}$$

$$\vec{\nabla} \times \underline{\vec{H}} - j\omega\varepsilon_0\underline{\vec{E}} = \underline{\vec{J}} \tag{3.2.7}$$

$$\vec{\nabla} \cdot \varepsilon_0\underline{\vec{E}} = \underline{\rho} \tag{3.2.8}$$

$$\vec{\nabla} \cdot \mu_0\underline{\vec{H}} = 0\,. \tag{3.2.9}$$

Our next step is to formulate Poynting's theorem within the framework of this notation. Before doing this it is important to point out that the complex product of two phasors provides information about the *time average* of the "real" quantities. In order to illustrate this fact, consider

$$G_1(\vec{r},t) = \mathrm{Re}\left\{\underline{G}_1(\vec{r})\, e^{j\omega t}\right\} \quad \text{and} \quad G_2(\vec{r},t) = \mathrm{Re}\left\{\underline{G}_2(\vec{r})\, e^{j\omega t}\right\}. \tag{3.2.10}$$

The product of the two is

$$\begin{aligned} G_1(\vec{r},t)G_2(\vec{r},t) &= G_{01}(\vec{r})\cos[\omega t + \theta_1(\vec{r})] \times G_{02}(\vec{r})\cos[\omega t + \theta_2(\vec{r})] \\ &= G_{01}G_{02}\tfrac{1}{2}\left[e^{j\omega t + j\theta_1} + e^{-j\omega t - j\theta_1}\right] \\ &\qquad \times \tfrac{1}{2}\left[e^{j\omega t + j\theta_2} + e^{-j\omega t - j\theta_2}\right] \\ &= \tfrac{1}{4}\,G_{01}G_{02}\left\{e^{2j\omega t + j(\theta_1+\theta_2)} + e^{-2j\omega t - j(\theta_1+\theta_2)}\right. \\ &\qquad \left. + e^{j(\theta_1-\theta_2)} + e^{-j(\theta_1-\theta_2)}\right\} \end{aligned} \tag{3.2.11}$$

hence

$$G_1(\vec r,t)G_2(\vec r,t) = \frac{1}{2}\,G_{01}G_{02}\cos[2\omega t+\theta_1+\theta_2] + \frac{1}{4}\;[\underline{G}_1\underline{G}_2^* + \underline{G}_1^*\underline{G}_2]\;. \tag{3.2.12}$$

Averaging over a time span T eliminates the first term on the right hand side yielding

$$\begin{aligned} <G_1(\vec r,t)G_2(\vec r,t)>_t &= \frac{1}{4}\;[\underline{G}_1(\vec r)\underline{G}_2^*(\vec r) + G_1^*(\vec r)G_2(\vec r)] \\ &= \frac{1}{2}\,\mathrm{Re}\;[\underline{G}_1(\vec r)\underline{G}_2^*(\vec r)] \\ &= \frac{1}{2}\,\mathrm{Re}\;[\underline{G}_2(\vec r)\underline{G}_1^*(\vec r)]\;; \end{aligned} \tag{3.2.13}$$

here $<\ldots.>_t \equiv (1/T)\displaystyle\int_0^T \mathrm{d}t\ldots.$ represents the averaging process. Based upon this result we conclude that in phasor notation, whenever a product of two amplitudes is involved, the *time average value* of this product is readily obtained by the above procedure. As an example we enquire about the value of the time-average energy density stored on the electric field ($\varepsilon_r = 1$):

$$\begin{gathered} \vec E(\vec r,t) = \mathrm{Re}\,\{\underline{\vec E}(\vec r)\,e^{j\omega t}\} \\ w_E = \frac{1}{2}\,\varepsilon_0\,\vec E(\vec r,t)\cdot\vec E(\vec r,t) \qquad <w_E>_t = \frac{1}{4}\,\varepsilon_0\,\underline{\vec E}(\vec r)\cdot\underline{\vec E}^*(\vec r) = \frac{1}{4}\,\varepsilon_0\,|\vec E(\vec r)|^2\,. \end{gathered} \tag{3.2.14}$$

We are now in a position to develop an expression for the complex Poynting theorem. We multiply (3.2.6) scalarly by $\underline{\vec H}^*$

$$\underline{\vec H}^*\cdot(\vec\nabla\times\underline{\vec E}) + j\omega\mu_0\underline{\vec H}^*\cdot\underline{\vec H} = 0 \tag{3.2.15}$$

and, similarly the *complex conjugate* of Eq. (3.2.7) is scalarly multiplied by $\underline{\vec{E}}$:

$$\underline{\vec{E}} \cdot (\vec{\nabla} \times \underline{\vec{H}}^*) + j\omega\varepsilon_0 \underline{\vec{E}} \cdot \underline{\vec{E}}^* = \underline{\vec{J}}^* \cdot \underline{\vec{E}} \, . \tag{3.2.16}$$

Subtracting (3.2.16) from (3.2.15) we obtain the relation

$$\underline{\vec{H}}^* \cdot (\vec{\nabla} \times \underline{\vec{E}}) - \underline{\vec{E}} \cdot (\vec{\nabla} \times \underline{\vec{H}}^*) - j\omega \left[\varepsilon_0 \underline{\vec{E}} \cdot \underline{\vec{E}}^* - \mu_0 \underline{\vec{H}} \cdot \underline{\vec{H}}^* \right] = -\underline{\vec{J}}^* \cdot \underline{\vec{E}} \tag{3.2.17}$$

hence

$$\vec{\nabla} \cdot (\underline{\vec{E}} \times \underline{\vec{H}}^*) - 4j\omega \left[\frac{1}{4} \varepsilon_0 \underline{\vec{E}} \cdot \underline{\vec{E}}^* - \frac{1}{4} \mu_0 \underline{\vec{H}} \cdot \underline{\vec{H}}^* \right] = -\underline{\vec{J}}^* \cdot \underline{\vec{E}} \tag{3.2.18}$$

or

$$\vec{\nabla} \cdot \left(\frac{1}{2} \underline{\vec{E}} \times \underline{\vec{H}}^* \right) - 2j\omega \left[\frac{1}{4} \varepsilon_0 \underline{\vec{E}} \cdot \underline{\vec{E}}^* - \frac{1}{4} \mu_0 \underline{\vec{H}} \cdot \underline{\vec{H}}^* \right] = -\frac{1}{2} \underline{\vec{J}}^* \cdot \underline{\vec{E}} \, . \tag{3.2.19}$$

We now *define* the complex Poynting vector

$$\underline{\vec{S}} \equiv \frac{1}{2} \underline{\vec{E}} \times \underline{\vec{H}}^* \, , \tag{3.2.20}$$

which determines the average *energy flux density* from or to the system. The time average *electric field energy density* is

$$< w_E >_t \equiv \frac{1}{4} \varepsilon_0 \underline{\vec{E}} \cdot \underline{\vec{E}}^* \tag{3.2.21}$$

whereas the time averaged *magnetic field energy density* is

$$< w_M >_t \equiv \frac{1}{4} \mu_0 \underline{\vec{H}} \cdot \underline{\vec{H}}^* \tag{3.2.22}$$

so that we may finally write for the complex Poynting theorem:

$$\boxed{\vec{\nabla} \cdot \underline{\vec{S}} - 2j\omega \left[< w_E >_t - < w_M >_t \right] = -\frac{1}{2} \underline{\vec{J}}^* \cdot \underline{\vec{E}} .} \tag{3.2.23}$$

We now review a few simple examples in light of this theorem (a somewhat more systematic discussion follows in the next chapters).

(a) Capacitor: For a capacitor $\underline{\vec{J}} = 0$; we may ignore the magnetic energy yet $< w_E > \neq 0$. Consequently, the *imaginary* part of the Poynting vector is *non zero*.

(b) Inductor: Similar to (a) except that now $< w_E >_t \simeq 0$ and $< w_M >_t \neq 0$. The phase of $\vec{S}$ is shifted by 180^o.

(c) Resonance: $< w_M > = < w_E > \Rightarrow$ power exchange takes place inside the system. Comparing (3.2.23) with (3.1.9) we observe that in the present formulation a minus sign occurs between the average magnetic energy density and its electric counterpart.

(d) Resistor:

$$\left.\begin{array}{l} <w_M>\sim 0 \\ <w_E>\sim 0 \\ \underline{\vec{J}} = \sigma \underline{\vec{E}} \end{array}\right\} \quad \vec{\nabla} \cdot \vec{S} = -\frac{1}{2}\sigma \underline{\vec{E}} \cdot \underline{\vec{E}}^* \ .$$

$\Rightarrow$ Only the *real* part of the Poynting vector is *non zero*.

3.3. Global Energy Conservation – Hydrodynamic Approximation

As observed in several cases so far, the electromagnetic field is a direct result of the presence of a source and it was tacitly assumed that the latter was not affected by the field. In practice this generally is not the case. In fact, since the source of any electromagnetic field is electric charge, the source itself is affected by the field since each individual charge is affected by the field through the Lorentz force $F_{\rm L} = -q[\vec{E} + \vec{V} \times \vec{B}]$; the charge here is assumed to be negative. Consequently, it is important to determine the energy conservation relation for a specific case when the source is affected by the field. For this purpose we shall consider the dynamics of electrons within the simplest framework that is – hydrodynamics. We assume that within an infinitesimal volume dv a uniform distribution of electrons exists all having a common velocity ($\vec{u}$) which may, in fact, be attributed to a single particle in the center of the infinitisimal volume. It is further assumed that the dynamics of this particular particle reflects the dynamics of all the electrons in the infinitisimal volume. The full set of equations describing such phenomenon is given by

$$\vec{\nabla} \cdot \vec{S} + \partial_t w_{EM} = -\vec{J} \cdot \vec{E} \tag{3.3.1}$$

$$\vec{\nabla} \cdot (n\vec{u}) + \partial_t n = 0 \tag{3.3.2}$$

$$\frac{d}{dt}\left[\frac{1}{2}mu^2\right] = -q\vec{u} \cdot \vec{E} \tag{3.3.3}$$

subject to the relation $\vec{J} = -q\, n\, \vec{u}$; here n represents the density of electrons, $-q$ is the charge of a single electron and m represents its rest mass.

Before we proceed it is important to clarify the meaning of the last operator in Eq. (3.3.3). For a function of three variables (x, y, z) varying in time (t), namely $F(x, y, z; t)$, the total time derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} F(x,y,z;t) = \frac{\partial F}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial F}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial F}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial F}{\partial z} \tag{3.3.4}$$

$$= \frac{\partial F}{\partial t} + u_x \frac{\partial F}{\partial x} + u_y \frac{\partial F}{\partial y} + u_z \frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} + (\vec{u} \cdot \vec{\nabla}) F\,.$$

Our goal is to determine the energy conservation associated with the dynamics of charges and fields as reflected by this set of equations. Multiplying now Eq. (3.3.3) by the particles density n we find that

$$n \frac{d}{dt} E_{\mathrm{kin}} = \vec{J} \cdot \vec{E} \tag{3.3.5}$$

therefore Eq. (3.3.1) has the following form

$$\vec{\nabla} \cdot \vec{S} + \partial_t w_{\mathrm{EM}} = -\vec{J} \cdot \vec{E} = -n \frac{d}{dt} E_{\mathrm{kin}}$$

$$= -n \left[\frac{\partial E_{\mathrm{kin}}}{\partial t} + \vec{u} \cdot \vec{\nabla}\, E_{\mathrm{kin}} \right], \tag{3.3.6}$$

where in the last expression we have used Eq. (3.3.5). The last term may be simplified since

$$
\begin{aligned}
&= -n \frac{\partial}{\partial t} E_{\mathrm{kin}} - (n\vec{u}) \cdot \vec{\nabla}\, E_{\mathrm{kin}} \\
&= -\frac{\partial}{\partial t}(n\, E_{\mathrm{kin}}) + E_{\mathrm{kin}} \frac{\partial}{\partial t} n - \vec{\nabla} \cdot [E_{\mathrm{kin}}\, n\, \vec{u}] + E_{\mathrm{kin}} \vec{\nabla} \cdot (n\vec{u}) \\
&= -\frac{\partial}{\partial t}(n\, E_{\mathrm{kin}}) - \vec{\nabla} \cdot (n\vec{u}\, E_{\mathrm{kin}}) + E_{\mathrm{kin}} \left[\underbrace{\frac{\partial}{\partial t} n + \vec{\nabla} \cdot (n\vec{u})}_{\equiv 0} \right] .
\end{aligned}
\tag{3.3.7}
$$

The last term vanishes by virtue of the continuity equation; hence

$$
\vec{\nabla} \cdot [\vec{S} + n\vec{u}\, E_{\mathrm{kin}}] + \partial_t \, [w_{\mathrm{EM}} + n\, E_{\mathrm{kin}}] = 0 \,. \tag{3.3.8}
$$

This expression has the form of a field conservation law: the term $n\vec{u}E_{\mathrm{kin}}$ represents the kinetic energy flux, whereas nE_{kin} stands for the kinetic energy density. Employing Gauss theorem we obtain that

$$
\oiint \mathrm{d}\vec{a} \cdot [\vec{S} + n\vec{u}E_{\mathrm{kin}}] + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathrm{v}} \mathrm{d}v\, [w_{\mathrm{EM}} + n\, E_{\mathrm{kin}}] = 0 \,, \tag{3.3.9}
$$

which implies that time-variations of kinetic and electromagnetic energies stored within a given volume are compensated by the electromagnetic energy flux and/or by the kinetic energy flux across the relevant boundaries.

3.4. Conservation of Linear Momentum

The conservation of linear momentum can be considered by a similar approach. The total electromagnetic force on a charged particle is

$$\vec{F} = -q(\vec{E} + \vec{v} \times \vec{B})\,. \tag{3.4.1}$$

If the sum of all the momenta of all the particles in the volume V is denoted by $\vec{P}_{\text{mech}}$, we can write, from Newton's second law,

$$\frac{d\vec{P}_{\text{mech}}}{dt} = \int\limits_V (\rho\vec{E} + \vec{J} \times \vec{B})\, dv\,. \tag{3.4.2}$$

Using Maxwell's equations to eliminate ρ and $\vec{J}$ from (3.4.2) i.e.,

$$\rho = \vec{\nabla} \cdot \vec{D}\,, \quad \vec{J} = \vec{\nabla} \times \vec{H} - \frac{\partial}{\partial t}\vec{D} \tag{3.4.3}$$

we obtain

$$\rho\vec{E} + \vec{J} \times \vec{B} = \left[\vec{E}(\vec{\nabla} \cdot \vec{D}) + \vec{B} \times \frac{\partial\vec{D}}{\partial t} - \vec{B} \times (\vec{\nabla} \times \vec{H})\right]\,.$$

Further writing

$$\vec{B} \times \frac{\partial\vec{D}}{\partial t} = -\frac{\partial}{\partial t}(\vec{D} \times \vec{B}) + \vec{D} \times \frac{\partial\vec{B}}{\partial t}$$

and adding $\vec{H}(\vec{\nabla} \cdot \vec{B}) = 0$ to the square bracket, we obtain

$$\rho\vec{E} + \vec{J} \times \vec{B} = \vec{E}(\vec{\nabla} \cdot \vec{D}) + \vec{H}(\vec{\nabla} \cdot \vec{B}) - \vec{D} \times (\vec{\nabla} \times \vec{E}) - \vec{B} \times (\vec{\nabla} \times \vec{H}) - \frac{\partial}{\partial t}(\vec{D} \times \vec{B})\,.$$

The rate of change of mechanical momentum (3.4.2) can now be written

$$\frac{d\vec{P}_{\rm mech}}{dt} + \frac{d}{dt}\int\limits_V (\vec{D} \times \vec{B})\,dv = \frac{1}{4\pi}\int\limits_V \Big[\vec{E}(\vec{\nabla} \cdot \vec{D}) - \vec{D} \times (\vec{\nabla} \times \vec{E}) + \vec{H}(\vec{\nabla} \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{H})\Big]\,dv\,. \quad (3.4.4)$$

We may identify the volume integral on the left as the total electromagnetic momentum $\vec{P}_{\rm field}$ in the volume v:

$$\vec{P}_{\rm field} = \int\limits_V (\vec{D} \times \vec{B})\,dv\,. \quad (3.4.5)$$

We interprete the integrand as representing a density of electromagnetic momentum; we further note that this momentum density is proportional to the energy-flux density $\vec{S}$, with the proportionality constant c^{-2} (in vacuum). In order to complete the identification of the volume integral of

$$\vec{g} = \vec{D} \times \vec{B} \quad (3.4.6)$$

as the electromagnetic momentum, and in order to postulate (3.4.4) as the conservation law for momentum, we must convert the volume integral on the right into a surface integral of the normal component of an expression which may be identified to represent a flow of momentum. Let the Cartesian coordinates be denoted by $x_{\alpha=1,2,3}$. The $\alpha = 1$ component of the electric part of the integrand in (3.4.4) is given explicitly by

$$[\vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E})]_1 = E_1\left(\frac{\partial D_1}{\partial x_1}+\frac{\partial D_2}{\partial x_2}+\frac{\partial D_3}{\partial x_3}\right) - D_2\left(\frac{\partial E_2}{\partial x_1}-\frac{\partial E_1}{\partial x_2}\right) + D_3\left(\frac{\partial E_1}{\partial x_3}-\frac{\partial E_3}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_1}(\varepsilon_0 E_1^2) + \frac{\partial}{\partial x_2}(\varepsilon_0 E_1 E_2) + \frac{\partial}{\partial x_3}(\varepsilon_0 E_1 E_3) - \frac{\varepsilon_0}{2}\frac{\partial}{\partial x_1}(E_1^2+E_2^2+E_3^2)\,.$$

This means that we can write the α-th component as

$$[\vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E})]_\alpha = \varepsilon_0\sum_{\beta=1}^{3}\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta - \frac{1}{2}\vec{E}\cdot\vec{E}\delta_{\alpha\beta}) \tag{3.4.7}$$

obtaining in the right-hand side the form of a divergence of a second rank tensor.

With the definition of the *Maxwell stress tensor* $T_{\alpha\beta}$ as

$$T_{\alpha\beta} = \varepsilon_0 E_\alpha E_\beta + \mu_0 B_\alpha B_\beta - \frac{1}{2}(\varepsilon_0\vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\,. \tag{3.4.8}$$

Eq. (3.4.4) may now be written in component form as

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, dv\,. \tag{3.4.9}$$

Application of the divergence theorem to the volume integral finally yields

$$\frac{d}{dt}\left(\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}\right)_\alpha = \oint\limits_s \sum_\beta T_{\alpha\beta} n_\beta \, da \,, \tag{3.4.10}$$

where $\vec{n}$ is the outward normal to the closed surface s. $\sum\limits_\beta T_{\alpha\beta} n_\beta$ is the α-th component of the flow per unit area of momentum across the surface s into the volume V. In other words, it is the force per unit area transmitted across the surface s and acting on the combined system of particles and fields inside V. Equation (3.4.10) may therefore be used in principle in order to calculate the forces acting on objects in electromagnetic fields by enclosing the objects within a boundary surface s and by adding up the total electromagnetic force according to the right-hand side of (3.4.10).

3.5. Regimes of Operation

In Section 1.8 we have postulated the equations describing a quasi-static system. We are now in a position to put forward a somewhat more quantitative classification of this division.

Both the potentials as well as the various components of the electromagnetic field satisfy the wave equation, i.e., denoting by $U(x, y, z, t)$ either one of the components of the electromagnetic field in a Cartesian coordinate system, we have, for matter-free space

$$\left[\nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right] U(x, y, z, t) = 0 \tag{3.5.1}$$

further, if the system oscillates at the monochromatic angular frequency ω i.e. $U(x, y, z, t) = \mathrm{Re}\left[\underline{U}(x, y, z, \omega)e^{j\omega t}\right]$ then, for a homogeneous, isotropic time-independent and linear non-conducting environment one has

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right] \underline{U}(x, y, z, \omega) = 0\,, \tag{3.5.2}$$

where

$$c = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}} \tag{3.5.3}$$

is the so-called phase-velocity of an electromagnetic plane wave in vacuum – also referred to as the "speed of light". The angular frequency ω and the speed of light c determine a characteristic wavelength

$$\lambda_0 = 2\pi \, \frac{c}{\omega} \tag{3.5.4}$$

in free space (subscript 0 indicates vacuum) or in a material medium

$$\frac{2\pi}{\lambda_m} = \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \, . \tag{3.5.5}$$

As a rule of thumb the criterion for a quasi-static regime can be formulated as

$$\lambda_m \gg \left\{ \begin{array}{l} \text{typical geometrical length of} \\ \text{the system.} \end{array} \right\} . \tag{3.5.6}$$

This criterion does not distinguish between magneto and electro-quasi-statics. For this purpose, we should compare energies in the system,

(a) Systems operating in the magneto-quasi-static (MQS) regime imply

$$\int \, < w_M > \, dv \gg \int \, < w_E > \, dv \tag{3.5.7}$$

(b) Systems operating in the electro-quasi-static (EQS) regime imply

$$\int \, < w_E > \, dv \gg \int \, < w_M > \, dv \, . \tag{3.5.8}$$

4. QUASI-STATICS

A variety of phenomena does not fall within the scope of statics or optics; these phenomena correspond to cases in which the "characteristic time" of the system is much longer than the time it takes the EM phenomena to traverse the system, or alternatively expressed, one of the energies (magnetic or electric) is much larger than the other. To quote an example, in statics or in a low frequency regime a capacitor stores energy which is predominantly electric, whereas the magnetic energy is taken to be zero. On the other hand, as it will be shown here that at high frequencies the capacitor itself becomes inductive. The opposite is valid for an inductor. These effects may have a dramatic impact on the design of fast circuits. Thus, many commercial software packages modeling various communication circuits utilize values of capacitance and inductance calculated for low frequencies ($< 100MHz$). With the increase in the requirements for bandwidth, the frequency also increases and the predictions of these models at 10 or 35GHz are definitely wrong: the parameters have to be re-adapted to the new operating regime or even better, the so-called "dynamic capacitance" and/or "inductance" must be resorted to.

In this chapter we shall start with a discussion of *Electro-Quasi-Statics* (EQS) which is a branch of electromagnetics treating phenomena characterized by the fact that the electric energy stored in the system is much larger than its magnetic counterpart. The discussion will be followed by a section introducing *Magneto-Quasi-Statics* (MQS) i.e. – the regime in which the magnetic energy predominates. Finally, in the last section, we investigate the case in which ohmic losses play a predominant role. The latter case is particularly important, since at high frequencies or for short electromagnetic impulses, the field distribution in metals differs substantially from its dc distribution.

4.1. Electro-Quasi Statics

In this section we investigate a structure that at low frequency behaves as a capacitor but at high frequency, its behavior changes dramatically. The system under investigation is schematically illustrated in Figure 4.1.

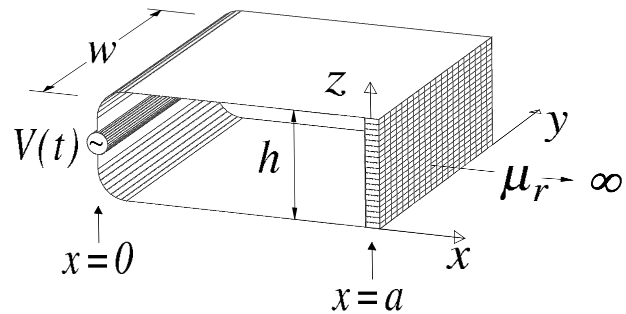

Figure 4.1: Model of a quasi-static capacitor.

Our goal is to determine the electromagnetic field in the volume confined between the electrodes, and further, to examine the values of the power and of the energy exchanged within the system.

4.1.1. Assumptions and Notation Conventions

1. From the geometric point of view the system is "infinite" along the $\pm y$-axis i.e. $\dfrac{\partial}{\partial y} \sim 0$.

2. Along the z-axis, the system is extremely narrow so that inherent variations may safely be neglected, i.e. $\dfrac{\partial}{\partial z} \simeq 0$.

3. In order to simplify the analysis and to exclude as well end effects, we introduce at $x = a$ a thin layer of material of "infinite" permeability ($\mu_r \to \infty$) at the end ($x = a$).

4. The *distributed* generator ($\partial_y = 0$) located at $x = 0$, generates a signal which oscillates at the monochromatic angular frequency ω i.e., we assume a steady-state operation $\sim e^{j\omega t}$.

5. No sources and/or charges are to be found in the inner space.

6. ε_r, μ_r stand for the relative permittivity and for the relative permeability of the material.

7. The electromagnetic field outside the device is assumed to be negligible.

4.1.2. Basic Approach

Step I: **Gauss' Charge Law**. In the y and z directions the spatial variations are negligibly small (see above) and therefore we can write

$$\vec{\nabla} \cdot \varepsilon_o \varepsilon_r \, \underline{\vec{E}} = 0 \Rightarrow \frac{\partial \underline{E}_x}{\partial x} \simeq 0 \Rightarrow \underline{E}_x \sim \text{ const.}$$

Furthermore since, $\underline{E}_x(z = 0, h) = 0$ [recall that $\partial_z \sim 0$] $\Rightarrow \underline{E}_x = 0$.

Step II: **Conservation of Magnetic Induction**

$$\vec{\nabla} \cdot \mu_0 \mu_r \; \underline{\vec{H}} = 0 \Rightarrow \frac{\partial}{\partial x} \underline{H}_x \sim 0 \Rightarrow \underline{H}_x \sim \text{ const.}$$

If this magnetic field is non-zero, we may apply an *external* magnetic field to compensate it. We therefore consider the case $\underline{H}_x = 0$.

Step III: **Faraday's Induction Law**

$$\begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_x & 0 & 0 \\ 0 & \underline{E}_y & \underline{E}_z \end{vmatrix} = -j\omega\mu_0\mu_r \; \underline{\vec{H}}$$

$$\begin{array}{lll} \vec{1}_x & : & 0 = 0 \\ \vec{1}_y & : - & \partial_x \underline{E}_z = -j\omega\mu_0\mu_r \; \underline{H}_y \, , \\ \vec{1}_z & : & \partial_x \underline{E}_y = -j\omega\mu_0\mu_r \; \underline{H}_z \, . \end{array}$$

Based on the boundary conditions, $\underline{E}_y(z = 0, h) = 0$ and $\partial_z \sim 0$ concluding that $\underline{E}_y = 0$. Consequently $\Rightarrow \underline{H}_z = 0$. Hence from Faraday's law one obtains the non-trivial relation

$$\partial_x \underline{E}_z = j\omega\mu_0\mu_r \; \underline{H}_y \, . \tag{4.1.1}$$

Step IV: **Ampere's Law**

$$\begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_x & 0 & 0 \\ 0 & \underline{H}_y & 0 \end{vmatrix} = j\omega\varepsilon_0\varepsilon_r\,\underline{\vec{E}}\,,$$

or explicitly

$$\begin{aligned} \vec{1}_x &: \; 0 = 0 \\ \vec{1}_y &: \; 0 = 0 \\ \vec{1}_z &: \; \partial_x \underline{H}_y = j\omega\varepsilon_0\varepsilon_r\,\underline{E}_z \end{aligned}$$

i.e. non-trivially

$$\partial_x \underline{H}_y = j\omega\varepsilon_0\varepsilon_r\,\underline{E}_z \tag{4.1.2}$$

substituting into (4.1.1) we have

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\underline{E}_z &= j\omega\mu_0\mu_r\,\frac{\partial}{\partial x}\underline{H}_y = j\omega\mu_0\mu_r\,(j\omega\varepsilon_0\varepsilon_r\,\underline{E}_z) \\ &= -\,\frac{\omega^2}{c^2}\,\varepsilon_r\mu_r\,\underline{E}_z \end{aligned} \tag{4.1.3}$$

or

$$\left[\frac{d^2}{dx^2} + \varepsilon_r\mu_r\,\frac{\omega^2}{c^2}\right]\,\underline{E}_z = 0\,. \tag{4.1.4}$$

The general solution of this equation is

$$\underline{E}_z = \underline{A}\, e^{-jx\,\frac{\omega}{c}\,\sqrt{\varepsilon_r \mu_r}} + \underline{B}\, e^{jx\,\frac{\omega}{c}\,\sqrt{\varepsilon_r \mu_r}} \tag{4.1.5}$$

and according to (4.1.1):

$$\underline{H}_y = \frac{1}{j\omega\mu_0\mu_r}\frac{d\underline{E}_z}{dx} = \frac{1}{j\omega\mu_0\mu_r}\left(-j\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right)\left[\underline{A}\, e^{-jx\,\frac{\omega}{c}\,\sqrt{\varepsilon_r \mu_r}} - \underline{B}\, e^{jx\,\frac{\omega}{c}\,\sqrt{\varepsilon_r \mu_r}}\right]. \tag{4.1.6}$$

This magnetic field component vanishes [prove that!!] at the high permeability wall ($\mu_r \to \infty$ at $x = a$); hence

$$\underline{H}_y(x = a) = 0 \Rightarrow \underline{A}\, e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r \mu_r}} - \underline{B}\, e^{ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r \mu_r}} = 0 \tag{4.1.7}$$

and therefore

$$\underline{B} = \underline{A}\, e^{-j2a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r \mu_r}}. \tag{4.1.8}$$

Consequently,

$$
\begin{aligned}
\underline{H}_y &= -\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}\left[\underline{A}\,e^{-jx\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}} - \underline{A}\,e^{-j2a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}\,e^{jx\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}\right] \\
&= \sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}\,\underline{A}\,e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}\,2j\sin\left[(x-a)\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right] \\
&= 2j\underline{A}\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}\,e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}\,\sin\left[(x-a)\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right] \qquad (4.1.9)
\end{aligned}
$$

and

$$
\begin{aligned}
E_z &= \frac{1}{j\omega\varepsilon_0\varepsilon_r}\,\partial_x H_y \\
&= \frac{1}{j\omega\varepsilon_0\varepsilon_r}\cdot 2j\underline{A}e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\cos\left[(x-a)\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right] \\
&= 2\underline{A}\,e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}\cos\left[(x-a)\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]\,. \qquad (4.1.10)
\end{aligned}
$$

Note that the expression

$$
Z_0 \equiv \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} \qquad (4.1.11)
$$

in (4.1.9) usually labeled by “wave-impedance” of the medium.

The distributed generator imposes at $x = 0$ a voltage V_0; therefore the z-component of the electric field between the two plates at the input is

$$\underline{E}_z(x=0) = -\frac{V_0}{h} \tag{4.1.12}$$

hence

$$\underline{E}_z(x=0) = 2\underline{A}\, e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}} \cos\left[a\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right] \tag{4.1.13}$$

or

$$\underline{A} = -\frac{1}{2}\, e^{ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}} \frac{\dfrac{V_0}{h}}{\cos\left[a\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]} \tag{4.1.14}$$

so that finally

$$\underline{E}_z = -\frac{V_0}{h}\,\frac{\cos\left[(x-a)\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]}{\cos\left[a\,\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]} \tag{4.1.15}$$

$$\underline{H}_y = -j\frac{V_0}{h}\,\frac{1}{Z_0}\,\frac{\sin\left[(x-a)\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\cos\left[a\,\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]}\,. \tag{4.1.16}$$

Equations (4.1.15) and (4.1.16) determine the electromagnetic field components confined within the structure.

4.1.3. Impedance Considerations

The magnetic field outside the system is assumed to be zero, therefore at $z = 0, h$ a surface current density is flowing accounting for the discontinuity in the magnetic field as per the relation [see Eq. (1.7.6)]

$$\vec{1}_n \times (\vec{H}_a - \vec{H}_b) = \vec{J}_s\,. \tag{4.1.17}$$

If we consider the upper plate:

$$\begin{aligned} \vec{1}_n &= \vec{1}_z\,, \\ \vec{H}_a &= 0\,, \\ \underline{\vec{H}}_b &= -\; j\frac{V_0}{h}\frac{1}{Z_0}\frac{\sin\left[(x-a)\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\cos\left[a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]}\,\vec{1}_y \end{aligned} \tag{4.1.18}$$

so that

$$\underline{J}_{s,x}(x) = -\; j\frac{V_0}{h}\frac{1}{Z_0}\frac{\sin\left[(x-a)\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\cos\left[a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]}\,. \tag{4.1.19}$$

The width of the system is denoted by w; accordingly, the input current is

$$\begin{aligned}
\underline{I}_{\rm in} &= \underline{J}_{s,x}(x=0)w \\
&= -\ j\frac{V_0}{h}\frac{w}{Z_0}\frac{(-)\sin\left[a\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\cos\left[a\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}\,.
\end{aligned} \tag{4.1.20}$$

With the aid of the exciting voltage we are able to define the input impedance:

$$\begin{aligned}
Z_{\rm in} &\equiv \frac{V_0}{\underline{I}_{\rm in}} = \frac{V_0}{j\frac{V_0}{h}\frac{w}{Z_0}\tan\left(\frac{\omega}{c}a\sqrt{\varepsilon_r\mu_r}\right)} \\
&= \frac{Z_0}{j\frac{w}{h}\tan\left(\frac{\omega}{c}a\sqrt{\varepsilon_r\mu_r}\right)}\,.
\end{aligned} \tag{4.1.21}$$

At the limit of a small argument of the trigonometric function

$$\frac{\omega}{c}a\sqrt{\varepsilon_r\mu_r} \ll 1\,, \tag{4.1.22}$$

we find that

$$Z_{\text{in}} \simeq \frac{\sqrt{\dfrac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}}{j\,\dfrac{w}{h} \times \dfrac{\omega}{c}\,a\sqrt{\varepsilon_r\mu_r}} \;\simeq\; \frac{1}{j\omega\left[\varepsilon_0\varepsilon_r\;\dfrac{wa}{h}\right]}$$

$$\simeq \frac{1}{j\omega C_0}\,, \tag{4.1.23}$$

wherein $C_0 \equiv \varepsilon_0\varepsilon_r\,wa/h$ represents the low frequency (static) capacitance. If (4.1.22) is not satisfied, the trigonometric function in (4.1.22) may become negative and the device rather than being a capacitor as per (4.1.23), becomes an *inductor*. As remarked, this conclusion may be of great practical importance in the design of fast circuits, as components designed to operate as capacitors at low frequency may behave inherently in a different form at high frequencies.

With this definition in mind we may express the input impedance as

$$Z_{\text{in}} = \frac{1}{j\omega C_0}\,\frac{1}{\left[\dfrac{\tan\left(\dfrac{\omega}{c}\,a\sqrt{\mu_r\varepsilon_r}\right)}{\dfrac{\omega}{c}\,a\sqrt{\mu_r\varepsilon_r}}\right]} \tag{4.1.24}$$

Resonance. If instead of the generator we locate at $x = 0$ a shunting plane, the input impedance reduces to zero ($Z_{\text{in}} = 0$) and resonances of the system are determined from the condition

$$\cot(a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}) = 0 \tag{4.1.25}$$

implying

$$\frac{\omega}{c}\,a\sqrt{\varepsilon_r\mu_r} = \frac{\pi}{2} \pm n\,\pi\,; \qquad n = 0, 1, 2, \ldots\,.$$

As opposed to the simplified lumped circuit theory, it is quite clear that infinitely many resonances occur.

Exercise: Expand the function $\dfrac{\tan(x)}{x}$ in Taylor series (up to the third term) and discuss the role of the various terms of Z_{in} when viewed as an electric circuit.

4.1.4. **Energy and Power Considerations**

With the components of the EM field established, we now take one further step and examine the power exchange in the system. This will be done in three stages: first we calculate the power entering into the system, then we calculate the energies stored in the system and finally we integrate Poynting's theorem taking into account the results from the previous two stages.

(a) Poynting vector and power flow.

Recall that the field components are

$$\underline{E}_z = -\frac{V_0}{h}\,\frac{\cos\left[(x-a)\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]}{\cos\left[a\,\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]}\,, \tag{4.1.26}$$

$$\underline{H}_y = -j\frac{V_0}{h}\,\frac{1}{Z_0}\,\frac{\sin\left[(x-a)\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\cos\left[a\,\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]}\,, \tag{4.1.27}$$

consequently, Poynting's vector at the input comprises one component only, namely

$$\begin{aligned}\underline{S}_x\Big|_{x=0} &= -\frac{1}{2}\,\underline{E}_z\underline{H}_y^*\Big|_{x=0} \\ &= -\frac{1}{2}\left[-\frac{V_0}{h}\right]\left[j\frac{V_0}{h}\frac{1}{Z_0}\tan\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)\right]^* \\ &= \frac{1}{2}\,|V_0|^2\,\frac{-j}{h^2Z_0}\,\tan\left[a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]\,; \qquad (4.1.28)\end{aligned}$$

$\underline{H}_y^*$ represents the complex conjugate of the magnetic field $\underline{H}_y$.

The total power flowing into the system

$$\begin{aligned}\underline{P} = \iint\, d\vec{a}\cdot\vec{S} &= whS_x\Big|_{x=0} \\ &= \frac{1}{2}\,|V_0|^2\,\frac{w}{jh}\,\frac{1}{Z_0}\,\tan\left[a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]\,, \qquad (4.1.29)\end{aligned}$$

or using (4.1.21) i.e.

$$Z_{\rm in} = \frac{Z_0}{j\,\dfrac{w}{h}\,\tan\left[a\,\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]} = \frac{1}{j\omega C_0}\,\frac{a\,\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}{\tan\left(a\,\dfrac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right)} \qquad (4.1.30)$$

we obtain

$$\underline{P} = \frac{1}{2}\,|V_0|^2\,\frac{1}{Z_{in}^*} = \frac{1}{2}\,V_0\,\underline{I}_{in}^*\ . \tag{4.1.31}$$

This result clearly reveals that the Poynting vector describes the energy flux to or from the system.

(b) Energy stored in the system.

Consider next the average (in time) magnetic energy stored in the system

$$
\begin{aligned}
<W_{\mathrm{M}}> &= \int dv <w_{\mathrm{M}}> = w \cdot h \int_0^a dx \, \frac{1}{4} \, \mu_0 \mu_r \, |H_y|^2 \\
&= \frac{w h \mu_0 \mu_r}{4} \, \frac{|V_0|^2}{h^2} \, \frac{1}{Z_0^2} \int_0^a \frac{\sin^2 \left[(x-a) \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right]}{\cos^2 \left[a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right]} \, dx \\
&= \frac{(w h \mu_0 \mu_r \,) \, |V_0|^2}{4 h^2 Z_0^2 \, \cos^2 \left[a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right]} \, \frac{1}{2} \left\{ a - \frac{\sin \left[2a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right]}{2 \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r}} \right\}
\end{aligned}
$$

i.e.,

$$
<W_{\mathrm{M}}> = \frac{1}{4} \, \varepsilon_0 \varepsilon_r \, \frac{|V_0|^2}{h^2} \, \frac{\frac{1}{2} \left[1 - \operatorname{sinc} \left(2a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right) \right]}{\cos^2 \left(a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right)} \, (wha) \tag{4.1.32}
$$

wherein $\operatorname{sinc}(x) \equiv \sin(x)/x$.

In a similar way the time-average electric energy stored within the system is given by

$$
\begin{aligned}
<W_{\rm E}> &= \int dv <w_{\rm E}>= wh \int_0^a dx \, \frac{1}{4} \varepsilon_0 \varepsilon_r \, |E_z|^2 \\
&= wh \frac{1}{4} \varepsilon_0 \varepsilon_r \, \frac{|V_0|^2}{h^2} \frac{1}{\cos^2 \left[a \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right]} \int_0^a \cos^2 \left[(x-a) \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right] \\
&= wh \frac{1}{4} \varepsilon_0 \varepsilon_r \, \frac{|V_0|^2}{h^2} \frac{1}{\cos^2 \left[a \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right]} \frac{1}{2} \left\{ a + \frac{\sin \left[2a \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right]}{2 \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r}} \right\} \\
<W_{\rm E}> &= \frac{1}{4} \varepsilon_0 \varepsilon_r \, \frac{|V_0|^2}{h^2} \frac{\frac{1}{2} \left[1 + \text{ sinc } \left(2a \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right) \right]}{\cos^2 \left(a \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right)} (wha) \, . \qquad (4.1.33)
\end{aligned}
$$

With the aid of the last two expressions we are able to examine the ratio between the two forms of energy stored in the system, namely

$$
\frac{<W_M>}{<W_E>} = \frac{1 - \text{ sinc } \left(2a \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right)}{1 + \text{ sinc } \left(2a \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right)} \, .
$$

At the quasi-static limit (4.1.22) i.e., $\left(u \equiv 2a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r} \ll 1\right)$, we find that

$$\frac{<W_{\rm M}>}{<W_{\rm E}>} = \left.\frac{1-\dfrac{\sin(u)}{u}}{1+\dfrac{\sin(u)}{u}}\right|_{u\,\equiv\,2a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}} \simeq \left.\frac{1-(u-\frac{1}{6}\,u^3)/u}{1+(u-\frac{1}{6}\,u^3)/u}\right|_{u\,=\,2\,\frac{\omega}{c}\,a\sqrt{\varepsilon_r\mu_r}} ,$$

i.e.

$$\frac{<W_{\rm M}>}{<W_{\rm E}>} \simeq \frac{\frac{1}{6}\,u^2}{2} \simeq \frac{1}{12}\,\left[2a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]^2 ,$$

$$\frac{<W_{\rm M}>}{<W_{\rm E}>} \simeq \frac{1}{3}\,\left[a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right]^2 \ll 1\,, \tag{4.1.34}$$

implying that the magnetic energy stored in the system is much smaller than its electric counterpart, as one may expect for a capacitor.

(c) Poynting's theorem in its integral form.

In subsection (a) we have found that

$$\underline{P} = \iint\, d\vec{a}\cdot\underline{\vec{S}} = \frac{1}{2}\,|V_0|^2\,\frac{1}{j\dfrac{h}{w}Z_0}\;\tan\,\left[a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right] \tag{4.1.35}$$

whereas in (b) we have found

$$<W_{\rm E}>= \frac{1}{4}\,\varepsilon_0\varepsilon_r\;\frac{|V_0|^2}{h^2}\;\frac{\frac{1}{2}\,\left[1+\,\mathrm{sinc}\,\left(2a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right)\,\right]}{\cos^2\left(a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right)}\,(wha)\;, \tag{4.1.36}$$

$$<W_{\rm M}>= \frac{1}{4}\,\varepsilon_0\varepsilon_r\;\frac{|V_0|^2}{h^2}\;\frac{\frac{1}{2}\,\left[1-\,\mathrm{sinc}\,\left(2a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right)\,\right]}{\cos^2\left(a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right)}\,(wha)\;. \tag{4.1.37}$$

With these quantities defined we may now formulate the complex Poynting theorem in its integral form. Using Gauss' mathematical theorem by analogy to (3.1.10)

$$\oiint d\vec{a}\cdot\underline{\vec{S}}+2j\omega\Big[\langle W_M\rangle-\langle W_E\rangle\Big]=-\frac{1}{2}\int\limits_V dv\underline{\vec{J}}^{\,*}\cdot\underline{\vec{E}}\,, \tag{4.1.38}$$

so that in a sourceless space

$$\oiint d\vec{a}\cdot\underline{\vec{S}}-2j\omega\,[<W_{\rm E}>-<W_{\rm M}>]=0\,. \tag{4.1.39}$$

Hence on substituting (4.1.35–39) we obtain

$$-\frac{1}{2}|V_0|^2\frac{1}{j\dfrac{h}{w}Z_0}\tan\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right) - 2j\omega\left\{\ \frac{1}{4}\ \ \varepsilon_0\varepsilon_r\,\frac{|V_0|^2}{h^2}\,\frac{\dfrac{1}{2}\left[1+\ \mathrm{sinc}\ \left(2a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)\ \right]}{\cos^2\left(a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)}\,(wha)\right.$$

$$\left. -\ \frac{1}{4}\ \ \varepsilon_0\varepsilon_r\,\frac{|V_0|^2}{h^2}\,\frac{\dfrac{1}{2}\left[1-\ \mathrm{sinc}\ \left(2a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)\ \right]}{\cos^2\left(a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)}\,(wha)\right\} = 0$$

$$-\frac{1}{2}|V_0|^2\frac{1}{j\dfrac{h}{w}Z_0}\tan\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right) - 2j\omega\frac{1}{4}\varepsilon_0\varepsilon_r\,\frac{|V_0|^2}{h^2}\,(wha)\,\frac{1}{2}\cdot 2\,\frac{\mathrm{sinc}\ \left(2a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)}{\cos^2\left(a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)} = 0$$

hence

$$-\frac{1}{2}|V_0|^2\frac{1}{j\dfrac{h}{w}Z_0}\tan\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right) - \frac{j}{2}|V_0|^2\frac{1}{\dfrac{h}{w}Z_0}\tan\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right) = 0$$

the solution clearly satisfies the integral complex Poynting theorem.

Conclusion. Within the framework of the electro-quasi-static (EQS) approximation, the energy stored in the electric field is much larger than that stored in the magnetic field ($< W_{\mathrm{M}} > \ll < W_{\mathrm{E}} >$). One possible way to interpret this fact is to recall that the average magnetic field is proportional to $\mu_0 H^2$ and since the energy stored in the magnetic field is negligible we may develop a simplified set of equations for the electro-quasi-static regime by taking "$\mu_0 \to 0$" (or "$c^2 \to \infty$"). Explicitly, within the framework of phasor notation these equations read

$$\boxed{\vec{\nabla} \times \underline{\vec{E}} \simeq 0} \tag{4.1.40}$$

$$\boxed{\vec{\nabla} \times \underline{\vec{H}} = \underline{\vec{J}} + j\omega \underline{\vec{D}}}, \tag{4.1.41}$$

$$\boxed{\vec{\nabla} \cdot \underline{\vec{D}} = \underline{\rho}}. \tag{4.1.42}$$

Thus the complex Poynting Theorem for the capacitor may be stated in the form

$$\boxed{\vec{\nabla} \cdot \underline{\vec{S}} - 2j\omega < W_E >_t = -\frac{1}{2}\, \underline{\vec{J}}^* \cdot \underline{\vec{E}}} \tag{4.1.43}$$

4.2. Magneto-Quasi-Statics

In this section we examine in detail a system in which the magnetic field plays a predominant role. For this purpose we consider the system reproduced in Fig. 4.2.

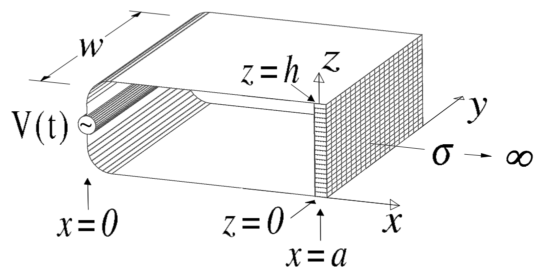

Figure 4.2: Model of a quasi-static inductor.

4.2.1. Assumptions and Notation Conventions

1. The system is "infinite" along the $\pm y$-axis $\Rightarrow \partial_y \sim 0$.
2. Along the $\pm z$-axis, the system is extremely narrow $\Rightarrow \partial_z \sim 0$.
3. At $x = a$ the plates are electrically short circuited.
4. The voltage at the input oscillates at an angular frequency $\omega \Rightarrow e^{j\omega t}$.
5. No sources exist within the inner space.
6. ε_r, μ_r stand respectively for the relative permittivity and for the relative permeability of the material.

7. The electromagnetic field external to the device is assumed to be zero.

4.2.2. Approach to Solution

The approach is entirely similar to that outlined in the previous section; thus from Gauss' charge law $[\vec{\nabla} \cdot \varepsilon_0 \varepsilon_r \underline{\vec{E}} = 0]$ we find

$$\underline{E}_x = 0 \tag{4.2.1}$$

and from the equation of continuity of the magnetic induction $[\vec{\nabla} \cdot \underline{\vec{B}} = 0]$ we find

$$\underline{H}_x \simeq 0\,. \tag{4.2.2}$$

The electric field component in the y-direction, vanishes since $E_y(z = 0, h) = 0$ and $\partial_z \simeq 0$, i.e.

$$\underline{E}_y \simeq 0\,. \tag{4.2.3}$$

From the z component of Faraday's law $[\vec{1}_z : \ \partial_x \underline{E}_y = -j\omega\mu_0\mu_r\, \underline{H}_z]$ we conclude, relying on (4.2.3) that

$$\underline{H}_z \simeq 0\,; \tag{4.2.4}$$

in addition, from the y component (of Faraday's law) we have

$$\vec{1}_y \ : \quad -\partial_x \underline{E}_z = -j\omega\mu_0\mu_r\, \underline{H}_y\,. \tag{4.2.5}$$

Finally the non-trivial component of Ampere's law yields

$$\vec{1}_z \ : \quad \partial_x \underline{H}_y = j\omega\varepsilon_0\varepsilon_r\, \underline{E}_z\,. \tag{4.2.6}$$

Substituting (4.2.6) in (4.2.5) we obtain the same expression as in (4.1.4):

$$\left[\frac{d^2}{dx^2} + \varepsilon_r \mu_r \frac{\omega^2}{c^2}\right] \underline{E}_z = 0 \tag{4.2.7}$$

The general solution of (4.2.7) is

$$\underline{E}_z = \underline{A}\, e^{-jx\,\frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}} + \underline{B}\, e^{jx\,\frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}} \tag{4.2.8}$$

and here that the *difference* from the previous case arises: as opposed to the vanishing of the tangential magnetic field, in this case the tangential electric field vanishes at $x = a$ ($\sigma \to \infty$) so that

$$\begin{aligned} \underline{E}_z(x = a) &= \underline{A}\, e^{-ja\,\frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}} + \underline{B}\, e^{ja\,\frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}} \\ &= 0 \end{aligned} \tag{4.2.9}$$

implying

$$\underline{B} = -\underline{A}\, e^{-2ja\,\frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}} \tag{4.2.10}$$

so that

$$\underline{E}_z = \underline{A}\, e^{-ja\,\frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}} (-2j) \sin\left[(x-a)\frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}\right]. \tag{4.2.11}$$

With Eq. (4.2.5) we obtain

$$\begin{aligned}\underline{H}_y &= \frac{1}{j\omega\mu_0\mu_r}\partial_x \underline{E}_z \\ &= \frac{1}{j\omega\mu_0\mu_r}(-2j)\underline{A}\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}\cos\left[(x-a)\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right] \\ = - &\quad 2\underline{A}\,\frac{e^{-ja\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}}}{\sqrt{\dfrac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}}\cos\left[(x-a)\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]\;. \end{aligned} \tag{4.2.12}$$

The generator imposes at $x = 0$ a voltage V_0 and therefore, the electric field at the input is

$$\underline{E}_z(x=0) = -\frac{V_0}{h}\,. \tag{4.2.13}$$

Now bearing in mind that

$$\underline{E}_z(x=0) = 2jA\,e^{-ja\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}}\sin\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right) \tag{4.2.14}$$

we find that

$$2jA\,e^{-ja\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}}\sin\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right) = -\frac{V_0}{h}$$

i.e.

$$\underline{A} = \frac{j}{2}\,\frac{V_0}{h}\,\frac{e^{ja\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}}}{\sin\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)} \tag{4.2.15}$$

hence, finally

$$\underline{E}_z \;=\; \frac{V_0}{h}\,\frac{\sin\left[(x-a)\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\sin\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}\,, \tag{4.2.16}$$

along with

$$\underline{H}_y \;=\; -j\,\frac{V_0}{h}\,\frac{1}{Z_0}\,\frac{\cos\left[(x-a)\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\sin\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}\,. \tag{4.2.17}$$

4.2.3. Impedance Considerations

The magnetic field in the region exterior to the system has been assumed to be zero, therefore at $z = 0, h$ a surface current density balances the discontinuity of the *tangential* magnetic field component excited: recall that

$$\vec{1}_n \times (\vec{H}_a - \vec{H}_b) = \vec{J}_s \, .$$

If we now consider the upper plate

$$\begin{aligned} &\vec{1}_n = \vec{1}_z \, , \\ &\vec{H}_a = 0 \, , \\ &\vec{H}_b = -j\,\frac{V_0}{h}\,\frac{1}{Z_0}\,\frac{\cos\left[(x-a)\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\sin\left[a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}\vec{1}_y \, , \end{aligned} \tag{4.2.18}$$

hence

$$\underline{J}_{s,x}(x) = -j\,\frac{V_0}{h}\,\frac{1}{Z_0}\,\frac{\cos\left[(x-a)\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\sin\left[a\,\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]} \, . \tag{4.2.19}$$

The width of the system is w and so the total current entering the system is

$$\begin{aligned} \underline{I}_{\rm in} &= \underline{J}_{s,x}(x=0)w \\ &= -j\,\frac{V_0}{h}\,\frac{w}{Z_0}\,\text{ctan}\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right) . \end{aligned} \tag{4.2.20}$$

With the input voltage introduced above, we define the input impedance of the system as

$$Z_{\rm in} \equiv \frac{V_0}{\underline{I}_{\rm in}} = \frac{V_0}{-j\,\frac{V_0}{h}\,\frac{w}{Z_0}\,{\rm ctan}\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)}$$

$$Z_{\rm in} = jZ_0\,\frac{h}{w}\,\tan\left(a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)\,. \qquad (4.2.21)$$

In the quasi-static limit $a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r} \ll 1$, and therefore

$$Z_{\rm in} \simeq j\,\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}\,\frac{h}{w}\,a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r} \sim j\omega L_0\,, \qquad (4.2.22)$$

wherein $L_0 = \mu_0\mu_r\,\dfrac{ah}{w}$ stands for the static (or low frequency) inductance of the system. Upon frequency increase, the sign of the impedance may change, a fact which implies that the system now exhibits capacitive characteristics. Based on Eq. (4.2.21) a resonator is obtained by either shortening the input $\Rightarrow Z_{\rm in} = 0 \Rightarrow$

$$a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r} = \pm\pi\,n \qquad (4.2.23)$$

or by opening the input $Z_{\rm in} \to \infty \Rightarrow$

$$a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r} = \frac{\pi}{2} \pm \pi n \qquad (4.2.24)$$

in both cases $n = 0, \pm 1, \pm 2 \cdots$; tacitly in the latter case radiation throughout the open region is discarded.

Exercise: Show that $Z_{\text{in}} = j\omega L_0 \dfrac{\tan\left(a\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right)}{a\dfrac{\omega}{c}\sqrt{\varepsilon_r\mu_r}}$.

4.2.4. Power and Energy Considerations

(a) Poynting's vector and power flow.

With

$$\underline{E}_z \;=\; \frac{V_0}{h}\,\frac{\sin\left[(x-a)\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\sin\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]} \quad , \qquad \underline{H}_y = -j\,\frac{V_0}{h}\,\frac{1}{Z_0}\,\frac{\cos\left[(x-a)\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\sin\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]} \qquad (4.2.25)$$

we have

$$\underline{S}_x \;=-\;\frac{1}{2}\,\underline{E}_z\,\underline{H}_y^* = -\frac{j}{2}\,\frac{|V_0|^2}{h^2}\,\frac{1}{Z_0}\,\frac{\sin\left[(x-a)\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]\;\cos\left[(x-a)\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]}{\sin^2\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]} \qquad (4.2.26)$$

so that

$$\begin{aligned}\underline{P} \;&=\; \iint d\vec{a}\,\vec{S}\Big|_{x=0} = (w\,h)\frac{j}{2}\,\frac{|V_0|^2}{h^2}\,\frac{1}{Z_0}\quad \mathrm{ctan}\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right] \\ &=\; \frac{1}{2}\,|V_0|^2\,\frac{w}{h}\cdot j\,\frac{1}{Z_0}\mathrm{ctan}\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right]\;. \end{aligned} \qquad (4.2.27)$$

Resorting to the definition of the input impedance, i.e.

$$Z_{\rm in} = j Z_0\,\frac{h}{w}\,\tan\left[a\,\frac{\omega}{c}\sqrt{\varepsilon_r\mu_r}\right] \qquad (4.2.28)$$

one may write

$$\underline{P} = \frac{1}{2}\, V_0\, \frac{V_0^*}{Z_{\rm in}^*} = \frac{1}{2}\, V_0\, I_{\rm in}^*\,, \tag{4.2.29}$$

an expression which represents the so-called complex power flow into the system.

(b) Energy stored in the system.

The average stored magnetic energy in the system is given by

$$\begin{aligned} < W_{\rm M} > &= \int dv < w_{\rm M} >= wh \int_0^a dx\, \frac{1}{4}\, \mu_0 \mu_r\, |H_y|^2 \\ &= \frac{1}{4}\, wh\, \mu_0 \mu_r \left[\frac{V_0}{h}\, \frac{1}{Z_0}\, \frac{1}{\sin\left(a\, \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r} \right)} \right]^2 \int_0^a dx \cos^2 \left[(x-a)\, \frac{\omega}{c}\, \sqrt{\varepsilon_r \mu_r} \right] \\ &= \frac{1}{4}\, \varepsilon_0 \varepsilon_r \left(\frac{|V_0|}{h} \right)^2 \frac{\frac{1}{2} \left[1 + \operatorname{sinc} \left(2a\, \frac{\omega}{c}\, \sqrt{\varepsilon_r \mu_r} \right) \right]}{\sin^2 \left(a\, \frac{\omega}{c}\, \sqrt{\varepsilon_r \mu_r} \right)} \, (wha)\,. \end{aligned} \tag{4.2.30}$$

In a similar way the time-average stored electric energy is

$$
\begin{aligned}
<W_{\mathrm{E}}> &= \int dv < w_{\mathrm{E}} > = wh \int_0^a dx \, \frac{1}{4} \, \varepsilon_0 \varepsilon_r \; |E_z|^2 \\
&= \frac{1}{4} \, wh \, \varepsilon_0 \varepsilon_r \; \left[\frac{V_0}{h} \, \frac{1}{\sin\left(a \, \frac{\omega}{c} \sqrt{\varepsilon_r \mu_r}\right)} \right]^2 \int_0^a dx \sin^2 \left[(x-a) \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right] \\
&= \frac{1}{4} \, \varepsilon_0 \varepsilon_r \; \left(\frac{|V_0|}{h} \right)^2 \, \frac{\frac{1}{2} \left[1 - \operatorname{sinc} \left(2a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right) \right]}{\sin^2 \left(a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right)} \, (wha) \, .
\end{aligned}
\tag{4.2.31}
$$

Note that the ratio

$$
\frac{<W_{\mathrm{E}}>}{<W_M>} = \frac{1 - \operatorname{sinc} \left(2a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right)}{1 + \operatorname{sinc} \left(2a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right)} \simeq \frac{1}{3} \left[a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right]^2 \ll 1 \tag{4.2.32}
$$

is much smaller than unity at low frequencies, implying that the system behaves as an inductor according to the prediction (4.2.22). Note also that the total time-average energy is given by

$$
<W_{\mathrm{total}}> = <W_{\mathrm{E}}> + <W_{\mathrm{M}}> = \frac{1}{4} \, \varepsilon_0 \varepsilon_r \; \left(\frac{|V_0|}{h} \right)^2 \, \frac{wha}{\sin^2 \left(a \, \frac{\omega}{c} \, \sqrt{\varepsilon_r \mu_r} \right)} \tag{4.2.33}
$$

whereas the ratio

$$\frac{<W_{\mathrm{M}}> - <W_{\mathrm{E}}>}{<W_{\mathrm{M}}> + <W_{\mathrm{E}}>} = \mathrm{sinc}\ \left(2a\,\frac{\omega}{c}\,\sqrt{\varepsilon_r\mu_r}\right)$$

may be either positive or negative, according to whether the system behaves as an inductor or as a capacitor.

Exercise:

Determine the reduced set of equations for MQS as well as the relevant complex Poynting's Theorem.

4.3. Steady–State Field in a Conductor

Within the realm of stationary conduction phenomena, we are used to think that the only manifestation of the finite conductivity of a metal is via its resistance, which is inversely proportional to the specific conductance and to the area (A) across which the current flows being, however directly proportional to the extension (l) of the conducting path, i.e. $R \propto l/\sigma A$. In this section we shall investigate a simple phenomenon associated with conductivity and EM fields at relatively high frequencies. For this purpose consider the system in Figure 4.3.

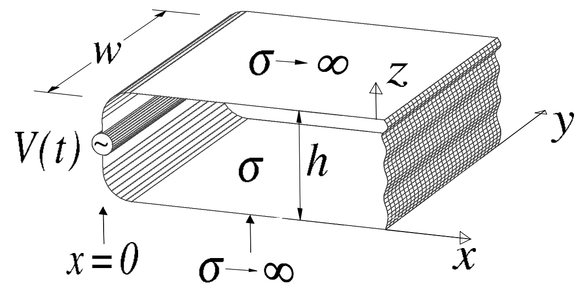

Figure 4.3: Model of a quasi-static resistor.

4.3.1. Assumptions and Notation Conventions

1. The system is "infinite" along the $\pm y$-axis $\Rightarrow \partial_y \sim 0$.
2. Along the z-axis the system is extremely narrow $\Rightarrow \partial_z \sim 0$.
3. The only possible variations are assumed along the x-axis.
4. The time dependence is $e^{j\omega t}$.
5. No sources reside within the metallic medium.
6. σ is the conductivity of the medium; its other electromagnetic characteristics are assumed to be represented by μ_0, ε_0.
7. Finally, $\sigma \gg \omega\varepsilon_0$.

4.3.2. Approach to Solution

1. Gauss' law $\Rightarrow \underline{E}_x = 0$.
2. $\vec{\nabla} \cdot \underline{\vec{B}} = 0 \Rightarrow \underline{B}_x = 0$.
3. $\underline{E}_y(z = 0, h) = 0$ and $\partial_z \sim 0 \Rightarrow \underline{E}_y = 0$.
4. $\underline{E}_y = 0$ and Faraday's law $\Rightarrow \underline{H}_z \simeq 0$.

5. Faraday's law

$$\vec{1}_y \; : \quad -\partial_x \underline{E}_z = -j\omega\mu_0 \underline{H}_y \,. \tag{4.3.1}$$

6. Ampere's law

$$\vec{1}_z \; : \quad -\partial_x \underline{H}_y = \underline{J}_z + j\omega\varepsilon_0 \underline{E}_z = (\sigma + j\omega\varepsilon_0)\, \underline{E}_z \,. \tag{4.3.2}$$

The goal is, as in the previous two cases, to determine the field distribution inside the device. For this purpose we substitute (4.3.2) into (4.3.1) and assume that $\sigma \gg \omega\varepsilon_0$

$$\frac{d^2}{dx^2}\underline{E}_z = +j\omega\mu_0 \frac{d\underline{H}_y}{dx} = j\omega\mu_0(\sigma\, \underline{E}_z)$$

so that

$$\left[\frac{d^2}{dx^2} - j\omega\mu_0\sigma\right] E_z = 0\,. \tag{4.3.3}$$

We assume a solution of the form e^{-sx}; for a non-trivial solution s satisfies

$$s^2 - j\omega\mu_0\sigma = 0 \tag{4.3.4}$$

hence

$$\begin{aligned} s &= \pm\sqrt{+j\omega\mu_0\sigma} \\ &= \pm\sqrt{e^{j\pi/2}\,\omega\mu_0\sigma} \\ &= \pm\, e^{+j\pi/4}\sqrt{\omega\mu_0\sigma} = \pm(1+j)\,\frac{1}{\sqrt{2}}\sqrt{\omega\mu_0\sigma}\,. \end{aligned} \tag{4.3.5}$$

We define a so-called *skin-depth* δ

$$\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}} \tag{4.3.6}$$

thus

$$s = \pm(1+j)\,\frac{1}{\delta}\,. \tag{4.3.7}$$

The general solution of (4.3.3.) is now stated to be

$$\underline{E}_z(x) = \underline{A}\,e^{-(1+j)\,x/\delta} + \underline{B}\,e^{+(1+j)\,x/\delta}\,. \tag{4.3.8}$$

In the present case the system is "infinite" along the x-axis, we therefore conclude that $\underline{B} = 0$ (otherwise the solution diverges as $x \to \infty$); hence

$$\underline{E}_z(x) = \underline{A}\,e^{-(1+j)\,x/\delta}\,. \tag{4.3.9}$$

Note that this type of solution is both *decaying as well as oscillating*. The second boundary condition is obtained looking at the generator, which imposes at $x = 0$ an electric field component

$$\underline{E}_z(x=0) = -\frac{V_0}{h} \tag{4.3.10}$$

i.e.

$$\underline{E}_z(x) = -\frac{V_0}{h}\,e^{-(1+j)\,x/\delta}\,. \tag{4.3.11}$$

The relevant magnetic field component is in turn

$$\begin{aligned}\underline{H}_y(x) &= \frac{1}{j\omega\mu_0}\frac{d}{dx}\underline{E}_z \\ &= \frac{1+j}{j\omega\mu_0\delta}\frac{V_0}{h}\,e^{-(1+j)\,x/\delta}\,. \end{aligned} \tag{4.3.12}$$

Consequently, the surface current density along the confining plates is

$$\underline{J}_{s,x}(x) = \frac{1+j}{j\omega\mu_0}\frac{1}{\delta}\frac{V_0}{h}\,e^{-(1+j)x/\delta}$$

hence the current in a specified section w is

$$\begin{aligned}\underline{I}_{\text{in}} &= \underline{J}_{s,x}(x=0)\,w \\ &= \frac{1+j}{j\omega\mu_0}\frac{w}{\delta}\frac{V_0}{h}\end{aligned}$$

implying that the input impedance is

$$Z_{\text{in}} = \frac{V_0}{\underline{I}_{\text{in}}} = \frac{V_0}{\frac{1+j}{j\omega\mu_0}\frac{w}{\delta}\frac{V_0}{h}} = \frac{j\omega\mu_0}{1+j}\frac{\delta h}{w} = (1+j)\,\frac{1}{2}\,\frac{\omega\mu_0\delta h}{w}\,. \tag{4.3.13}$$

Note that in spite the fact that the conductance is assumed to be frequency independent, the impedance is nevertheless frequency dependent. This fact is of particular importance at high frequencies.

4.3.3. Power and Energy Considerations

Once the EM field components have been established, we are able to calculate the relevant energies:

$$\begin{aligned}
\underline{S}_x &= -\frac{1}{2}\,\underline{E}_z\,\underline{H}_y^* \\
&= -\frac{1}{2}\left[-\frac{V_0}{h}\,e^{-(1+j)\dfrac{x}{\delta}}\right]\left[\frac{1-j}{-j\omega\mu_0}\,\frac{1}{\delta}\,\frac{V_0^*}{h}\,e^{-(1-j)\dfrac{x}{\delta}}\right] \\
&= -\frac{1}{2}\,\frac{|V_0|^2}{h^2}\,\frac{1-j}{j\omega\mu_0}\,\frac{1}{\delta}\,e^{-2\dfrac{x}{\delta}}\,, \qquad (4.3.14) \\
\underline{P} &= \iint d\vec{a}\,\vec{S}\Big|_{x=0} = -\frac{1}{2}\,\frac{|V_0|^2}{h^2}\,\frac{1-j}{j\omega\mu_0}\,\frac{1}{\delta}\,w\,h\,, \qquad (4.3.15) \\
<W_{\mathrm{E}}> &= \int dv <w_{\mathrm{E}}>= wh\int_0^\infty dx\,\frac{1}{4}\,\varepsilon_0\,|E_z|^2 \\
&= wh\,\frac{1}{4}\,\varepsilon_0\left(\frac{V_0}{h}\right)^2\,(\delta/2)\,, \qquad (4.3.16) \\
<W_{\mathrm{M}}> &= \int dv <w_{\mathrm{M}}>= wh\int_0^\infty dx\,\frac{1}{4}\,\mu_0\,|H_y|^2 \\
&= wh\,\frac{1}{4}\,\mu_0\frac{2}{(\omega\mu_0\delta)^2}\left(\frac{V_0}{h}\right)^2\,(\delta/2)\,, \qquad (4.3.17) \\
\underline{P}_D &= \int dv\,\frac{1}{2}\,\underline{\vec{J}}^*\cdot\underline{\vec{E}} = \frac{1}{2}\,wh\sigma\left(\frac{V_0}{h}\right)^2\,(\delta/2)\,. \qquad (4.3.18)
\end{aligned}$$

Some specific conclusions:

$$1. \qquad \frac{<W_{\rm E}>}{<W_{\rm M}>} = \frac{wh\frac{1}{4}\varepsilon_0 \left(\frac{V_0}{h}\right)^2 (\delta/2)}{wh\frac{1}{4}\mu_0\frac{2}{(\omega\mu_0\delta)^2}\left(\frac{V_0}{h}\right)^2 (\delta/2)} = \frac{1}{2}\,\frac{\omega^2\mu^2\cdot 2}{\omega\mu_0\sigma} = \frac{\varepsilon_0\omega}{\sigma} \ll 1\,,$$

$$2. \qquad \frac{\underline{P}_D}{2\omega <W_{\rm M}>} = \frac{\frac{1}{2}\,wh\sigma\left(\frac{V_0}{h}\right)^2 (\delta/2)}{\omega\left[wh\frac{1}{4}\,\mu_0\,\frac{2}{(\omega\mu_0\delta)^2}\left(\frac{V_0}{h}\right)^2 (\delta/2)\right]} = \frac{1}{2}\frac{\sigma(\omega^2\mu_0^2)\frac{2}{\omega\mu_0\sigma}}{\omega\mu_0} \sim 1\,.$$

The electric energy is much smaller than its magnetic counterpart, or than the power dissipated in one period, the last two being of the same order of magnitude indicating that this is basically an MQS regime with some of the power dissipated due to presence of conducting medium.

It is important to remember that for an oscillating field the EM field penetrates only to the extent of a characteristic depth δ, which may be quite small. As a practical example let us assume a 200MHz signal for copper $\sigma \sim 5\times 10^7\Omega^{-1}{\rm m}^{-1}$ so that

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu_0}} = \sqrt{\frac{2}{2\pi\,\times(200\times 10^6)\times(\,5\times 10^7)\times(4\pi\times 10^{-7})}} \sim 3\mu{\rm m}\,.$$

4.4. Transient Conduction Phenomena

The behavior of quasi-static magnetic field differs significantly from that of a d.c. excited magnetic field. One of the most obvious phenomena is the "diffusion" of the magnetic field into a metallic slab. Quite a substantial amount of time is required for such a field to *penetrate* into a metal. This fact has several implications with regard to many electromagnetic phenomena: (1) Let us assume that it is required to transfer a signal along a wire from point ⓐ to point ⓑ. Conducting the experiment in a matter-free space, one finds that the time required is of the order of the distance to be covered divided by c. If, however, a metallic piece is in the close proximity of the wire, the signal will have to penetrate the metal, and this may cause a substantial delay in the propagation time of the signal. (2) Another case arises when the finite penetration time may be of importance occurs in the grounding of a fast circuit: one is tempted to think that the charge associated with an impulse "dissolves" immediately into the ground. As we shall shortly see this is definitely not the case.

In order to examine transients associated with the penetration of a magnetic fields into metal, let us consider an impulse of current carrying a charge q injected into a metallic layer. Our goal is to investigate the field as well as energy penetrating the whole space. The basic configuration to be analyzed is illustrated in Figure 4.4.

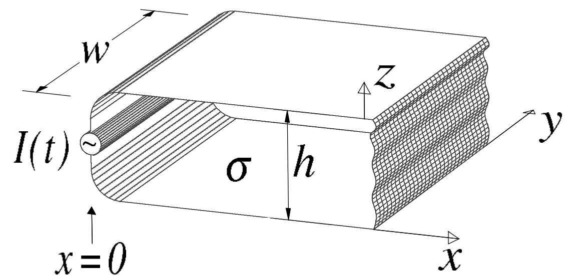

Figure 4.4: Basic setup for demonstration of a transient phenomena in a metallic medium.

4.4.1. Assumptions and Notation Conventions

1. The system is "infinite" along the $\pm y$-axis $\Rightarrow \partial_y \sim 0$.
2. Along the z-axis the system is extremely narrow $\Rightarrow \partial_z \sim 0$.
3. Variations in the x-direction.
4. The system is driven by an impulse of current i.e. $I(t) = q\delta(t)$.
5. Electric properties of the medium: $\sigma, \varepsilon_0, \mu_0$.
6. $\dfrac{1}{\sigma}\sqrt{\dfrac{\varepsilon_0}{\mu_0}} \ll h \longleftrightarrow \sigma \gg \varepsilon_0 \dfrac{c}{h}$.
7. The electromagnetic field outside the device is zero.

4.4.2. Approach to Solution

1. Gauss' charge law $\rightsquigarrow E_x \simeq 0$

2. $\vec{\nabla} \cdot \vec{B} \simeq 0 \Rightarrow B_x \simeq 0$

3. $E_y(z=0,h) \simeq 0$ and $\partial_z \sim 0 \Rightarrow E_y \simeq 0$

4. $E_y \simeq 0$ and Faraday's law $\Rightarrow H_z \simeq 0$

5. Faraday's law

$$\vec{1}_y \ : \quad -\partial_x E_z = \mu_0 \partial_t H_y \, . \tag{4.4.1}$$

6. Ampere's law

$$\vec{1}_z \ : \quad -\partial_x H_y = J_z + \varepsilon_0 \partial_t E_z \simeq \sigma \, E_z \, . \tag{4.4.2}$$

With the last two equations we shall determine the EM field in the entire space. Let us first substitute (4.4.1) into (4.4.2)

$$\frac{\partial^2}{\partial x^2} \, H_y = \sigma \mu_0 \, \frac{\partial}{\partial t} \, H_y \ . \tag{4.4.3}$$

It will be convenient to define

$$\xi = \frac{x}{h} \, , \qquad \tau = \frac{t}{\sigma \mu_0 \, h^2} \, , \tag{4.4.4}$$

a notation which enables one to rewrite the differential equation

$$\frac{\partial^2}{\partial \xi^2} H_y = \frac{\partial}{\partial \tau} H_y \tag{4.4.5}$$

along with the initial condition

$$\begin{aligned} H_y(x=0,t) &= \frac{I(t)}{w} = \frac{q}{w}\,\delta(t) = \frac{q}{w\sigma\mu_0 h^2}\,\delta(\tau) \\ &= H_0 \delta(\tau)\,, \end{aligned} \tag{4.4.6}$$

$$H_0 = \frac{q}{w\sigma\mu_0 h^2}\,.$$

At this point we may take advantage of the integral representation of the Dirac delta function i.e., $\delta(\tau) \equiv \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\Omega\, e^{+j\Omega\tau}$ hence

$$H_y(\xi = 0, \tau) = \frac{H_0}{2\pi} \int_{-\infty}^{\infty} d\Omega e^{j\Omega\tau}\,.$$

For a solution in the domain $\xi \geq 0$ one may include in the integral the spatial variation of the form

$$H_y(\xi, \tau) = \frac{H_0}{2\pi} \int_{-\infty}^{\infty} d\Omega\, e^{j\Omega\tau}\, e^{-K\xi}\,,$$

which being substituted in (4.4.5) entails that $K^2 = j\Omega$; consequently

$$H_y(\xi,\tau) = \frac{H_0}{2\pi}\int_{-\infty}^{\infty} d\Omega\, e^{j\Omega\tau}\, e^{-\sqrt{j\Omega}\xi}\,.$$

Note that in this integral we chose only the solution that ensures convergence at $\xi \to \infty$.
This integral may be evaluated analytically resulting in

$$H_y(\xi,\tau) = H_0\frac{\xi}{\tau}\,\frac{1}{\sqrt{4\pi\tau}}\,e^{-\xi^2/4\tau}\,;$$

substitute in (4.4.5) and show that indeed the last expression is a solution.

The next two figures illustrate the diffusion of the dimensionless (H_y/H_0) magnetic field into the metallic medium.

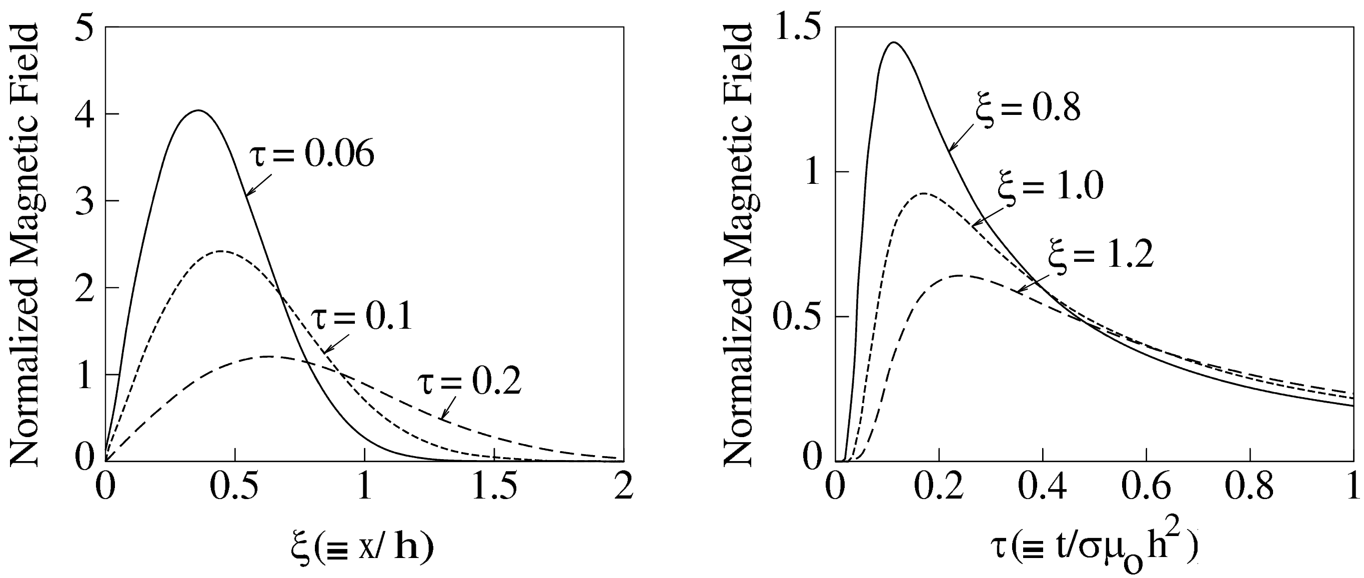

Figure 4.5: Normalized magnetic field at various times (left) and locations (right).

4.4.3. Power and Energy Considerations

Our next step is to investigate the power flow:

$$S_x = - \; E_z H_y = - \left[\frac{1}{\sigma} \frac{\partial H_y}{\partial x} \right] H_y = -\frac{1}{2\sigma} \frac{\partial}{\partial x} H_y^2 = -\frac{1}{2} \frac{1}{\sigma\, h} \frac{\partial}{\partial \xi} H_y^2 \tag{4.4.7}$$

$$P(\xi,\tau) = h\, w\, S_x = -\frac{1}{2} \frac{hw}{\sigma h} \frac{\partial}{\partial \xi} \left[\left(\frac{\xi}{\tau} \right)^2 \frac{1}{4\pi\tau} e^{-\xi^2/2\tau} \right] H_0^2$$

$$\overline{P} \equiv \frac{P(\xi,\tau)}{\frac{1}{2} \frac{w}{\sigma} H_0^2} = -\frac{2\xi^2}{4\pi\tau^2} \left(1 - \frac{\xi}{2\tau} \right) e^{-\xi^2/2\tau} \, . \tag{4.4.8}$$

This quantity ($\overline{P}$) is plotted in the next two figures.

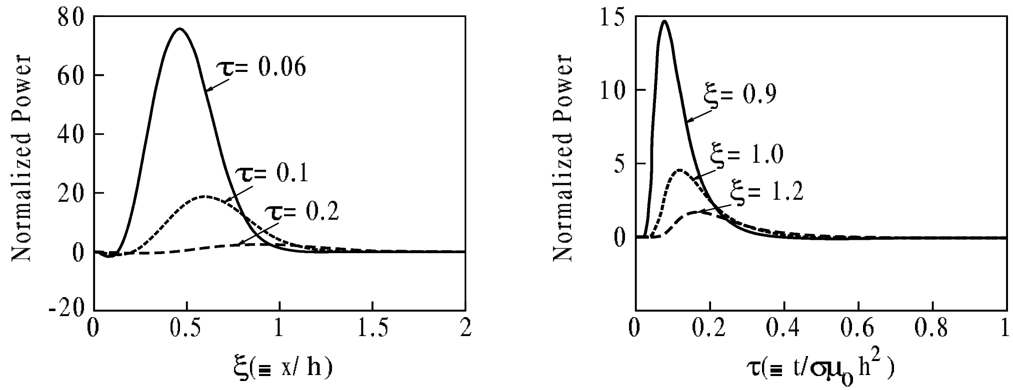

Figure 4.6: Penetration of the dimensionless power in space (left) and in time (right).

The next stage consists in investigation of the energies. The energy stored in the magnetic field is given by

$$
\begin{aligned}
W_{\mathrm{M}} &= \frac{1}{2}\,\mu_0\, wh \int_0^\infty dx\, H_y^2 = \frac{1}{2}\,\mu_0\, wh^2 H_0^2 \int_0^\infty d\xi \frac{\xi^2}{4\pi\tau^3}\, e^{-\xi^2/2\tau} \\
&= \frac{1}{2}\,\mu_0\, wh^2 H_0^2\, \frac{1}{4\pi}\, \frac{1}{\tau^{1.5}}\, \underbrace{\int_0^\infty du\, u^2 e^{-u^2/2}}_{\sqrt{\pi/2}} \qquad (4.4.9) \\
&= \frac{\mu_0 wh^2}{\sqrt{32\pi^3\tau^3}}\, H_0^2 = \frac{\mu_0 wh^2}{4\pi\sqrt{2\pi t^3}}\, (\sigma\mu_0 h^2)^{3/2}\, \left[\frac{q}{w\mu_0\sigma h^2}\right]^2 \\
W_{\mathrm{M}} &= \frac{q^2}{4\pi\varepsilon_0 w}\sqrt{\frac{1}{2\pi(\sigma Z_0 h)}\, \left(\frac{h}{ct}\right)^3}\,, \qquad (4.4.10)
\end{aligned}
$$

where as before

$$Z_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

is the so-called wave impedance of the vacuum. Note that the dissipation of the magnetic field intensity in the ground decays *algebraically*, in spite the fact that at each frequency the field decays *exponentially*!!

Similarly, for the electric field:

$$\begin{aligned}
W_{\rm E} &= \frac{1}{2}\,\varepsilon_0\,wh\int_0^\infty dx\,\frac{1}{\sigma^2}\left(\frac{\partial H_y}{\partial x}\right)^2 = \frac{1}{2}\,\frac{\varepsilon_0\,wh}{\sigma^2 h}\int_0^\infty d\xi\left(\frac{\partial H_y}{\partial \xi}\right)^2 \\
&= \frac{1}{2}\,\frac{\varepsilon_0 w}{\sigma^2}\int_0^\infty d\xi\left[\frac{\partial}{\partial\xi}\left(H_y\,\frac{\partial H_y}{\partial\xi}\right) - H_y\,\frac{\partial^2 H_y}{\partial\xi^2}\right] \\
&= -\frac{1}{2}\,\frac{\varepsilon_0 w}{\sigma^2}\left\{\underbrace{H_y\frac{\partial H_y}{\partial\xi}\Big|_0^\infty}_{=0} - \int_0^\infty d\xi\,H_y\,\frac{\partial H_y}{\partial\tau}\right\} \\
&= -\frac{1}{2}\,\frac{\varepsilon_0 w}{\sigma^2}\,\frac{d}{d\tau}\int_0^\infty d\xi H_y^2 = -\frac{\varepsilon_0}{\mu_0}\,\frac{1}{\sigma^2}\,\frac{1}{h^2}\,\frac{d}{d\tau}\,w_{\rm M}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{W_{\rm E}}{W_{\rm M}} &= -\frac{1}{(\eta_0\sigma h)^2}\,\frac{d}{d\tau}\,\ln\,w_{\rm M} = -\frac{1}{(\eta_0\sigma h)^2}\,\frac{d}{d\tau}\,\ln\left(\tau^{-3/2}\right) = \frac{3}{2}\,\frac{1}{(\eta_0\sigma h)^2}\,\frac{1}{\tau} \\
\frac{W_{\rm E}}{W_{\rm M}} &= \frac{3}{2}\,\frac{1}{(\eta_0\sigma h)^2}\,\frac{1}{\tau} = \frac{3}{2}\,\frac{1}{(\eta_0\sigma h)^2}\,\frac{1}{\dfrac{t}{\sigma\mu_0 h^2}} = \frac{3}{2}\,\frac{1}{\dfrac{\mu_0}{\varepsilon_0}\,\sigma^2 h^2}\,\frac{\sigma\mu_0 h^2}{t} \\
\frac{W_{\rm E}}{W_{\rm M}} &= \frac{3}{2}\,\frac{1}{\sigma\eta_0(ct)}\,. \qquad (4.4.11)
\end{aligned}$$

On very short time scales the electric energy stored in the system is dominant however, special attention should be paid to assumptions which may be violated in these circumstances. This may be understood in terms of the electric field linked to the charge deposited by the external source in the system. When this charge starts to "dissolve", current flows thus the magnetic energy starts to dominate.

Finally we shall calculate the power dissipated:

$$
\begin{aligned}
P_D &= wh \int_0^\infty dx\, J_z E_z = wh\sigma \int_0^\infty dx E_z^2 \\
&= \frac{2}{\varepsilon_0}\sigma \left[\frac{1}{2}\varepsilon_0 \int_0^\infty dx\, E_z^2\right] = 2\,\frac{\sigma}{\varepsilon_0}\,\frac{(-1)}{(\eta_0\sigma h)^2}\,\frac{d}{dt} W_{\rm M} \\
\frac{P_D}{W_{\rm M}} &= \frac{-2\,\dfrac{\sigma}{\varepsilon_0}}{(\eta_0\sigma h)^2}\,\frac{d}{dt}\ln W_{\rm M} \\
&= \frac{-2\sigma/\varepsilon_0}{(\eta_0\sigma h)^2}\left(-\frac{3}{2}\right)\frac{1}{\tau} = 3\,\frac{\sigma/\varepsilon_0}{\dfrac{\mu_0}{\varepsilon_0}\,\sigma^2 h^2}\,\frac{\sigma\mu_0 h^2}{t} = \frac{3}{t}
\end{aligned}
$$

hence

$$
\frac{P_D\, t}{W_{\rm M}} = 3\,. \tag{4.4.12}
$$

Consequently, the energy dissipated in the metal is three times the magnetic energy stored in the system. It is important to emphasize again that the energy which was injected in the metal "dissolves" quite slowly on a time scale of $t^{-3/2}$ - see (4.4.10).

4.4.4. Input Voltage

A clear manifestation of the finite penetration time of the electromagnetic field is revealed when measuring the voltage, as the current impulse "hits" the metallic "ground". From pure dc arguments one will be tempted to say that since this is a so-called "ground" the voltage must be zero. This is clearly not the case, since the electric field is given by

$$E_z = -\frac{1}{\sigma}\frac{\partial H_y}{\partial x} = -\frac{1}{h\sigma}\frac{\partial H_y}{\partial \xi}\,. \tag{4.4.13}$$

Based on this expression the voltage at the input may be defined as $V_{\rm in} = -hE_z(x = 0)$ and therefore

$$V_{\rm in} = -h\left(-\frac{1}{h\sigma}\frac{\partial H_y}{\partial \xi}\right)_{\xi=0}\,, \tag{4.4.14}$$

which finally implies [recall that $H_0 = q/w\sigma\mu_0 h^2$ and $\tau = t/\sigma\mu_0 h^2$] that

$$V_{\rm in} = \frac{H_0}{\sigma\tau}\left(\frac{\partial}{\partial \xi}\xi\, e^{-\xi^2/4\tau}\right)_{\xi=0} = \frac{q}{w\sigma}\frac{1}{t}\,. \tag{4.4.15}$$

This result shows that the current impulse injected into the "ground" generates a *long voltage transient* which decays *algebraically* in time. For $q \sim 10^{-4}$C, $w \sim 10^{-2}$m, $\sigma \sim 10^{7}$Ohm^{-1}m^{-1} then $V_{\rm in}$[V] $\sim$ $1/t$[nsec] implying that there is a non-zero voltage for a period that is of the order of a few nano-seconds.

4.5 Transient Phenomena in a Capacitor

In an early analysis it has been demonstrated that in steady-state the impedance of a capacitor of dimensions similar to the typical wavelength is

$$Z_c(\omega) = \frac{1}{j\omega C} \frac{1}{\frac{\tan(\omega\tau)}{\omega\tau}}, \qquad 4.5.1$$

wherein $\tau = \frac{a}{c}\sqrt{\varepsilon_r \mu_r}$ and C is the "dc" capacitance. It is therefore natural to examine now the effect associated with the term $\tan(\omega\tau)/\omega\tau$ as expressed in the frequency domain, but "translated" into the time-domain. For this purpose consider the circuit

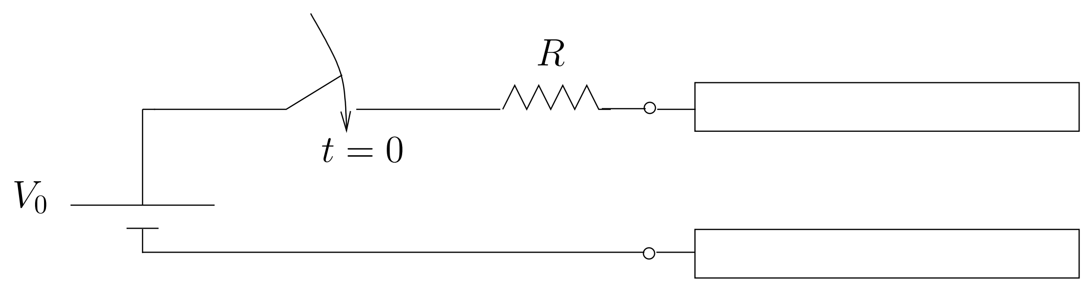

Figure 4.7: Charging of a lumped capacitor.

If we assume $\omega\tau \ll 1$ (static approach) it is well known that

$$V_c(t) = V_0 \left[1 - e^{-t/RC}\right] h(t) \qquad 4.5.2$$

and

$$I = C\,\frac{dV_c}{dt} = V_0\,C\,\frac{1}{RC}\,e^{-t/RC} = \frac{V_0}{R}\,e^{-t/RC}\,. \qquad 4.5.2$$

Let us now account for the effect of τ. It is convenient to solve the problem in the frequency domain. The voltage source may be represented by

$$\overline{V}_s(\omega) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dt\,V_0\,h(t)\,e^{-j\omega t} \qquad 4.5.3$$

therefore the current in the circuit is

$$\overline{I}(\omega) = \frac{\overline{V}_s(\omega)}{R + Z_c(\omega)} = \frac{\overline{V}_s(\omega)}{R + \dfrac{1}{j\omega C}\,\dfrac{1}{\dfrac{\tan(\omega\tau)}{\omega\tau}}}\,. \qquad 4.5.4$$

The current is given by

$$I(t) = \int\limits_{-\infty}^{\infty} d\omega\overline{I}(\omega)\,e^{j\omega t}$$

$$= \int\limits_{-\infty}^{\infty} d\omega\,e^{j\omega t}\,\frac{\overline{V}_s(\omega)}{R + \dfrac{1}{j\omega C}\,\dfrac{\omega\tau}{\tan(\omega\tau)}}\,. \qquad (4.5.5)$$

Using the explicit expression for $\overline{V}_s(\omega)$ [Eq. (4.5.3)]

$$I(t) = \frac{1}{R} \int\limits_{-\infty}^{\infty} d\omega\, e^{j\omega t} \frac{1}{1 - j\, \dfrac{\tau}{CR}\, \text{ctan}(\omega\tau)} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} dt' V_s(t')\, e^{-j\omega t'}$$

$$= \frac{1}{R} \int\limits_{-\infty}^{\infty} dt'\, V_s(t')\, \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\omega\, \frac{e^{j\omega(t-t')}}{1 - j\, \dfrac{\tau}{RC}\, \text{ctan}(\omega\tau)}\,. \tag{4.5.6}$$

Let us now define the function

$$G(t|t') = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\omega\, \frac{e^{j\omega(t-t')}}{1 - j\, \dfrac{\tau}{RC}\, \text{ctan}(\omega\tau)} \tag{4.5.7}$$

with it

$$I(t) = \frac{1}{R} \int\limits_{-\infty}^{\infty} dt'\, G(t|t')\, V_s(t')\,. \tag{4.5.8}$$

The next step is to simplify $G(t|t')$:

$$G(t|t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{e^{j\omega(t-t')}}{1 - j\,\dfrac{\tau}{RC}\,\dfrac{\frac{1}{2}\left(e^{j\omega\tau} + e^{-j\omega\tau}\right)}{\frac{1}{2}\left(e^{j\omega\tau} - e^{-j\omega\tau}\right)}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{e^{j\omega(t-t')}\left(1 - e^{-2j\omega\tau}\right)}{1 - e^{-2j\omega\tau} + \dfrac{\tau}{RC}\left(1 + e^{-2\omega\tau}\right)}$$

$$= \frac{1}{1 + \dfrac{\tau}{RC}} \frac{1}{2\pi} \int d\omega \, \frac{e^{j\omega(t-t')}\left(1 - e^{-2j\omega\tau}\right)}{1 - \left(\dfrac{1-\tau/RC}{1+\tau/RC}\right) e^{-2j\omega\tau}} \,. \tag{4.5.9}$$

In terms of geometric series

$$\frac{1}{1-\xi} = \sum_{\nu=0}^{\infty} \xi^{\nu} \tag{4.5.10}$$

(provided $|\xi| < 1$) entails

$$G(t|t') = \frac{1}{1+\dfrac{\tau}{RC}} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{j\omega(t-t')} \left(1 - e^{-2j\omega\tau}\right) \sum_{\nu=0}^{\infty} \left[\left(\frac{1-\tau/RC}{1+\tau/RC} \right) e^{-2j\omega\tau} \right]^{\nu}$$

$$= \frac{1}{1+\dfrac{\tau}{RC}} \sum_{\nu=0}^{\infty} \left[\frac{1-\tau/RC}{1+\tau/RC} \right]^{\nu} \left\{ \delta(t-t'-2\nu\tau) - \delta[t-t'-2(\nu+1)\tau \right\} . \qquad (4.5.11)$$

Substituting in Eq. (4.5.8) we get

$$I(t) = R \frac{1}{1+\dfrac{\tau}{RC}} \sum_{\nu=0}^{\infty} \left[\frac{1-\tau/RC}{1+\tau/RC} \right]^{\nu} \left\{ V_s(t-2\nu\tau) - V_s\Big[t-2(\nu+1)\tau\Big] \right\} \qquad (4.5.12)$$

$$V_c(t) = V_0 - RI\,(t) = V_s(t) - \frac{1}{1+\dfrac{\tau}{RC}} \sum_{\nu=0}^{\infty} \left(\frac{1-\tau/RC}{1+\tau/RC} \right)^{\nu} \left\{ V_s(t-2\nu\tau) - V_s\Big[t-2(\nu+1)\tau\Big] \right\} . \qquad (4.5.13)$$

These two quantities are illustrated for a source corresponding to a step function i.e. $V_s(t) = V_0\, h(t)$.

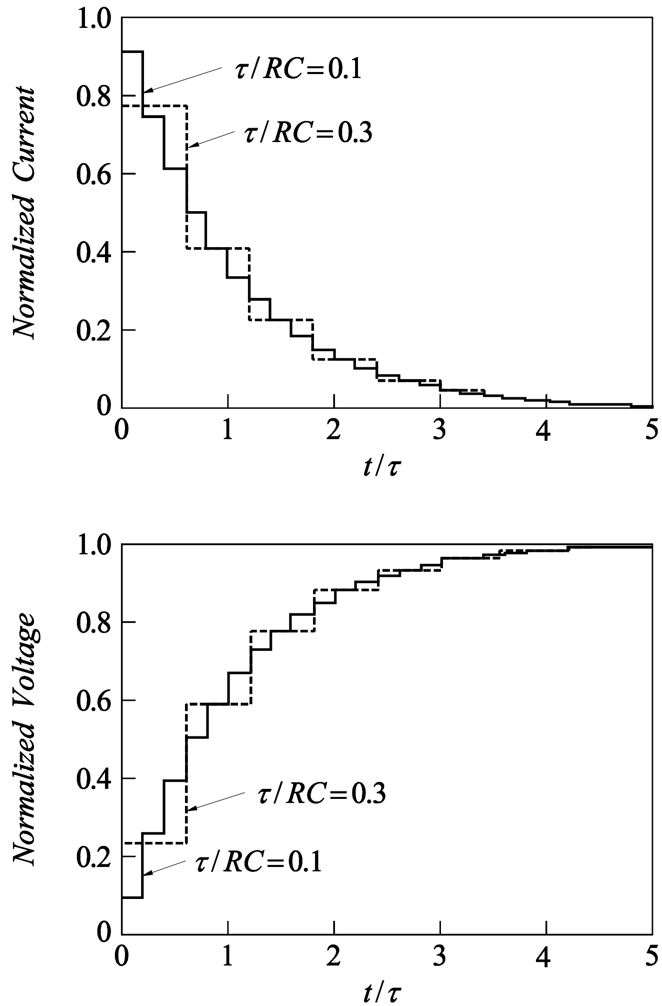

Figure 4.8: Time variation of the current and voltage at the input of the lumped capacitor.

5. COUPLING PHENOMENA

One of the main differences between a particle and a field is that the former is highly localized in space whereas the latter may cover a large volume and, in fact, in the case of radiating field it may occupy an infinite volume. In the case of an antenna, for example, we use this fact in order to transmit information. However, there are cases in which the field property to permeate large distances may alter the performance of the system. For example, if we wish to dispatch a signal along a wire from point A to point B, and the entire information content were to flow *within the wire* there would have been no problem. However, since as indicated in the past, the wire actually serves only to guide the current, whereas the power flows outside the wire (see discussion on Poynting's theorem), this signal may reach points that are not the designated ones. This problem becomes quite severe as the frequency range is increased. In order to gain insight into some problems inherent to signal transfer we consider in the present chapter three coupling phenomena: electro-quasi-static coupling, conduction coupling and finally magneto-quasi-static coupling.

5.1. Electro-Quasi-Static Coupling

In many fast circuits the current is low, primarily, in order to keep the dissipated power to a minimum. Consequently, the typical magnetic energy stored in the structure is significantly lower in comparison to the electric energy. In the present section we examine the coupling process associated with such systems.

5.1.1. General Formulation

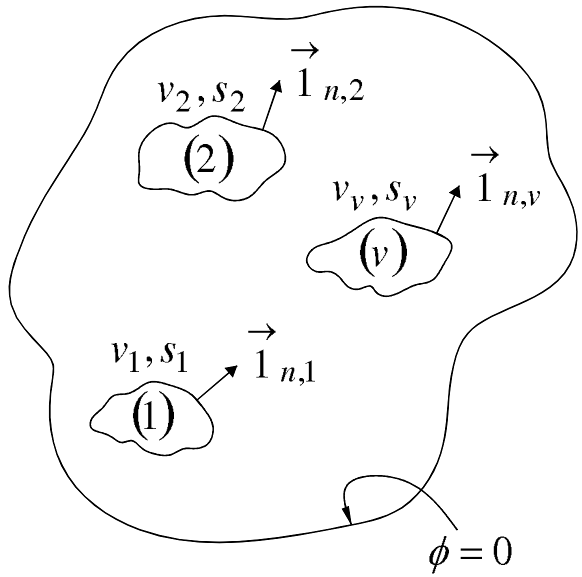

Figure 5.1: Schematics of a system characterized by electro-quasi-static coupling.

Let us commence by examining the arrangement reproduced in Figure 5.1; it consists of a set of metallic surfaces s_i (electrodes) on each one of which the voltage v_i is imposed. The volume in which the electrodes reside is confined by means of a grounded conducting surface s_0. Our first goal is to determine the electric field permeating in the enclosed space; for this purpose we have to solve Laplace's equation

$$\nabla^2 \phi = 0 \tag{5.1.1}$$

subject to

$$\phi = v_j \qquad \text{on} \qquad s_j\ , \tag{5.1.2}$$

where $j = 1, 2 \ldots n$.

By virtue of the *linearity* of Maxwell's equations we are able to subdivide the solution into a series of solutions, ϕ_j, each satisfying the equations

$$\nabla^2 \phi_j = 0$$

$$\phi_j = \begin{cases} v_j & \text{on} \quad s_j \\ 0 & \text{on} \quad s_{i \neq j} \,. \end{cases} \tag{5.1.3}$$

By superposition, these solutions yield the complete solution, i.e.

$$\phi = \sum_j \phi_j \,. \tag{5.1.4}$$

Let us assume (for the moment) that ϕ_j is known and we wish to calculate the charge upon each surface. Since within the volume of each electrode considered the electric field vanishes, the surface charge density on the i'th electrode is

$$\vec{1}_{n,i} \cdot \epsilon_0 \vec{E}\,; \tag{5.1.5}$$

the unit vector $\vec{1}_{n,i}$ being the normal to the surface pointing inside the volume where the field exists. Its total charge being therefore given by the relation

$$q_i = \oiint d\vec{a} \cdot \varepsilon_0 \, \vec{E} = -\varepsilon_0 \oiint_{s_i} d\vec{a} \cdot \nabla \Phi \tag{5.1.6}$$

and based upon (5.1.4) we obtain:

$$\begin{aligned} q_i &= -\ \varepsilon_0 \oiint_{s_i} d\vec{a} \cdot \nabla \sum_{j=1}^{n} \phi_j \\ &= -\ \varepsilon_0 \sum_{j=1}^{n} \oiint_{s_i} d\vec{a} \cdot \nabla \phi_j \,. \end{aligned} \tag{5.1.7}$$

We are now able to define the *capacitance matrix*

$$q_i = \sum_{j=1}^{n} C_{ij} v_j \,, \tag{5.1.8}$$

where

$$\boxed{C_{ij} \equiv -\frac{\varepsilon_0 \oiint_{s_i} d\vec{a} \cdot \nabla \phi_j}{v_j}} \tag{5.1.9}$$

The result in (5.1.8) indicates that the charge at the i^{th} port depends on the voltage at the j^{th} port. In other words, the off-diagonal terms of the capacitance matrix represent the coupling coefficients of the system.

With the capacitance matrix defined in (5.1.9) it is possible to evaluate the total energy in the system:

$$\begin{aligned}
W_E &= \int d^3x \frac{1}{2}\varepsilon_0 \vec{E}\cdot\vec{E} = \frac{1}{2}\varepsilon_0 \int dv \left[(-\nabla\phi)\cdot(-\nabla\phi)\right] \\
&= \frac{1}{2}\varepsilon_0 \int dv \left[\nabla(\phi\nabla\phi) - \phi \underbrace{\nabla^2\phi}_{\equiv 0}\right] = -\frac{1}{2}\sum_i \varepsilon_0 v_i \oiint_{s_i} d\vec{a}\cdot\nabla\phi \\
&= -\frac{1}{2}\sum_i \varepsilon_0 v_i \oiint_{s_i} d\vec{a}\cdot\sum_j \nabla\phi_j = -\frac{1}{2}\sum_{i,j} v_i \varepsilon_0 \oiint_{s_i} d\vec{a}\cdot\nabla\phi_j \\
&= \frac{1}{2}\sum_{i,j} v_i \underbrace{\frac{-\varepsilon_0 \oiint_{s_i} d\vec{a}\cdot\nabla\phi_j}{v_j}}_{\equiv C_{ij}} v_j = \frac{1}{2}\sum_{i,j} C_{ij}\, v_i v_j
\end{aligned}$$

$$\boxed{W_E = \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\, v_i v_j} \tag{5.1.10}$$

Comment #1. Based on this last result we shall demonstrate that the capacitance matrix is symmetric. Intuitively, this seems reasonable since it is well known from algebra that a quadratic form which involves a symmetric matrix is always positive and also, that the electric energy is defined positive. So, let us assume that on one of the ports (k) the voltage is increased by dv_k without changing the charge. Such a conceptual increment entails a change in the stored energy

$$dW_E = \frac{\partial W_E}{\partial v_k}\, dv_k = Q_k dv_k = \left[\sum_{j=1}^{n} C_{kj} v_j\right] dv_k\,. \tag{5.1.11}$$

According to (5.1.10) the charge in the stored energy is

$$dW_E = \left[\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\,\frac{\partial}{\partial v_k}\,(v_i v_j)\right] dv_k\,, \tag{5.1.12}$$

which may be simplified bearing in mind that

$$\frac{\partial v_i}{\partial v_k} = \delta_{ik}\,,$$

wherein δ_{ij} is the Kronicker delta function hence

$$dW_E = \left[\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\left(\delta_{ik}v_j + \delta_{jk}v_i\right)\right] dv_k$$

$$= \left[\frac{1}{2}\sum_{j=1}^{n} C_{jk}v_j + \frac{1}{2}\sum_{i=1}^{n} C_{ki}v_i\right] dv_k$$

$$= \left[\frac{1}{2}\sum_{j=1}^{n} C_{jk}v_j + \frac{1}{2}\sum_{j=1}^{n} C_{kj}v_j\right] dv_k$$

$$= \left[\sum_{j=1}^{n} \frac{1}{2}\left(C_{jk} + C_{kj}\right)\right] dv_k\,. \tag{5.1.13}$$

Comparing the right term of Eq. (5.1.11) with the last term in Eq. (5.1.13) leads to

$$\frac{1}{2}\left(C_{jk} + C_{kj}\right) = C_{kj}$$

and finally

$$\boxed{C_{j,i} = C_{i,j}} \tag{5.1.14}$$

Comment #2. The diagonal terms of the capacitance are also called the self-capacitance. Clearly, it is defined as the ratio of the charge and voltage at a given port when all the other ports are grounded.

Applying a positive voltage implies presence of positive charge on the specific electrode and negative charge on all the grounded electrodes. Consequently, the self-capacitance is always *positive* whereas the mutual-capacitance ($C_{i,\,j \neq i}$) is always *negative*.

5.1.2. A Simple Configuration

In order to obtain an intuitive feeling about the physical processes involved we consider a set of plates as illustrated in Figure 5.2 and calculate the capacitance matrix by ignoring edge effects.

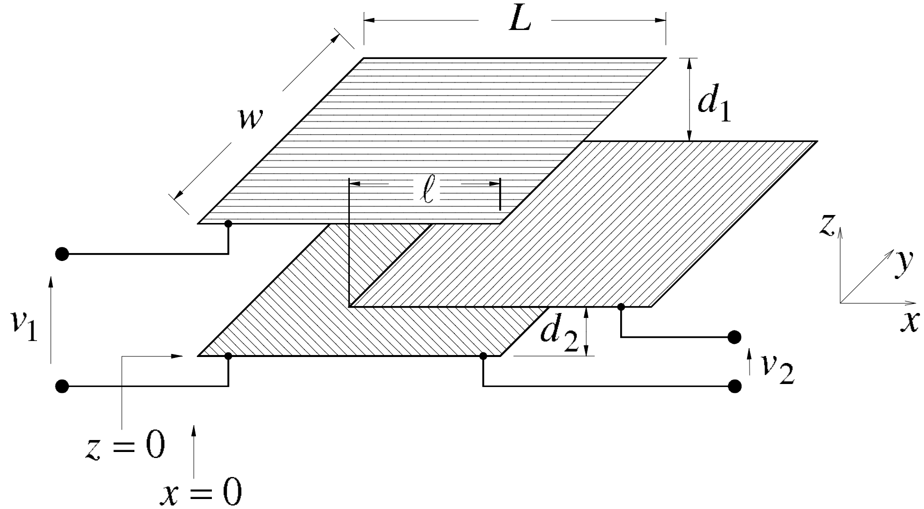

Figure 5.2: Simple electro-static coupling example.

The basic assumptions resorted to for the solution of our problem to a first order approximation are:

1. The system is "infinite" along the y-axis i.e. $\partial_y \sim 0$.

2. Time variations are much lower than the time it takes a plane electromagnetic wave to traverse the structure; this may be formulated as $\left|\frac{\partial^2\psi}{\partial t^2}\right| \ll \frac{c^2}{L^2}|\psi|$, defining $\tau = \frac{L}{c} \sim \frac{3\times 10^{-2}}{3\times 10^{8}} = 10^{-10}\,\text{sec} \sim$ 0.1 nsec, this assumption is valid as long as the highest frequency involved is much smaller than

$$\frac{1}{0.1 \times 10^{-9}} \simeq 10 \text{ GHz}.$$

3. Fringe-effects are neglected.

The first order field solution is taken as

$$E_z = \begin{cases} -\dfrac{v_1}{d_1 + d_2} & \begin{cases} 0 < x < L - l \\ 0 < z < d_1 + d_2 \end{cases} \\ -\dfrac{v_1 - v_2}{d_1} & \begin{cases} L - l < x < L \\ d_2 < z < d_1 + d_2 \end{cases} \\ -\dfrac{v_2}{d_2} & \begin{cases} L - l < x < L \\ 0 < z < d_2 \,. \end{cases} \end{cases} \tag{5.1.15}$$

The energy stored in the electric field

$$\begin{aligned} W_E &= \frac{1}{2}\,\varepsilon_0 \left\{ \frac{v_1^2}{(d_1 + d_2)^2}\, w(d_1 + d_2)(L - l) + \left(\frac{v_1 - v_2}{d_1}\right)^2 w\, d_1 l + \left(\frac{v_2}{d_2}\right)^2 w d_2\, l \right\} \\ &= \frac{1}{2}\, v_1^2 \left[\varepsilon_0 w \left(\frac{L - l}{d_1 + d_2} + \frac{l}{d_1}\right)\right] - \frac{1}{2}\, v_1 v_2 \left[2\varepsilon_0 w \frac{l}{d_1}\right] + \frac{1}{2} v_2^2 \left[\varepsilon_0 w \left(\frac{l}{d_1} + \frac{l}{d_2}\right)\right] . \end{aligned} \tag{5.1.16}$$

Bearing in mind that $C_{12} = C_{21}$ we conclude that

$$\begin{aligned} C_{11} &= \varepsilon_0 \, w \left(\frac{L-l}{d_1+d_2} + \frac{l}{d_1} \right) \\ C_{12} &= C_{21} = -\varepsilon_0 \, w \, \frac{l}{d_1} \\ C_{22} &= \varepsilon_0 \, w \left(\frac{l}{d_1} + \frac{l}{d_2} \right) . \end{aligned} \tag{5.1.17}$$

Note that the off-diagonal term of the capacitance matrix is *linear* in l; it will be shown that it is this term which accounts for the cross-coupling. Let us stress, once more, the fact that a very simple configuration along with a much simplified solution were assumed, in order to obtain source insight into capacitive cross-coupling.

5.1.3. Coupling and Fringe Effect

In the first section we have developed the capacitance matrix by ignoring fringe effect. In the present section we shall revisit this problem by taking into consideration this effect.

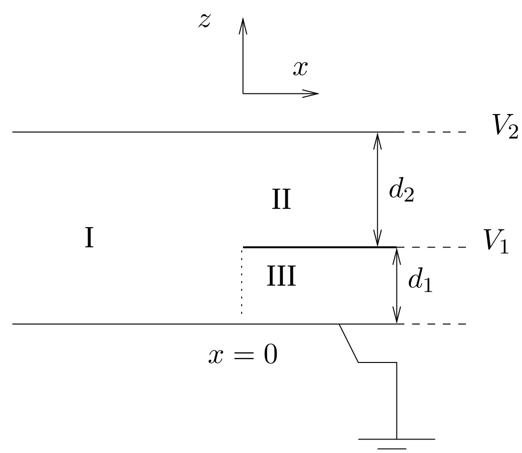

Figure 5.3: A system illustrating the electrostatic coupling.

Consider a three electrodes system. The lower plate is grounded, the inner electrode is connected to a voltage source V_1 being placed at an elevation d_1 above the ground plate, extending from $0 < z < \infty$; finally, the third plate is placed at an elevation $d_1 + d_2$ being kept at an imposed voltage V_2. Formally the solution of the Laplace equation reads

$$\Phi(x,z) = \begin{cases} V_1 \dfrac{z}{d_1} + \displaystyle\sum_{n=1}^{\infty} A_n \sin\left(\dfrac{\pi n z}{d_1}\right) e^{-\dfrac{\pi n x}{d_1}} & \begin{cases} x > 0 \\ 0 < z < d_1 \end{cases} \\ V_1 + (V_2 - V_1)\dfrac{z - d_1}{d_2} + \displaystyle\sum_{n=1}^{\infty} B_n \sin\left[\dfrac{\pi n}{d_2}(z - d_1)\right] e^{-\dfrac{\pi n}{d_2}x} & \begin{cases} x > 0 \\ d_1 < z < d_1 + d_2 \end{cases} \\ V_2 \dfrac{z}{d_1 + d_2} + \displaystyle\sum_{n=1}^{\infty} C_n \sin\left[\dfrac{\pi n z}{d_1 + d_2}\right] e^{+\dfrac{\pi n}{d_1 + d_2}x} & \begin{cases} x < 0 \\ 0 \le z \le d_1 + d_2 \end{cases} \end{cases} \quad (5.1.18)$$

Within the context of the previous formulation we take first $V_2 = 0$, $V_1 \neq 0$ and determine $\Phi^{(1)}(x,z)$:

$$\Phi^{(1)}(x,z) = \begin{cases} V_1 \dfrac{z}{d_1} + \displaystyle\sum_{n=1}^{\infty} A_n^{(1)} \sin\left(\dfrac{\pi n z}{d_1}\right) e^{-\dfrac{\pi n x}{d_1}} & \begin{cases} x > 0 \\ 0 < z < d_1 \end{cases} \\ V_1 - V_1\dfrac{z - d_1}{d_2} + \displaystyle\sum_{n=1}^{\infty} B_n^{(1)} \sin\left[\dfrac{\pi n}{d_2}(z - d_1)\right] e^{-\dfrac{\pi n}{d_2}x} & \begin{cases} x > 0 \\ d_1 < z < d_1 + d_2 \end{cases} \\ \displaystyle\sum_{n=1}^{\infty} C_n^{(1)} \sin\left[\dfrac{\pi n z}{d_1 + d_2}\right] e^{+\dfrac{\pi n}{d_1 + d_2}x} & \begin{cases} x < 0 \\ 0 \le z \le d_1 + d_2 \end{cases} \end{cases} \quad (5.1.19)$$

further, we establish $\Phi^{(2)}(x,z)$ by taking $V_1 = 0$ and $V_2 \neq 0$:

$$\Phi^{(2)}(x,z) = \begin{cases} \sum_{n=1}^{\infty} A_n^{(2)} \sin\left(\frac{\pi n z}{d_1}\right) e^{-\frac{\pi n x}{d_1}} & \begin{cases} x > 0 \\ 0 < z < d_1 \end{cases} \\ V_2 \frac{z - d_1}{d_2} + \sum_{n=1}^{\infty} B_n^{(2)} \sin\left[\frac{\pi n}{d_2}(z - d_1)\right] e^{-\frac{\pi n}{d_2}x} & \begin{cases} x > 0 \\ d_1 < z < d_1 + d_2 \end{cases} \\ V_2 \frac{z}{d_1 + d_2} + \sum_{n=1}^{\infty} C_n^{(2)} \sin\left[\frac{\pi n z}{d_1 + d_2}\right] e^{+\frac{\pi n}{d_1 + d_2}x} & \begin{cases} x < 0 \\ 0 \leq z \leq d_1 + d_2 \end{cases} \end{cases} \tag{5.1.20}$$

the overall electrostatic potential being the superposition of the two solutions, i.e. $\Phi = \Phi^{(1)} + \Phi^{(2)}$. It should be pointed out that $C_n^{(1)}$ and $C_n^{(2)}$ are amplitudes of the electrostatic potential and net capacitances.

In order to determine the amplitudes we impose the boundary conditions. Continuity of the *potential* at $x = 0$ and $0 < z < d_1 + d_2$:

$$V_2 \frac{z}{d_1 + d_2} + \sum_{n=1}^{\infty} C_n \sin\left[\frac{\pi n z}{d_1 + d_2}\right] = \begin{cases} V_1 \frac{z}{d_1} + \sum_{n=1}^{\infty} A_n \sin \frac{\pi n z}{d_1} & 0 < z < d_1 \\ V_1 + (V_2 - V_1)\frac{z - d_1}{d_2} + \sum_{n} B_n \sin\left[\frac{\pi n}{d_2}(z - d_1)\right] & d_1 < z < d_1 + d_2\,. \end{cases} \tag{5.1.21}$$

Resorting to the orthogonality of the trigonometric function $\boxed{\sin\frac{\pi n z}{d_1+d_2}}$.

$$C_n = \frac{2}{d_1+d_2} \quad \left\{\int_0^{d_1} dz \left[V_1\frac{z}{h} - V_2\frac{z}{d_1+d_2}\right]\sin\left(\frac{\pi n z}{d_1+d_2}\right)\right.$$
$$+ \sum_{m=1}^{\infty} A_m \int_0^{d_1} dz \sin\left(\frac{\pi m z}{d_1}\right)\sin\left(\frac{\pi n z}{d_1+d_2}\right)$$
$$+ \int_{d_1}^{d_1+d_2} dz \left[V_1 + (V_2 - V_1)\frac{z-d_1}{d_2} - V_2\frac{z}{d_1+d_2}\right]\sin\left(\frac{\pi n z}{d_1+d_2}\right) \tag{5.1.22}$$
$$\left.+ \sum_{m=1}^{\infty} B_m \int_{d_1}^{d_2+d_1} dz \sin\left[\frac{\pi m}{d_2}(z-d_1)\right]\sin\left(\frac{\pi n z}{d_1+d_2}\right)\right\}.$$

Continuity of D_x in *each* aperture $0 < z < d_1$:

$$\sum_n \frac{\pi n}{d_1} A_n \sin\left(\frac{\pi n}{d_1} z\right) = -\sum_n \frac{\pi n}{d_2+d_2} C_n \sin\left(\frac{\pi n}{d_1+d_2} z\right) \tag{5.1.23}$$

and $d_1 < z < d_2 + d_1$:

$$\sum_n \frac{\pi n}{d_2} B_n \sin\left[\frac{\pi n}{d_2}(z-d_1)\right] = -\sum_n \frac{\pi n}{d_1+d_2} C_n \sin\left(\frac{\pi n}{d_1+d_2} z\right). \tag{5.1.24}$$

We resort to the orthogonality of $\boxed{\sin\left(\frac{\pi n z}{d_1}\right)}$ in the first expression and to that of $\boxed{\sin\left[\frac{\pi n(z-d_1)}{d_2}\right)}$ in the second:

$$A_n = -\frac{2}{d_1}\cdot\frac{d_1}{\pi n}\cdot\sum_{m=1}^{\infty}\frac{\pi m}{d_1+d_2}C_m\int_0^{d_1}dz\,\sin\left(\frac{\pi m z}{d_1+d_2}\right)\sin\left(\frac{\pi n z}{d_1}\right) \tag{5.1.25}$$

$$B_n = -\frac{2}{d_2}\cdot\frac{d_2}{\pi n}\cdot\sum_{m=1}^{\infty}\frac{\pi m}{d_2+d_1}C_m\int_{d_1}^{d_2+d_1}dz\,\sin\left(\frac{\pi m z}{d_1+d_2}\right)\sin\left[\frac{\pi n}{d_2}(z-d_2)\right]\,. \tag{5.1.26}$$

It is convenient to define the source term $\vec{S}$ and the matrices $\underline{\underline{M}}$ and $\underline{\underline{N}}$

$$S_n \equiv \frac{2}{d_1+d_2} \left\{ \int_0^{d_1} dz \left[V_1 \frac{z}{h} - V_2 \frac{z}{d_1+d_2} \right] \sin\left(\frac{\pi n z}{d_1+d_2} \right) \right.$$

$$\left. + \int_{d_1}^{d_1+d_2} dz \left[V_1 + (V_2 - V_1) \frac{z-d_1}{d_2} - V_2 \frac{z}{d_1+d_2} \right] \sin\left(\frac{\pi n z}{d_1+d_2} \right) \right\} \quad (5.1.27)$$

$$M_{nm} \equiv \frac{2}{d_1+d_2} \int_0^{d_1} dz \sin\left(\frac{\pi m z}{d_1} \right) \sin\left(\frac{\pi n z}{d_1+d_2} \right) \quad (5.1.28)$$

$$N_{nm} \equiv \frac{2}{d_1+d_2} \int_{d_1}^{d_1+d_2} dz \sin\left[\frac{\pi m}{d_2} (z - d_1) \right] \sin\left(\frac{\pi n z}{d_1+d_2} \right) . \quad (5.1.29)$$

With these definitions we are able to write the boundary conditions as

$$\begin{aligned} C_n &= S_n + \sum_{m=1}^{\infty} M_{nm} A_m + \sum_{m=1}^{\infty} N_{nm} B_m \simeq S_n \\ A_n &= - \sum_{m=1}^{\infty} \frac{m}{n} M_{mn} C_m \\ B_n &= - \sum_{m=1}^{\infty} \frac{m}{n} N_{mn} C_m . \end{aligned} \quad (5.1.30)$$

Substituting the last two relations $A_n = -\sum_{m=1}^{\infty} \frac{m}{n} M_{mn} C_m$ and $B_n = -\sum_{m=1}^{\infty} \frac{m}{n} N_{mn} C_m$ we obtain

$$\begin{aligned} C_n &= S_n + \sum_{m=1}^{\infty} M_{nm}(-) \sum_{n'} \frac{n'}{m} M_{n'm} C_{n'} \\ &\quad + \sum_{m=1}^{\infty} N_{nm}(-) \sum_{n'} \frac{n'}{m} N_{n'm} C_{n'} , \end{aligned} \tag{5.1.31}$$

thus defining

$$U_{nn'} \equiv \delta_{nn'} + \sum_{m=1}^{\infty} M_{nm} \frac{n'}{m} M_{n'm} + \sum_{m=1}^{\infty} N_{nm} \frac{n'}{m} N_{n'm} \tag{5.1.32}$$

we obtain

$$\boxed{\underline{\underline{U}}\, \vec{C} = \vec{S}} \tag{5.1.33}$$

and consequently

$$\boxed{\vec{C} = \underline{\underline{U}}^{-1} \vec{S}}\,. \tag{5.1.34}$$

Figure 5.4 illustrates the contours of constant potentials for a given geometry, as well as several applied voltages.

Once the amplitudes A, B and C have been established, we proceed to the calculation of the capacitance matrix, trying to estimate, in particular, the impact of the edge effect. For this purpose, let us assume that the electrodes have a total length $2\Delta_x$ i.e. $-\Delta_x < x < \Delta_x$, thus bearing in mind that the x-component of the electric field is

$$E_x = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi n}{d_1}\right) A_n \sin\left(\pi n \frac{z}{d_1}\right) e^{-\pi n \dfrac{x}{d_1}} & \begin{cases} x > 0 \\ 0 < z < d_1 \end{cases} \\ \displaystyle\sum_{n=1}^{\infty} \frac{\pi n}{d_2} B_n \sin\left[\frac{\pi n}{d_2}\left(z - d_1\right)\right] e^{-\pi n \dfrac{x}{d_2}} & \begin{cases} x > 0 \\ d_1 < z < d_1 + d_2 \end{cases} \\ \displaystyle\sum_{n=1}^{\infty} (-1) \left(\frac{\pi n}{d_1 + d_2}\right) C_n \sin\left[\frac{\pi n z}{d_1 + d_2}\right] e^{\pi n \dfrac{x}{d_1 + d_2}} & \begin{cases} x < 0 \\ 0 < z < d_1 + d_2 \end{cases} \end{cases} , \tag{5.1.35}$$

whereas its z directed counterpart reads

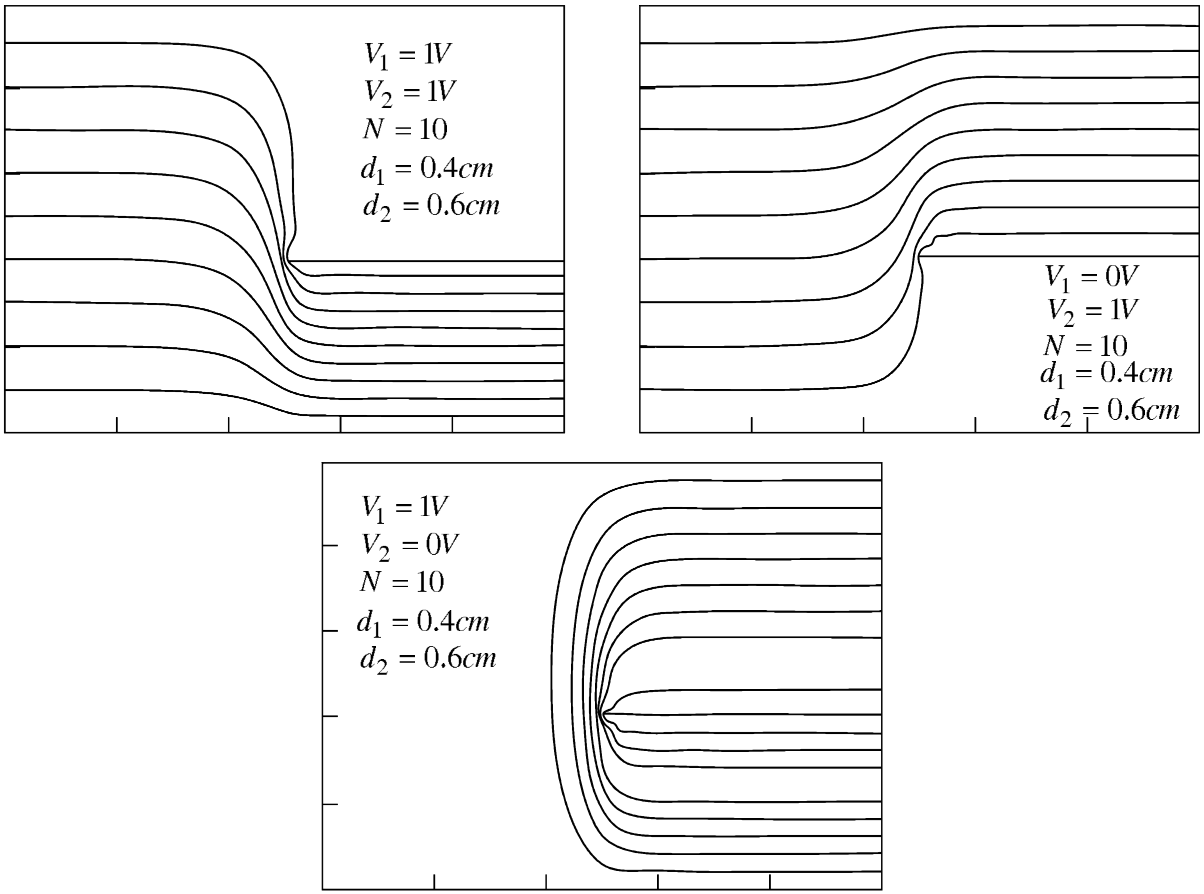

Figure 5.4: The potential contours occurring between the metallic plates.

$$E_z = -\begin{cases} \dfrac{V_1}{d} + \displaystyle\sum_{n=1}^{\infty} \left(\dfrac{\pi n}{d_1}\right) A_n \cos\left(\pi n \dfrac{z}{d_1}\right) e^{-\pi n \dfrac{x}{d_1}} & \begin{cases} x > 0 \\ 0 < z < d_1 \end{cases} \\ \dfrac{(V_2 - V_1)}{d_2} + \displaystyle\sum_{n=1}^{\infty} \left(\dfrac{\pi n}{d_2}\right) B_n \cos\left[\dfrac{\pi n}{d_2}(z - d_1)\right] e^{-\pi n \dfrac{x}{d_2}} & \begin{cases} x > 0 \\ d_1 < z < d_1 + d_2 \end{cases} \\ \dfrac{V_2}{d_1 + d_2} + \displaystyle\sum_{n=1}^{\infty} \left(\dfrac{\pi n}{d_1 + d_2}\right) C_n \cos\left[\dfrac{\pi n z}{d_1 + d_2}\right] e^{\pi n \dfrac{x}{d_1 + d_2}} & \begin{cases} x < 0 \\ 0 < z < d_1 + d_2 \end{cases} \end{cases} . \tag{5.1.36}$$

The total electric energy stored is given by

$$\begin{aligned} W &= \frac{1}{2}\,\varepsilon_0 \Delta_y \int_0^{d_1+d_2} \int_{-\Delta_x}^{\Delta_x} dx \left[E_\alpha^2(x,z) + E_z^2(x,z)\right] \\ &= \frac{1}{2}\,\varepsilon_0 \Delta_y \left\{\frac{\pi}{4} \sum_{n=1}^{\infty} n\left[A_n^2 + B_n^2 + C_n^2\right] + \left(\frac{V_1^2}{d_1} + \frac{(V_2 - V_1)^2}{d_2} + \frac{V_2^2}{d_1 + d_2}\right)\Delta_x + \frac{\pi}{4} \sum_{n=1} n\left[A_n^2 + B_n^2 + C_n^2\right]\right\} \\ &= \frac{1}{2}\,\varepsilon_0 \Delta_y \left\{V_1^2 \left(\frac{1}{d_1} + \frac{1}{d_2}\right)\Delta_x - 2V_1 V_2 \frac{\Delta_x}{d_2} + V_2^2 \left(\frac{1}{d_2} + \frac{1}{d_1 + d_2}\right)\Delta_x + \frac{\pi}{2} \sum_{n=1}^{\infty} n\left[A_n^2 + B_n^2 + C_n^2\right]\right. . \end{aligned} \tag{5.1.37}$$

Ignoring the edge effects one has therefore

$$C_{11}^{(0)} = \varepsilon_0 \Delta_x \Delta_y \left(\frac{1}{d_1} + \frac{1}{d_2} \right) \qquad C_{22}^{(0)} = \varepsilon_0 \Delta_x \Delta_y \left(\frac{1}{d_2} + \frac{1}{d_1 + d_2} \right)$$

$$C_{12}^{(0)} = C_{21}^{(0)} = -\varepsilon_0 \Delta_x \Delta_y \frac{1}{d_2} \, . \tag{5.1.38}$$

However, when the edge effects are considered,

$$C_{11} = 2W(V_1 = 1V,\, V_2 = 0) \qquad C_{22} = 2W(V_1 = 0,\, V_2 = 1)$$

$$C_{12} = \left[W(V_1,\, V_2) - \frac{1}{2} C_{11} V_1^2 - \frac{1}{2} C_{22} V_2^2 \right]_{V_1 = 1V,\, V_2 = 1V} \, . \tag{5.1.39}$$

With these capacitances we may establish the relative error associated with the calculation of the capacitance matrix, i.e.

$$\delta C_{11} \equiv \frac{C_{11} - C_{11}^{(0)}}{C_{11}^{(0)}} \times 100$$

$$\delta C_{22} \equiv \frac{C_{22} - C_{22}^{(0)}}{C_{22}^{(0)}} \times 100$$

$$\delta C_{12} \equiv \frac{C_{12} - C_{12}^{(0)}}{C_{12}^{(0)}} \times 100 \, . \tag{5.1.40}$$

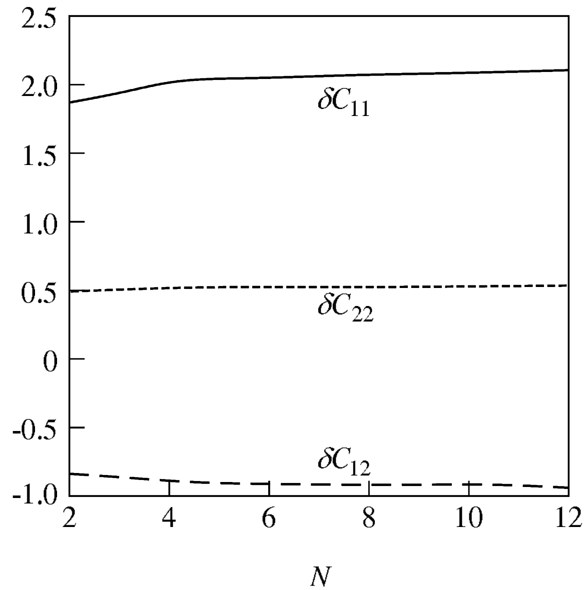

Figure 5.5: Error as a function of N.

Figure 5.5 shows this error as a function of N – the number of harmonics used. Evidently, for the chosen parameters this error is typically less than 3%; a different set of parameters leads towards a different error percentage.

Exercise: Repeat the analysis in Section (5.1.3) for the configuration illustrated in Figure 5.6.

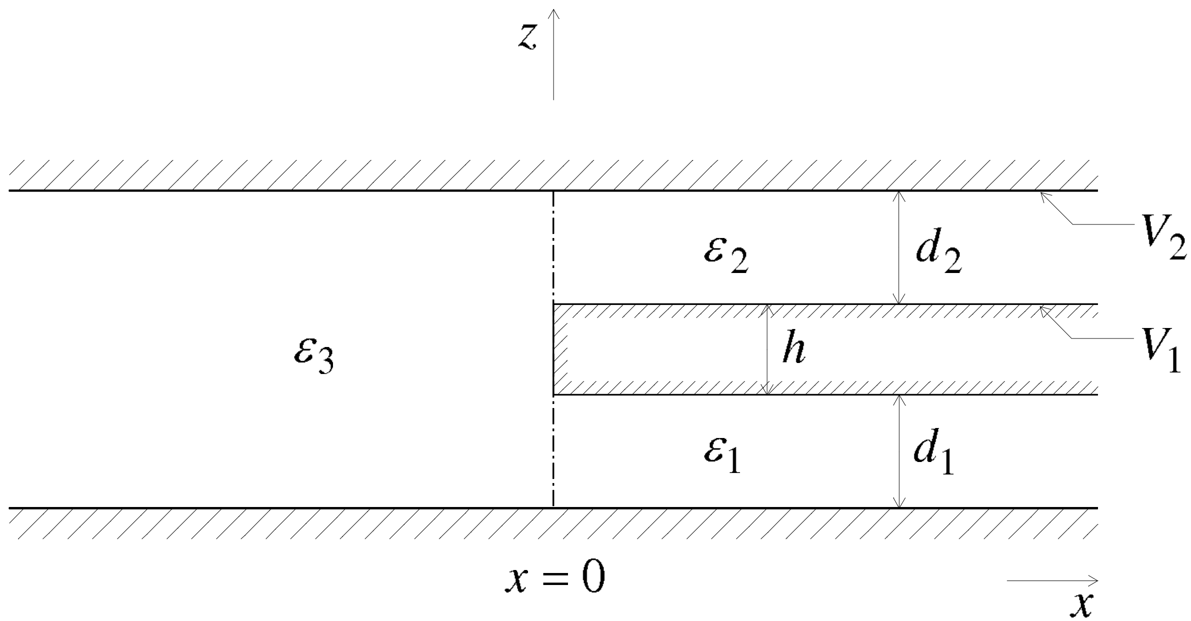

Figure 5.6: A three electrode system: the lower one is grounded on the central one at voltage V_1 is imposed and V_2 on the topelectrode. Note that in each region the dielectric coefficient is different.

5.1.4. Dynamics of Coupling Process

We now briefly examine now the implications of the off-diagonal term ($C_{12} = -\chi$) in the capacitance matrix; we refer to this term as the "coupling term" or the so-called "mutual capacitance". The other terms $C_{11} = C_{22} = C_g$.

Consider two gates ① and ② as illustrated in Figure 5.7

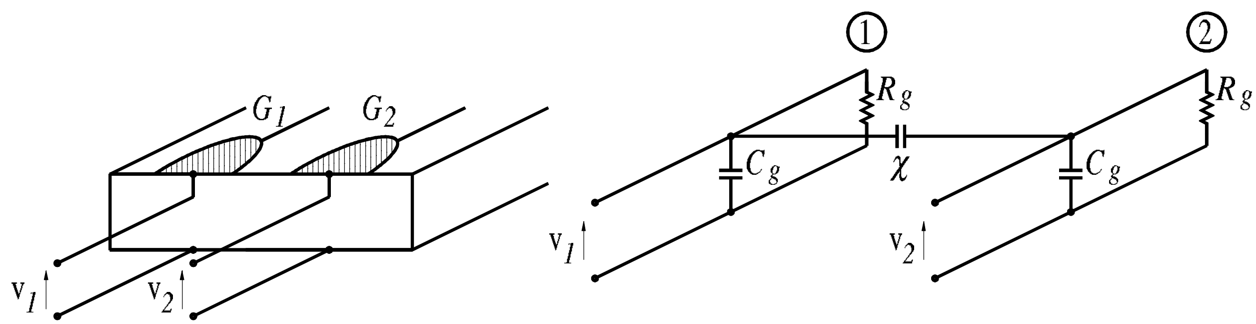

Figure 5.7: Two gates that may become coupled (left) and the equivalent circuit that describes their operation.

In principle, the two gates should not be coupled (see left frame); however due to the *proximity* of the two there exists a mutual capacitance ($-\chi$) which means that the capacitance matrix is given by

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} C_g & -\chi \\ -\chi & C_g \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} . \tag{5.1.41}$$

Each gate is characterized here by its capacitance C_g and by its resistance R_g; q_1 and q_2 being the charge on each gate. When considering the coupling between the gates, the equivalent circuit is represented by the right-hand frame in Figure 5.7. Let us now examine the equivalent circuit of the system for two different regimes of operation.

Case 1: DC condition. $I_1 = I$, current is injected into the first gate. The voltage on the resistor which represents gate ① is $V = iR_g$. On the second resistor the current and the voltage are both zero; however, on the coupling capacitor $(-\chi)$ the voltage is V; this implies that charge is being stored on that capacitor which may eventually affect the operation of the second gate.

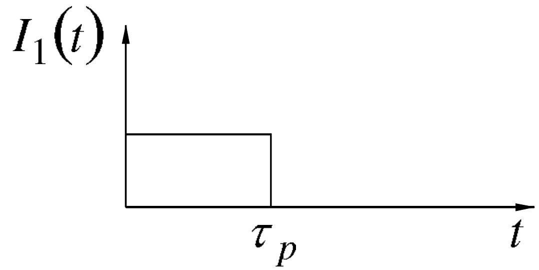

Figure 5.8: Current form to Gate 1.

Case 2: Transient condition. No current is being injected into the second gate;however, current is being injected into the first one – see Figure 5.8. Then, *ignoring* coupling, the voltage on the capacitor satisfies the relation

$$I = I_C + I_R = C_g \frac{dV}{dt} + \frac{V}{R_g} \tag{5.1.42}$$

as can be concluded from the simple equivalent circuit shown in Figure **??**.

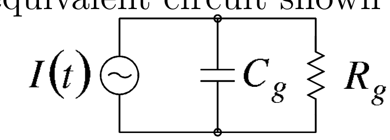

Figure 5.9: Equivalent circuit.

The voltage is schematically shown in Figure 5.10.

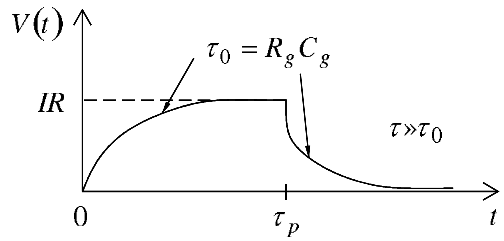

Figure 5.10: Voltage dynamics.

When taking into consideration the coupling capacitor we may expect it to drain part of the current from Gate 1; in order to examine this process we have to solve the circuit equations.

$$\begin{aligned}\text{Gate 1}: \quad I &= I_{C_1} + I_{R_1} + I_\chi = C_1 \frac{dV_1}{dt} + \frac{V_1}{R_g} - \chi \frac{d(V_2 - V_1)}{dt} \\ \text{Gate 2}: \quad I_\chi &= I_{C_2} + I_{R_2} = C_2 \frac{dV_2}{dt} + \frac{V_2}{R_g} \\ &= - \chi \frac{d(V_2 - V_1)}{dt} .\end{aligned}$$

In the above equations the indices 1 and 2 were introduced in order to emphasize the relation to the specific gate; they will be omitted in what follows since $C_1 = C_2 = C$ and $R_1 = R_2 = R$. From the first equation we obtain

$$(C_g + \chi) \frac{dV_1}{dt} - \chi \frac{dV_2}{dt} + \frac{V_1}{R} = I , \tag{5.1.43}$$

whereas the second implies the relation

$$(C_g + \chi)\,\frac{dV_2}{dt} - \chi\,\frac{dV_1}{dt} + \frac{V_2}{R_g} = 0\,. \tag{5.1.44}$$

From the last expression we see that V_2 is driven by the derivative of V_1, the driving term being proportional to χ:

$$(C_g + \chi)\,\frac{dV_2}{dt} + \frac{V_2}{R_g} = \chi\,\frac{dV_1}{dt}\;. \tag{5.1.45}$$

In order to solve equations (5.1.43-44) it is convenient to define $\overline{\chi} = \chi/C_g$, $\tau = t/R_gC_g$, $\bar{V}_{1,2} = \dfrac{V_{1,2}}{I_oR_g}$, so that these equations may be rewritten as

$$\left[(1+\overline{\chi})^2 - \overline{\chi}^2\right]\frac{d\overline{V}_1}{dt} \;+\; \overline{V}_1(1+\overline{\chi}) + \overline{\chi}\,\overline{V}_2 = \overline{I}\,(1+\overline{\chi}) \tag{5.1.46}$$

$$\left[(1+\overline{\chi})^2 - \overline{\chi}^2\right]\frac{d\overline{V}_2}{dt} \;+\; \overline{V}_2(1+\overline{\chi}) + \overline{\chi}\,\overline{V}_1 = \overline{I}\,\overline{\chi}\,. \tag{5.1.47}$$

For a solution of this set we assume the driving current to be given by

$$\overline{I}(\tau) = \frac{I(\tau)}{I_o} = \tau\,e^{-(\tau-2)}\,.$$

A set of solutions is illustrated in Figure 5.11, for three values of the coupling coefficient ($\overline{\chi}$ = 0.5, 1.0, 2.0). We conclude that capacitive coupling may generate a substantial voltage on the adjacent gate. Note that this coupling is inversely proportional to the distance between the gates as shown in (5.1.17) i.e.

$$\chi \propto \frac{1}{\text{distance between gates}}$$

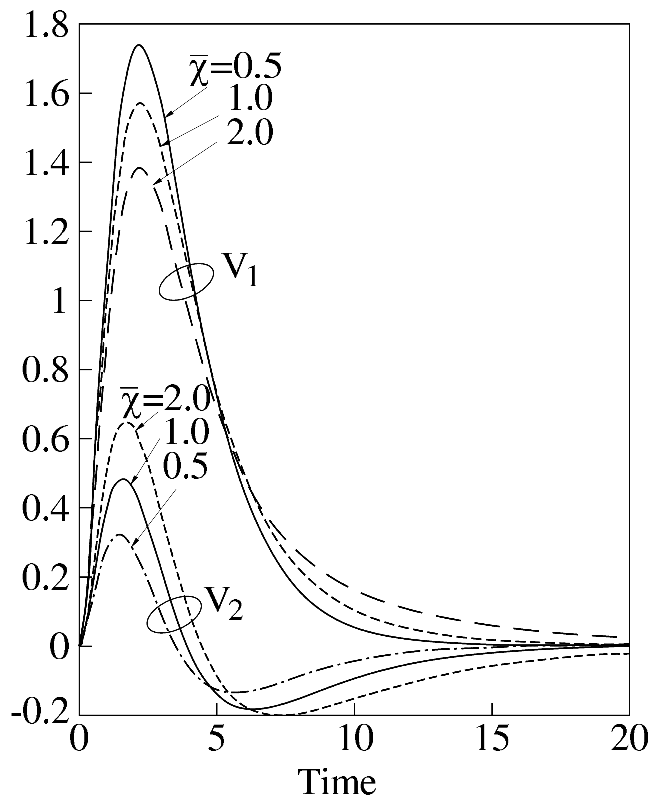

Figure 5.11: Voltage dynamics for coupled ports.

5.2. Conductive Coupling

In this section we examine the coupling process associated with the flow of electric currents due to the presence of electric fields. We shall formulate this coupling in terms of a conductance matrix similar to capacitance matrix introduced in the previous section.

5.2.1. General Formulation

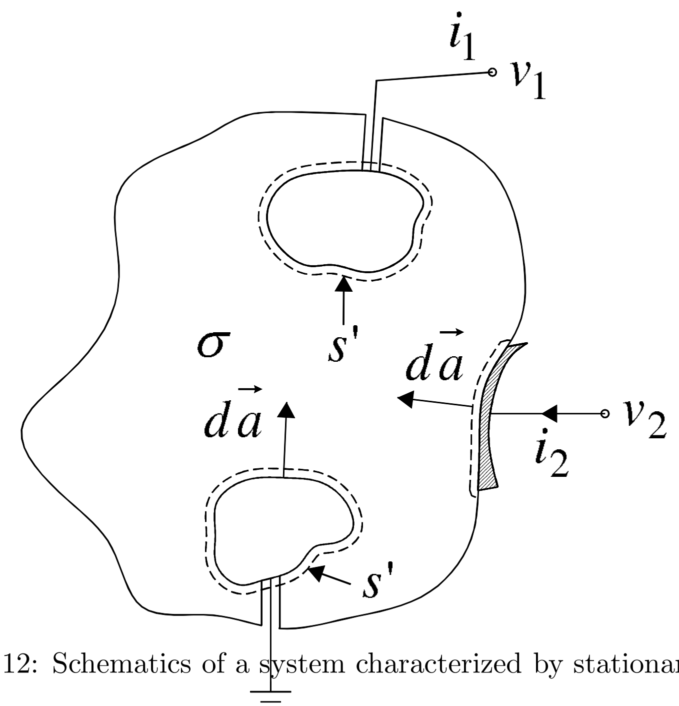

Figure 5.12: Schematics of a system characterized by stationary conductive coupling.

Assumptions:

1. $\left|\frac{\partial^2\psi}{\partial t^2}\right| \ll \frac{c^2}{L^2}\,|\psi|$.

2. σ is uniform in space.

3. s'_j surface of electrodes at which v_j is imposed.

4. s''_j denotes the remainder of the surface at which the normal current density is specified. If this boundary is insulated, then the normal current component is zero.

5. No sources reside inside the medium.

As in the electro-quasi-static case, we rely upon the fact that $\vec{\nabla}\cdot\vec{J} = 0$ and thus $\vec{\nabla}\cdot\sigma\vec{E} = 0$ implying

$$\vec{\nabla}\cdot[\sigma\nabla\phi] = 0\,, \tag{5.2.1}$$

or if σ is uniform

$$\nabla^2\phi = 0 \tag{5.2.2}$$

subject to

$$\begin{aligned} \phi &= V_i \quad &&\text{on} \quad s'_i \\ -\vec{1}_n\cdot\sigma\nabla\phi &= J_i \quad &&\text{on} \quad s''_i\,. \end{aligned} \tag{5.2.3}$$

By virtue of the superposition principle we may once more write

$$\phi = \sum_{j=1}^{n} \phi_j \,, \tag{5.2.4}$$

where each ϕ_j satisfies the requirements

$$\begin{aligned} \nabla^2 \phi_j &= 0 \\ \phi_j &= \begin{cases} v_j & \text{on } s_i' \quad i = j \\ 0 & \text{on } s_i' \quad i \neq j \,. \end{cases} \end{aligned} \tag{5.2.5}$$

Recall that $s_j^{'}$ was defined as the area over which the i'th electrode contacts the conducting material, and therefore the current at the ith electrode is given by

$$I_i = \sigma \int_{s_i'} \nabla\phi \cdot d\vec{a} \,. \tag{5.2.6}$$

In terms of (5.2.4) we may write

$$I_i = \sigma \int_{s_i} \nabla \sum_{j=1} \phi_j \, d\vec{a} = \sigma \sum_{j=1}^{n} \int_{s_i} d\vec{a} \cdot \nabla\phi_j = \sum_{j=1}^{n} G_{ij} V_j \,, \tag{5.2.7}$$

where the *conductance matrix* is given by

$$\boxed{G_{ij} = \sigma \, \frac{\int_{s_i'} d\vec{a} \cdot \nabla\phi_j}{V_j}} \tag{5.2.8}$$

Note that this relation is similar to the expression for $C_{i,j}$ in (5.1.9) with σ replacing ϵ_0. The power dissipated therefore reads

$$P = \sum I_i V_i = \sum_i^n V_i \sum_{j=1}^n G_{ij} V_j = \sum_{i,j} G_{ij} V_i V_j \,. \tag{5.2.9}$$

5.2.2. Example of Mutual Conductance

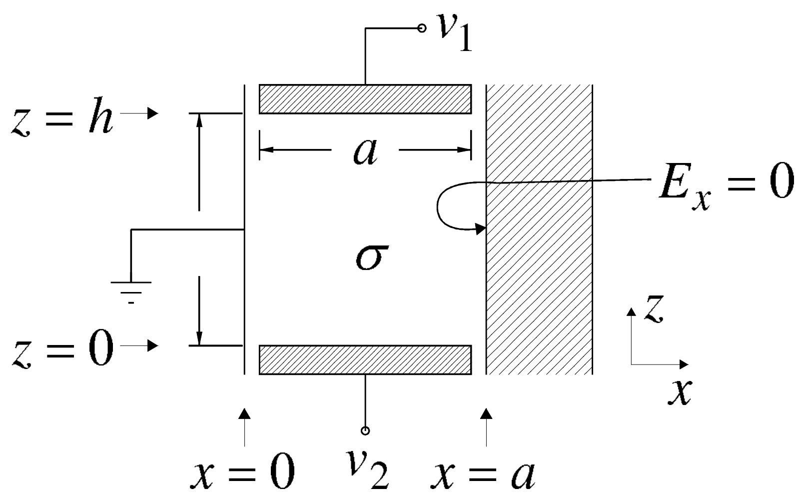

Figure 5.13: Simple model for conductive coupling.

Resorting to the superposition principle we establish the field in this system, assuming $\underline{v_2 = 0}$:

$$\nabla^2 \phi_1 = 0 \tag{5.2.10}$$

$$\begin{aligned} \phi_1 \quad (x = 0) &= 0 \\ \frac{\partial}{\partial x}\phi_1 \quad (x = a) &= 0 \\ \phi_1 \quad (x, z = h) &= V_1 \\ \phi_1 \quad (x, z = 0) &= 0 \end{aligned} \tag{5.2.11}$$

$$\phi_1 \propto \sinh(kz) \left[A\, e^{-jkx} + B\, e^{+jkx} \right] . \tag{5.2.12}$$

From the boundary conditions:

$$\begin{aligned}\phi_1(x=0) &= 0 \Rightarrow A + B = 0 \\ \left.\frac{\partial \phi_1}{\partial x}\right|_{x=a} &= 0 \Rightarrow A\, e^{-jka} - B\, e^{+jka} = 0\,. \end{aligned} \tag{5.2.13}$$

The latter is a direct result of the fact that $J_x(x=a)=0 \Rightarrow E_x(x=a)=0$. These expressions imply that

$$\cos(ka) = 0 \Rightarrow \; ka = \frac{\pi}{2} + \pi n \; ; \quad n = 0, \pm 1, \pm 2 \cdots \tag{5.2.14}$$

Thus, after having imposed three out of four boundary conditions, we have

$$\phi_1(x,z) = \sum_{n=0}^{\infty} A_n \sinh\left[\frac{z}{a}\left(\frac{\pi}{2} + \pi n\right)\right] \sin\left[\frac{x}{a}\left(\frac{\pi}{2} + \pi n\right)\right] \; . \tag{5.2.15}$$

On the electrode a voltage V_1 is imposed, thus $\phi_1(x, z=h) = V_1$, or explicitly

$$V_1 = \phi_1(x, z=h) = \sum_{n=0}^{\infty} A_n \sinh\left[\frac{h}{a}\left(\frac{\pi}{2} + \pi n\right)\right] \sin\left[\frac{\pi x}{2a}(2n+1)\right] \; . \tag{5.2.16}$$

Taking advantage of the orthogonality of the trigonometric functions one has

$$
\begin{aligned}
V_1 \int_0^a dx \sin\left[\frac{\pi x}{2a}(2n'+1)\right] &= \sum_{n=0}^{\infty} A_n \sin\left[\frac{\pi h}{2a}(2n+1)\right] \\
&\quad \times \int_0^a dx \sin\left[\frac{\pi x}{2a}(2n+1)\right] \sin\left[\frac{\pi x}{2a}(2n'+1)\right] \\
&= \sum_{n=0}^{\infty} A_n \sinh\left[\frac{\pi h}{2a}(2n+1)\right] \frac{a}{2}\, \delta_{nn'} \\
-V_1 \frac{\cos\left[\frac{\pi a}{2a}(2n'+1)\right] - 1}{\frac{\pi}{2a}(2n'+1)} &= A_{n'} \sinh\left[\frac{\pi h}{2a}(2n'+1)\right] \frac{a}{2}
\end{aligned}
$$

$$
A_n = \frac{V_1}{2n+1} \frac{4/\pi}{\sinh\left[\pi \frac{h}{2a}(2n+1)\right]} \tag{5.2.17}
$$

so that

$$
\phi_1(x,z) = V_1 \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{\sinh\left[\pi \frac{z}{2a}(2n+1)\right]}{\sinh\left[\pi \frac{h}{2a}(2n+1)\right]} \sin\left[\frac{\pi x}{2a}(2n+1)\right] . \tag{5.2.18}
$$

By a similar approach, one may calculate the potential for the case $\underline{V_1 = 0}$. The result is actually straightforward since one has only to replace v_1 with v_2, and instead of z one takes the difference, $h - z$, so that

$$\phi_2(x,z) = V_2 \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{\sinh\left[\pi \frac{(h-z)}{2a}(2n+1)\right]}{\sinh\left[\pi \frac{h}{2a}(2n+1)\right]} \sin\left[\frac{\pi x}{2a}(2n+1)\right] . \tag{5.2.19}$$

Based upon the definition of the conductance matrix in (5.2.8), we conclude that the mutual conductance is given by

$$\begin{aligned} G_{1,2} &= \frac{\sigma \Delta_y}{V_2} \int_0^a dx \left. \frac{\partial \phi_2}{\partial z} \right|_{z=h} = \frac{\sigma \Delta_y}{V_2} V_2 \frac{4}{\pi} \sum_{n=0}^{\infty} \int_0^a dx \frac{\frac{\pi}{2a}(2n+1)}{(2n+1)\sinh\left[\pi \frac{h}{2a}(2n+1)\right]} \sin\left[\frac{\pi x}{2a}(2n+1)\right] \\ &= \frac{4\sigma\Delta_y}{\pi} \frac{\pi}{2a} \sum_{n=0}^{\infty} \frac{1}{\sinh\left[\pi \frac{h}{2a}(2n+1)\right]} \times \frac{1}{\frac{\pi}{2a}(2n+1)} (-) \left[\cos \frac{\pi}{2}(2n+1) - 1\right] \end{aligned} \tag{5.2.20}$$

i.e.

$$G_{1,2} = \frac{4\sigma\Delta_y}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)\sinh\left[\pi \frac{h}{2a}(2n+1)\right]} . \tag{5.2.21}$$

The reasons we have decided to provide an explicit expression for $G_{1,2}$ are:

1. The result includes edge effects (infinite number of eigen-functions).

2. If only the first term is considered, i.e. (h/a) is sufficiently large, then

$$G_{1,2} \simeq \frac{8\sigma\Delta_y}{\pi} \frac{1}{\sinh\left(\pi \dfrac{h}{2a}\right)},$$

   which implies that the mutual conductance decays exponentially with the ratio $\dfrac{h}{a}$.

Comment #1. In analogy to the case analyzed under the heading of "mutual capacitance", the mutual conductance may couple between two or more ports.

Comment #2. The self-conductance G_{ii} is always positive.

Comment #3. The mutual-conductance $[G_{i,j\neq i}]$ is also always positive. In order to understand this statement envision the case that on one port we apply a positive voltage all the others being grounded. With no exception the current at all ports must be positive, therefore $G_{i,j\neq i} > 0$.

5.3. Magneto-Quasi-Static-Coupling

Consider a distribution of N wires, Figure 5.14 each one carrying a current I_n, located at (x_n, y_n) and having a radius R_n. The total current carried by the wires is zero

$$\sum_{n=1}^{N} I_n = 0\,, \tag{5.3.1}$$

otherwise the magnetic vector potential describing the field components in the system, diverges. In other words these wires form a series of loops, or whatever distribution of wires we may have there is at least one wire that ensures the flow of the return current. The *current density* linked to this current distribution is given by

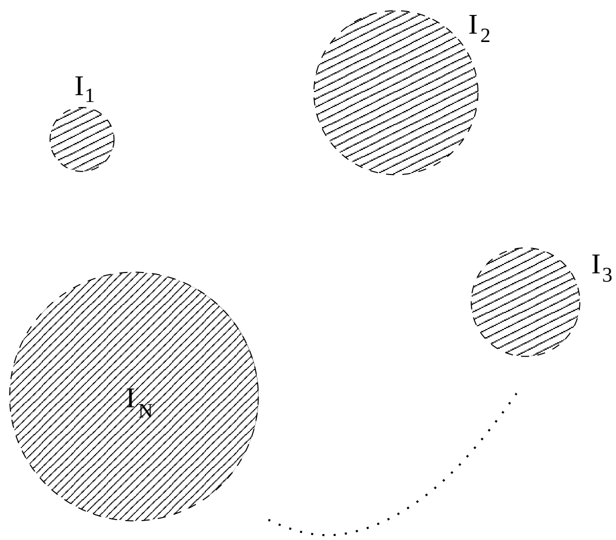

Figure 5.14: Distribution of N wires each one carrying a current I_n.

$$J_z(x,y) = \sum_{n=1}^{N} \frac{I_n}{\pi R_n^2}\, h_n(x,y) \tag{5.3.2}$$

wherein

$$h_n(x,y) = \begin{cases} 1 & \sqrt{(x-x_n)^2+(y-y_n)^2} < R_n \\ 0 & \sqrt{(x-x_n)^2+(y-y_n)^2} > R_n \,. \end{cases}$$

The magnetic vector potential generated by this current distribution is

$$A_z(x,y) = -\sum_{n=1}^{N} \frac{\mu_0 I_n}{\pi R_n^2}\, \frac{R_n^2}{4} \begin{cases} \dfrac{(x-x_n)^2+(y-y_n)^2}{R_n^2} & 0 \le \sqrt{(x-x_n)^2+(y-y_n)^2} < R_n \\ 1 + \ln\left[\dfrac{(x-x_n)^2+(y-y_n)^2}{R_n^2}\right] & R_n \le \sqrt{(x-x_n)^2+(y-y_n)^2} < \infty \end{cases} \tag{5.3.3}$$

here we resorted to the well-known result of a single wire

$$A_z(r) = -\frac{\mu_0 I}{\pi a^2} \begin{cases} \dfrac{1}{4}\, r^2 & 0 \le r < a \\ \dfrac{1}{4}\, a^2 + \dfrac{1}{2}\, a^2 \ln \dfrac{r}{a} & a \le r < \infty \,, \end{cases}$$

a being the radius of the wire after imposing $A_z(r=0)=0$ and assuming that A_z at $r=a$ is continuous.

Before we proceed to the evaluation of the inductance matrix it is important to emphasize the implications of the condition in (5.3.1). Firstly, it is evident that in case of a single "infinite" wire the magnetic vector potential diverges for $r \to \infty$. Consequently, any arbitrary distribution of wires that *do not* satisfy (5.3.1) leads to a diverging magnetic vector potential. In order to demonstrate this fact let us examine A_z as prescribed by (5.3.3) at the limit $r \to \infty$ namely, where $r \gg R_n$ and $r \gg r_n \equiv \sqrt{x_n^2 + y_n^2}$:

$$
\begin{aligned}
A_z &= -\sum_{n=1}^{N} \frac{\mu_0 I_n}{\pi R_n^2} \frac{R_n^2}{4} \left\{1 + \ln\left[\frac{x^2 - 2xx_n + x_n^2 + y^2 - 2yy_n + y_n^2}{R_n^2}\right]\right\} \\
&= -\sum_{n=1}^{N} \frac{\mu_0 I_n}{\pi R_n^2} \frac{R_n^2}{4} \left\{1 + \ln\left[\frac{r^2 - 2rr_n \cos(\phi - \phi_n) + \cancel{r_n^2}}{R_n^2}\right]\right\} \\
&\simeq -\sum_{n=1}^{N} \frac{\mu_0 I_n}{\pi R_n^2} \frac{R_n^2}{4} \left\{1 + \ln\left(\frac{r^2}{R_n^2}\right)\right\} = -\frac{\mu_0}{4\pi} \sum_{n=1}^{N} I_n \left\{1 + \ln\frac{r^2}{R_n^2}\right\} .
\end{aligned}
$$

Clearly from the last line we conclude that if $\sum_{n=1}^{N} I_n \neq 0$ the solution diverges at $r \to \infty$ and vice versa, the solution converges if (5.3.1) is satisfied.

With the magnetic vector potential established, we may proceed to the evaluation of the magnetic energy stored in an element Δ_z

$$W_M = \Delta_z \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \frac{\mu_0}{2} \left[H_x^2 + H_y^2 \right] . \tag{5.3.4}$$

Since the magnetic field components are given by

$$\begin{aligned} H_x &= \frac{1}{\mu_0} \frac{\partial A_z}{\partial y} \\ H_y &= \frac{-1}{\mu_0} \frac{\partial A_z}{\partial x} , \end{aligned} \tag{5.3.5}$$

the total magnetic energy may be evaluated by substituting in (5.3.4)

$$\begin{aligned} W_M &= \frac{\Delta_z}{2\mu_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[\left(\frac{\partial A_z}{\partial y} \right)^2 + \left(\frac{\partial A_z}{\partial x} \right)^2 \right] \\ &= \frac{1}{2\mu_0} \int dv \left\{ \vec{\nabla}_\perp \cdot (A_z \nabla_\perp A_z) - A_z \nabla_\perp^2 A_z \right\} , \end{aligned} \tag{5.3.6}$$

wherein $\nabla_\perp$ is the corresponding 2D operator. As indicated above, the condition in (5.3.1) implies that the contribution of the first term in the curled brackets is zero.

For proving this statement we use Gauss' mathematical theorem

$$\int dv\, \vec{\nabla}_{\perp} \cdot (A_z \nabla_{\perp} A_z) = \oiint_{r} d\vec{a} \cdot (A_z \nabla_{\perp} A_z)\,.$$

The surface integral in the right hand side has two contributions; the first is associated with longitudinal variations

$$\left\{ \int_0^{2\pi} d\phi \int_0^{\infty} drr \left[A_z \underbrace{\frac{\partial A_z}{\partial z}}_{\equiv 0} \right] \right\}_{z=\Delta_z} - \left\{ \int_0^{2\pi} d\phi \int_0^{\infty} drr \left[A_z \underbrace{\frac{\partial A_z}{\partial z}}_{\equiv 0} \right] \right\}_{z=0} ,$$

and it is identically zero. The second term

$$\left\{ r\Delta_z \int_0^{2\pi} d\phi \left[A_z \frac{\partial A_z}{\partial r} \right] \right\}_{r\to\infty}$$

vanishes since $\sum_n I_n = 0$ implying that $A_z(r \to \infty) = 0$. Consequently, (5.3.4) is simplified to read

$$W_M = \frac{\Delta_z}{2\mu_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy A_z(-)\nabla_{\perp}^2 A_z \tag{5.3.7}$$

and with Poisson equation in mind, $\nabla_\perp^2 A_z = -\mu_0 J_z$, we finally obtain

$$W_M = \frac{\Delta_z}{2} \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy \, A_z J_z \, . \tag{5.3.8}$$

Substituting now Eqs. (5.3.2)-(5.3.3) we obtain

$$\begin{aligned} W_M = \frac{1}{2} \Delta_z \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy & \left[\sum_{n=1}^{N} (-1) \frac{\mu_0 I_n}{\pi R_n^2} \frac{R_n^2}{4} F_n(x,y) \right] \\ & \times \left[\sum_{m=1}^{N} \frac{I_m}{\pi R_m^2} h_n(x,y) \right] , \end{aligned} \tag{5.3.9}$$

wherein

$$F_n(x,y) \equiv \begin{cases} \dfrac{(x-x_n)^2 + (y-y_n)^2}{R_n^2} & 0 \le (x-x_n)^2 + (y-y_n)^2 < R_n^2 \\ 1 + \ln \left[\dfrac{(x-x_n)^2 + (y-y_n)^2}{R_n^2} \right] & R_n^2 \le (x-x_n)^2 + (y-y_n)^2 < \infty \, . \end{cases}$$

In terms of the currents in the system the total energy stored in the system is

$$W_M = \frac{1}{2} \sum_{n,m=1}^{N} U_{nm} I_n I_m \, , \tag{5.3.10}$$

wherein

$$U_{n,m} = \frac{-\mu_0 \Delta_z}{(2\pi R_m)^2} \int\limits_0^{2\pi} d\phi \int_0^{R_m} drr F_n(x_m + r\cos\phi,\, y_m + r\sin\phi)\,. \tag{5.3.11}$$

For $n = m$ the integral can be calculated analytically the result being

$$U_{n,n} = \frac{-\mu_0 \Delta_z}{(2\pi R_n)^2} \int\limits_0^{2\pi} d\phi \int\limits_0^{R_n} drr\,\frac{r^2}{R_n^2} = \frac{-\mu_0 \Delta_z}{(2\pi R_n)^2}\, 2\pi\,\frac{R_n^4}{4R_n^2} = \frac{-\mu_0 \Delta_z}{8\pi}\,.$$

In the Appendix 5.A it is shown that the off-diagonal terms ($n \neq m$) are given by

$$U_{n,m} = -\frac{\mu_0 \Delta_z}{8\pi}\left[2 + 4\ln\left(\frac{D_{nm}}{R_n}\right)\right]$$

D_{nm} denoting the distance between the center of two wires, i.e. $D_{nm} = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}$ or explicitly

$$U_{n,m} = -\frac{\mu_0 \Delta_z}{8\pi}\begin{cases} 1 & n = m \\ 2 + 4\ln\left(\frac{D_{nm}}{R_n}\right) & n \neq m\,. \end{cases}$$

With the exception of the general statement regarding convergence of the magnetic vector potential (A_z) we have not used the condition (5.3.1). This is the stage that we may calculate the self as well as the mutual inductance of a realistic system. Consider a three wires system as illustrated in Figure 5.15

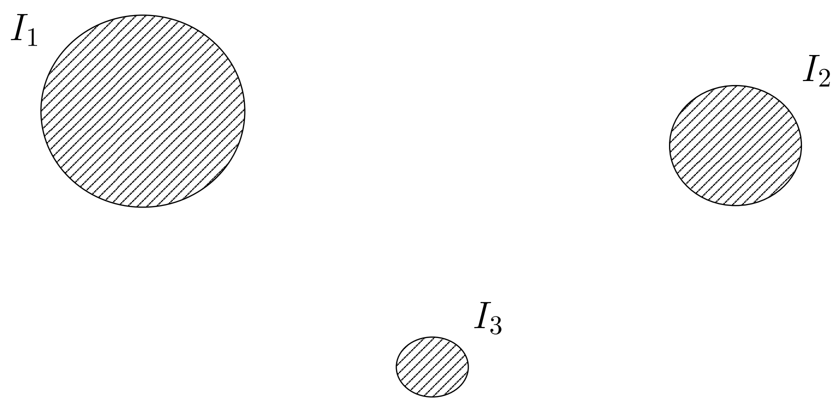

Figure 5.15: Three wires with the third providing the return path to the other two currents.

and without significant loss of generality let us assume that the wire #3 is the return path of the current i.e. $I_3 = -I_1 - I_2$ thus

$$\begin{aligned} W_M &= \frac{1}{2}\left[U_{11}I_1^2 + U_{22}I_2^2 + U_{33}I_3^2 + (U_{12}+U_{21})I_1I_2 + (U_{13}+U_{31})I_1I_3 + (U_{23}+U_{32})I_2I_3\right] \\ &\equiv \frac{1}{2}\left[\mathcal{L}_{11}I_1^2 + 2\mathcal{L}_{12}I_1I_2 + \mathcal{L}_{22}I_2I_2\right] \end{aligned} \tag{5.3.14}$$

hence

$$
\begin{aligned}
\mathcal{L}_{11} &= U_{11} + U_{33} - (U_{13} + U_{31}) \\
\mathcal{L}_{22} &= U_{22} + U_{33} - (U_{23} + U_{32}) \\
2\mathcal{L}_{12} &= 2U_{33} + U_{12} + U_{21} - (U_{13} + U_{31}) - (U_{23} - U_{32})\,.
\end{aligned}
\tag{5.3.15}
$$

In our special case

$$
\begin{aligned}
U_{11} = &\; U_{22} = U_{33} = -L_0 = -\frac{\mu_0 \Delta_z}{8\pi}\ , \\
U_{13} = &\; -2L_0 \left[1 + 2\ln\left(\frac{D_{13}}{R_1}\right)\right]\ , \qquad U_{31} = -2L_0 \left[1 + 2\ln\left(\frac{D_{31}}{R_3}\right)\right]\ , \\
U_{23} = &\; -2L_0 \left[1 + 2\ln\left(\frac{D_{23}}{R_2}\right)\right]\ , \qquad U_{32} = -2L_0 \left[1 + 2\ln\left(\frac{D_{32}}{R_3}\right)\right]\ , \\
U_{12} = &\; -2L_0 \left[1 + 2\ln\left(\frac{D_{12}}{R_1}\right)\right]\ , \qquad U_{21} = -2L_0 \left[1 + 2\ln\left(\frac{D_{21}}{R_2}\right)\right]
\end{aligned}
\tag{5.3.16}
$$

hence

$$\begin{aligned}
\mathcal{L}_{11} &= \frac{\mu_0 \Delta_z}{\pi} \left[\frac{1}{4} + \ln\left(\frac{D_{13}}{\sqrt{R_1 R_3}} \right) \right] , \\
\mathcal{L}_{22} &= \frac{\mu_0 \Delta_z}{\pi} \left[\frac{1}{4} + \ln\left(\frac{D_{23}}{\sqrt{R_2 R_3}} \right) \right] , \\
2\mathcal{L}_{12} &= \frac{\mu_0 \Delta_z}{\pi} \left[\frac{1}{4} + \ln\left(\frac{D_{13} D_{23}}{R_3 D_{12}} \right) \right] .
\end{aligned} \tag{5.3.17}$$

The matrix

$$\underset{=}{\mathcal{L}} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \tag{5.3.18}$$

is the *inductance matrix* with $\mathcal{L}_{12} = \mathcal{L}_{21}$. Note that in this case the mutual inductance depends only on the geometry of the return wire (R_3).

Appendix 5.A: Off-diagonal terms of the matrix $\underline{\underline{U}}$

The off-diagonal term is given by

$$U_{n,m} = -\frac{\mu_0 \Delta_z}{2(2\pi)^2} \int_0^{2\pi} d\phi \, \frac{2}{R_m^2} \int_0^{R_m} drr \, F_n(x_m + r\cos\phi, \; y_m + r\sin\phi)$$

$$= \frac{-\mu_0 \Delta_z}{8\pi^2} \int_0^{2\pi} d\phi \, \frac{2}{R_m^2} \int_0^{R_m} drr \left\{ 1 + \ln\left[\frac{(x_m - x_n + r\cos\phi)^2 + (y_m - y_n + r\cos\phi)^2}{R_n^2} \right] \right\}$$

$$= \frac{-\mu_0 \Delta_z}{8\pi^2} \int_0^{2\pi} d\phi \, \frac{2}{R_m^2} \int_0^{R_m} drr \left\{ 1 + \ln\left[\frac{D_{nm} + r^2 + 2r\cos(\phi - \phi_{nm})}{R_n^2} \right] \right\}, \qquad (5.A.1)$$

wherein $D_{nm} = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}$ and $\phi_{n,m} = \arctan\left(\frac{y_n - y_m}{x_n - x_m}\right)$.

Bearing in mind that

$$\ln\left[\frac{1}{\sqrt{1 + \xi^2 - 2\xi\cos\psi}} \right] = \sum_{\nu=1}^{\infty} \frac{\cos(\nu\psi)}{\nu} \, \xi^\nu \qquad (5.A.2)$$

we conclude that

$$\ln\left\{\frac{D_{nm}^2}{R_n^2}\left[1+\frac{r^2}{D_{nm}^2}-2\,\frac{r}{D_{nm}}\cos(\phi-\phi_{nm})\right]\right\}$$

$$=\ln\left(\frac{D_{nm}^2}{R_n^2}\right)-2\sum_{\nu=1}^{\infty}\frac{\cos[\nu(\phi-\phi_{nm})]}{\nu}\left(\frac{r}{D_{nm}}\right)^{\nu}. \tag{5.A.3}$$

The integration over ϕ is straightforward therefore,

$$U_{n,m}=\frac{-\mu_0\Delta_z}{8\pi}\left[2+4\ln\left(\frac{D_{nm}}{R_n}\right)\right]. \tag{5.A.4}$$

6. SOME FUNDAMENTAL THEOREMS

In this chapter we investigate two fundamental concepts associated with Maxwell's equations: the first is the *linearity* of Maxwell's equations allowing one to resort to the principle of *superposition* by representing each source in the form of a distribution of point-like sources leading, in turn to the total solution being represented by the sum of solutions of each individual point-source. The second basic topic to be considered, is *uniqueness* of a given solution within the framework of a static or quasi-static regime. The question in this case is the following: let us assume that by some means we have found a certain solution. How do we know it to be the only one? The uniqueness theorem shows that once a solution has been found, it is unique and any other solution is just a different *representation* of the same solution. In this chapter we focus our attention to the Poisson equation and not the wave equation.

6.1. Green's Theorem

In this section we shall examine some implications of the linearity of Maxwell's equations. The first step is to note that they are *first order* partial vector differential equations

$$\begin{aligned} &\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\,\mu_0 \vec{H} && \vec{\nabla} \cdot \mu_0 \vec{H} = 0 \\ &\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\,\varepsilon_0 \vec{E} && \vec{\nabla} \cdot \varepsilon_0 \vec{E} = \rho\,, \end{aligned} \tag{6.1.1}$$

which may be written in the form of a *second order* differential equation by following the next steps:

a)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

i.e.,

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= -\mu_0 \frac{\partial}{\partial t}\left[\vec{J} + \varepsilon_0 \frac{\partial}{\partial t}\vec{E}\right] \\ &= -\mu_0 \frac{\partial}{\partial t}\vec{J} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}\,. \end{aligned} \tag{6.1.2}$$

b) Now, in a Cartesian system of coordinates

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \tag{6.1.3}$$

hence

$$\nabla(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial}{\partial t}\vec{J} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}\,. \tag{6.1.4}$$

c) Introducing now Gauss' law ($\nabla \cdot \varepsilon_0 \vec{E} = \rho$) one obtains the relation

$$\nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial \vec{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} \tag{6.1.5}$$

d) so that finally

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E} = +\mu_0 \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon_0}\nabla\rho\,. \tag{6.1.6}$$

In a similar way one obtains the second order equation for the magnetic field

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{H} = -\vec{\nabla}\times\vec{J}\,. \tag{6.1.7}$$

These are non-homogeneous vector wave equations. In the past we have found for the potential functions similar equations, one being a vector equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0\,\vec{J}\,, \tag{6.1.8}$$

whereas the other being a scalar equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi = -\frac{\rho}{\varepsilon_0}\,. \tag{6.1.9}$$

Within the framework of quasi-static approximations we may limit the discussion to

$$\nabla^2\vec{E} = \mu_0\,\frac{\partial\vec{J}}{\partial t} + \frac{1}{\varepsilon_0}\nabla\rho \qquad \text{and} \qquad \nabla^2\vec{H} = -\vec{\nabla}\times\vec{J} \tag{6.1.10}$$

or

$$\nabla^2\vec{A} = -\mu_0\vec{J} \qquad \text{and} \qquad \nabla^2\phi = -\frac{\rho}{\varepsilon_0}\,, \tag{6.1.11}$$

which are all equations of the Poisson type, i.e.

$$\nabla^2\psi = -s(x,y,z)\,, \tag{6.1.12}$$

wherein $\psi(x,y,z)$ represents either one of the Cartesian components of the vectors $\vec{E}, \vec{H}$ or $\vec{A}$ or the scalar electric potential ϕ; $s(x,y,z)$ stands for the corresponding source term of the Poisson equation. Before turning to the general approach, we illustrate a simple case: consider a single point charge q_i located at $\vec{r}_i$ i.e., $\rho = -q\delta(\vec{r} - \vec{r}_i)$. Its potential in free space is

$$\phi_i(\vec{r}) = \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_i|}, \tag{6.1.13}$$

and therefore a distribution of discrete particles generates a potential

$$\phi(\vec{r}) = \sum_i \phi_i(\vec{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_i|}. \tag{6.1.14}$$

If in a unit volume Δv there is an infenitisimal charge Δq such that we may postulate the charge density $\rho = \frac{\Delta q}{\Delta v}$ one may replace the sum by an integral.

$$\sum_i q_i .... \to \int \mathrm{d}v \frac{\mathrm{d}q}{\mathrm{d}v} ..... = \int \mathrm{d}v \rho .....$$

or explicitly, for a continuous distribution $\rho(\vec{r})$, the summation is replaced by

$$\phi(\vec{r}) = \frac{1}{\epsilon_0} \int d\vec{r}' \rho(\vec{r}') \frac{1}{4\pi|\vec{r} - \vec{r}'|}, \tag{6.1.15}$$

wherein we represent the volume of integration by $\mathrm{d}\vec{r}'$, a notation which will turn out to be more convenient for the approach that follows.

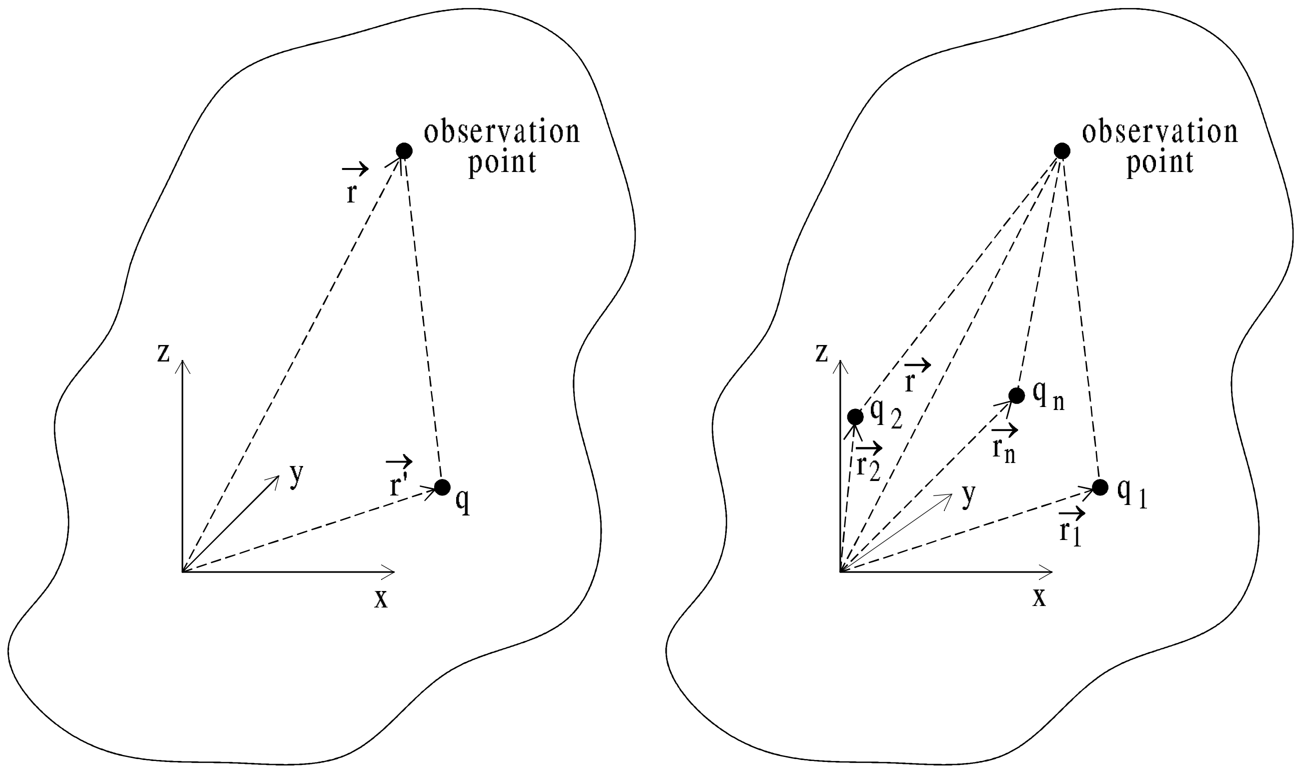

Figure 6.1: Potential generated by a point-charge and several point-charges at a given place in space.

The last expression is the so-called Green's function associated with Poisson's equation [(6.1.11)] in case of *no boundaries*, i.e.

$$G(\vec{r}|\vec{r}') = \frac{1}{4\pi|\vec{r} - \vec{r}'|} \, .$$

Let us now return to the general approach. We employ the superposition principle in the sense that any source $s(x, y, z)$ may be "divided" into a distribution of point-sources. We denote by G the field "generated" by such a point charge subject to the *boundary conditions* of the original problem. In other words if we denote by $\vec{r}_s$ the location of the source then G satisfies the equation

$$\nabla^2 G(\vec{r}|\vec{r}_s) = -\delta(\vec{r}-\vec{r}_s)\,. \tag{6.1.13}$$

Two comments are of importance:

1 Integration of the right-hand side of (6.1.13) over the entire volume leads to the result -1.

2 According to its definition in (6.1.13), G is symmetric with respect to the two variables i.e.

$$G(\vec{r}|\vec{r}_s) = G(\vec{r}_s|\vec{r})\,. \tag{6.1.14}$$

We now multiply (6.1.12) by G and (6.1.13) by ψ. The resulting two expressions are subtracted, i.e.

$$G(\vec{r}_s|\vec{r})\nabla_s^2\psi(\vec{r}_s) - \psi(\vec{r}_s)\nabla_s^2 G(\vec{r}_s|\vec{r}) = -G(\vec{r}_s|\vec{r})s(\vec{r}_s) + \psi(\vec{r}_s)\delta(\vec{r}_s-\vec{r}) \tag{6.1.15}$$

Integrating now over the whole space (with respect to the variable with index s - source), one obtains the relation

$$\begin{aligned}\int dv_s[G(\vec{r}_s|\vec{r})\nabla_s^2\psi(\vec{r}_s) - \psi(\vec{r}_s)\nabla_s^2 G(\vec{r}_s|\vec{r})] &= -\int dv_s G(\vec{r}_s|\vec{r})s(\vec{r}_s) \quad + \int dv_s \psi(\vec{r}_s)\delta(\vec{r}-\vec{r}_s) \\ &= -\int dv_s G(\vec{r}_s|\vec{r})s(\vec{r}_s) + \psi(r)\,.\end{aligned}$$

Using the fact that $G\nabla_s^2\psi = \nabla_s \cdot (G\nabla_s\psi) - \nabla_s G \cdot \nabla_s\psi$ we now obtain that

$$\begin{aligned}\int dv_s\Big[G(\vec{r}_s|\vec{r})\nabla_s^2\psi(\vec{r}_s) - \psi(\vec{r}_s)\nabla_s^2 G(\vec{r}_s|\vec{r})\Big] &= \int dv_s\Big\{\vec{\nabla}_s \cdot [G(\vec{r}_s|\vec{r})\nabla_s\psi(\vec{r}_s)] \\ &\quad -\underline{\nabla_s G(\vec{r}_s|\vec{r}) \cdot \nabla_s\psi(\vec{r}_s)} \\ &\quad -\vec{\nabla}_s \cdot [\psi(\vec{r}_s)\nabla_s G(\vec{r}_s|\vec{r})] \\ &\quad +\underline{\nabla_s\psi(\vec{r}_s) \cdot \nabla_s G(\vec{r}_s|\vec{r})}\Big\} \\ &= \int dv_s\vec{\nabla}_s\left\{G(\vec{r}_s|\vec{r})\nabla_s\psi(\vec{r}_s) - \psi(\vec{r}_s)\nabla_s G(\vec{r}_s|\vec{r})\right\} ,\end{aligned} \tag{6.1.16}$$

or the second and the fourth terms cancel; resorting now to Gauss' mathematical theorem, i.e.

$$\begin{aligned}\int dv_s[G(\vec{r}_s|\vec{r})\nabla_s^2\psi(\vec{r}_s) - \psi(\vec{r}_s)\nabla_s^2 G(\vec{r}_s|\vec{r})] &= \int dv_s\vec{\nabla}_s[G(\vec{r}_s|\vec{r})\nabla_s\psi(\vec{r}_s) \\ &\quad -\psi(\vec{r}_s)\nabla_s G(\vec{r}_s|\vec{r})] \\ &= \oiint d\vec{a}_s \cdot [G(\vec{r}_s|\vec{r})\nabla_s\psi(\vec{r}_s) \\ &\quad -\psi(\vec{r}_s)\nabla_s G(\vec{r}_s|\vec{r})]\end{aligned} \tag{6.1.17}$$

we obtain the relation

$$\psi(\vec{r}) = \int dv_s G(\vec{r}_s|\vec{r})s(\vec{r}_s) + \oiint d\vec{a}_s\Big[G(\vec{r}_s|\vec{r})\nabla_s\psi(\vec{r}_s) - \psi(\vec{r}_s)\nabla_s G(\vec{r}_s|\vec{r})\Big] \tag{6.1.18}$$

and with (6.1.14) one finally finds that

$$\psi(\vec{r}) = \int dv_s G(\vec{r}|\vec{r}_s)s(\vec{r}_s) + \oiint d\vec{a}_s \Big[G(\vec{r}|\vec{r}_s)\nabla_s\psi(\vec{r}_s) - \psi(\vec{r}_s)\nabla_s G(\vec{r}|\vec{r}_s) \Big] . \tag{6.1.19}$$

This is the scalar Green's theorem and several comments are now in order:

#1 $G(\vec{r}|\vec{r}_s)$ is called Green's function of the specific problem.

#2 In a sense, $G(\vec{r}|\vec{r}_s)$ is the equivalent of a matrix since if we identify the continuum integration with a discrete summation we can write

$$\psi_i = \psi(r_i) = \sum_j G(r = r_i|r' = r_j)s(r' = r_j) \equiv \sum_i G_{ij}s_j$$

#3 Even if the source term is zero $(s = 0)$, ψ is not necessarily zero provided that at some boundary ψ or $\nabla\psi$ are nonzero [Neumann or Derichlet conditions].

#4 We may generalize (6.1.19) to include slow time-variation of the source terms. Since the geometric parameters are typically much smaller than the characteristic wavelength of the electromagnetic field, the "information" traverses the device/system instantaneously; therefore, in the quasi-static case (6.1.19) reads

$$\boxed{\psi(\vec{r},t) = \int dv_s G(\vec{r}|\vec{r}_s)s(\vec{r}_s,t) + \oiint d\vec{a} \Big[G(\vec{r}|\vec{r}_s)\nabla_s\psi(\vec{r}_s,t) - \psi(\vec{r}_s,t)\nabla_s G(\vec{r}|\vec{r}_s) \Big]}.$$

Here the time t is no more than a parameter.

6.2. Magneto-Quasi-Static Example

In order to illustrate the concepts presented in the previous section we examine the magnetic field linked to a current distribution imposed along the z axis. The magnetic field is confined by means of two plates consisting of a fictitious material of "infinite" permeability – see Figure 6.2. Our starting point is to evaluate the field distribution due to a single line current.

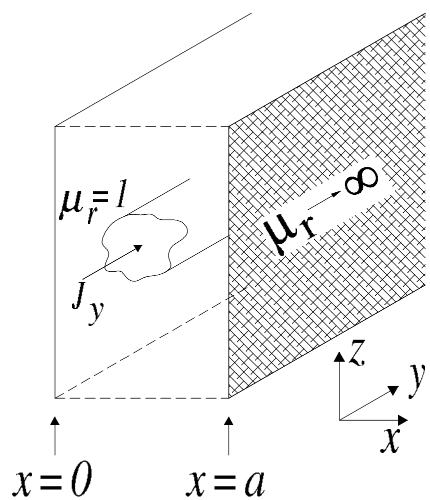

Figure 6.2: Magneto-quasi-static example for the use of Green's function.

The basic assumptions, relevant to the model

#1 A current density $J_y(x, z, t)$ is imposed.

#2 No variations along the y axis, i.e. $\partial_y \simeq 0$.

#3 $\mu_r \to \infty$ on the two walls implies zero tangential magnetic field along the z-axis, i.e. $H_z(x = 0, a) = 0$.

#4 Magneto-quasi-statics $\Rightarrow \frac{\omega}{c} a \ll 1$.

A current density in the y direction is linked to a y-component of the magnetic vector potential

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right] A_y(x, z, t) = -\mu_0 J_y(x, z, t)\,. \qquad (6.2.1)$$

Following Green's function approach we shall calculate the function $\alpha_y(x, z)$ which is a solution of

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right] \alpha_y(x, z) = -\delta(x - x_s)\delta(z - z_s)\,; \qquad (6.2.2)$$

α_y corresponds to a magnetic vector potential excited by the localized unit current; we further denote by $h_z = \frac{\partial \alpha_y}{\partial x}$ the corresponding longitudinal magnetic field. Since the z-component magnetic field vanishes at $x = 0$ and $x = a$ we conclude that

$$\alpha_y(x, z) = \sum_{n=0}^{\infty} A_n(z) \cos\left(\frac{\pi n x}{a}\right) . \tag{6.2.3}$$

Substituting in (6.2.2) we obtain

$$\sum_n \left[\frac{d^2}{dz^2} - \left(\frac{\pi n}{a}\right)^2\right] A_n(z) \cos\left(\frac{\pi n x}{a}\right) = -\delta(x - x_s)\delta(z - z_s) . \tag{6.2.4}$$

Use of the orthogonality of the trigonometric functions leads to

$$\sum_n \left[\frac{d^2}{dz^2} - \left(\frac{\pi n}{a}\right)^2\right] A_n(z) \int_0^a dx \cos\frac{\pi n' x}{a} \cos\frac{\pi n x}{a} = -\int_0^a dx \cos\left(\frac{\pi n' x}{a}\right) \delta(x - x_s)\delta(z - z_s)$$

or

$$\sum_n \left[\frac{d^2}{dz^2} - \left(\frac{\pi n}{a}\right)^2\right] A_n(z) a g_n \delta_{nn'} = (-1) \cos\left(\frac{\pi n' x_s}{a}\right) \delta(z - z_s)$$

wherein

$$g_n = \begin{cases} 1 & n = 0 \\ 0.5 & n \neq 0 \end{cases}$$

hence

$$\left[\frac{d^2}{dz^2} - \left(\frac{\pi n}{a}\right)^2\right] A_n(z) = -\frac{\cos(\pi n x_s / a)}{a g_n} \delta(z - z_s) . \tag{6.2.5}$$

For $z \neq z_s$, $A_n(z)$ is a solution of

$$\left[\frac{d^2}{dz^2} - \left(\frac{\pi n}{a}\right)^2\right] A_n(z) = 0\,. \tag{6.2.6}$$

The general solution of (6.2.6) is

$$A_n = u_n e^{-\frac{\pi n}{a}(z - z_s)} - v_n e^{+\frac{\pi n}{a}(z - z_s)} \tag{6.2.7}$$

however, for a converging solution at $z \to \infty$ one must assume that $v_n = 0$, similarly for $z \to -\infty$ one must take $u_n = 0$. Hence

$$A_n(z > z_s) = u_n e^{-\frac{\pi n}{a}(z - z_s)}\,, \tag{6.2.8}$$

whereas for $(z < z_s)$ the solution is

$$A_n(z < z_s) = v_n e^{+\frac{\pi n}{a}(z - z_s)}\,. \tag{6.2.9}$$

In order to establish u_n and v_n we assume that α_y is continuous at $z = z_s$ even though its derivative is not. The discontinuity in the derivative of α_y is established by integrating (6.2.5) in the vicinity of z_s:

$$\begin{aligned}\int_{z=z_s-0}^{z=z_s+0} dz \left[\frac{d^2}{dz^2} - \left(\frac{\pi n}{a}\right)^2\right] A_n(z) &= \left.\frac{dA_n}{dz}\right|_{z=z_s+0} - \left.\frac{dA_n}{dz}\right|_{z=z_s-0} \\ &= -\frac{\cos(\pi n x_s/a)}{a g_n}\,,\end{aligned} \tag{6.2.10}$$

where the second line is the result of the integration over the right hand side of (6.2.5). From the continuity of α_y we conclude that

$$u_n = v_n \,, \tag{6.2.11}$$

whereas from (6.2.10) we have

$$-\frac{\pi n}{a}u_n - \frac{\pi n}{a}v_n = -\frac{\cos \pi n x_s/a}{a g_n} \,, \tag{6.2.12}$$

hence

$$u_n = v_n = \frac{1}{2}\frac{\cos(\pi n x_s/a)}{\pi n \; g_n} \,. \tag{6.2.13}$$

Finally we obtain

$$\alpha_y(x,z) = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi n x}{a}\right)\cos\left(\frac{\pi n x_s}{a}\right)}{2\pi n \; g_n} e^{-\frac{\pi n}{a}|z-z_s|} \,, \tag{6.2.14}$$

which is now identified to be Green's function

$$G(x,z|x',z') = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi n x}{a}\right)\cos\left(\frac{\pi n x'}{a}\right)}{2\pi n \; g_n} e^{-\frac{\pi n}{a}|z-z'|} \tag{6.2.15}$$

and consequently

$$A_y(x,z,t) = \mu_0 \int_0^a dx' \int_{-\infty}^{\infty} dz' G(x,z|x',z') J_y(x',z',t) \,. \tag{6.2.16}$$

With this expression in mind we are able to examine a simple but interesting case, namely a narrow and uniform layer of current

$$J_y(x,z,t) = I(t)\delta(z)\frac{1}{b}U\left[\left(x-\frac{a}{2}\right)\frac{2}{b}\right] \tag{6.2.17}$$

$$U(\xi) = \begin{cases} 1 & |\xi| < 1 \\ 0 & |\xi| > 1 \end{cases}$$

$$\begin{aligned} A_y &= \frac{\mu_0 I(t)}{b}\sum_{n=0}^{\infty}\frac{\cos\left(\frac{n\pi x}{a}\right)}{2\pi n\; g_n} e^{-\frac{\pi n}{a}|z|}\int_{\frac{a}{2}-\frac{b}{2}}^{\frac{a}{2}+\frac{b}{2}} dx' \cos\left(\frac{\pi n x'}{a}\right) \\ &= \frac{\mu_0 I(t)}{b}\sum_{n=0}^{\infty}\frac{\cos\left(\frac{n\pi x}{a}\right)}{2\pi n\; g_n} e^{-\frac{\pi n|z|}{a}}\left\{\frac{\sin\left[\frac{\pi n}{a}\left(\frac{a}{2}+\frac{b}{2}\right)\right]-\sin\left[\frac{\pi n}{a}\left(\frac{a}{2}-\frac{b}{2}\right)\right]}{\frac{\pi n}{a}}\right\} \qquad (6.2.18) \\ &= \sum_{n=0}^{\infty}\mathcal{A}_n \cos\frac{\pi n x}{a}\, e^{-\pi n\frac{|z|}{a}}, \end{aligned}$$

where

$$\mathcal{A}_n = \frac{\mu_0 I(t)}{b}\frac{1}{2\pi n}\frac{1}{g_n}\left\{\frac{a+b}{2}\,\mathrm{sinc}\left[\frac{\pi n}{2a}(a+b)\right]-\frac{a-b}{2}\,\mathrm{sinc}\left[\frac{\pi n}{a}(a-b)\right]\right\}. \tag{6.2.19}$$

Now that we established $A_y(x,z,t)$, we may calculate the inductance of the metallic strip carrying this current in the vicinity of the magnetic walls. For this purpose we first calculate the magnetic field.

$$H_x = \frac{-1}{\mu_0}\frac{\partial A_y}{\partial z} = \frac{1}{\mu_0}\sum_{n=0}\left(\frac{\pi n}{a}\right)\mathcal{A}_n \cos\left(\frac{\pi n x}{a}\right) e^{-\pi n \frac{|z|}{a}} \begin{cases} 1 & z>0 \\ -1 & z<0 \end{cases} \quad (6.2.20)$$

$$H_z = \frac{1}{\mu_0}\frac{\partial A_y}{\partial x} = \frac{1}{\mu_0}\sum_{n=0}^{\infty}\left(\frac{\pi n}{a}\right)\mathcal{A}_n \sin\left(\frac{\pi n x}{a}\right) e^{-\pi n \frac{|z|}{a}}, \quad (6.2.21)$$

or

$$H_x = \sum_{n=0}^{\infty}\mathcal{H}_n \cos\left(\frac{\pi n x}{a}\right) e^{-\pi n \frac{|z|}{a}} \,\mathrm{sgn}\,(z) \quad (6.2.22)$$

$$H_z = \sum_{n=0}^{\infty}\mathcal{H}_n \sin\left(\frac{\pi n x}{a}\right) e^{-\pi n \frac{|z|}{a}} \quad (6.2.23)$$

$$\mathcal{H}_n = \frac{I(t)}{4 g_n b}\left\{\left(1+\frac{b}{a}\right)\mathrm{sinc}\left[\frac{\pi}{2}n\left(1+\frac{b}{a}\right)\right] - \left(1-\frac{b}{a}\right)\mathrm{sinc}\left[\frac{\pi}{2}n\left(1-\frac{b}{a}\right)\right]\right\}. \quad (6.2.24)$$

We are now able to calculate the total energy stored in the magnetic field in a unit length Δ_y

$$
\begin{aligned}
W_M &= \frac{1}{2}\mu_0\Delta_y \int_0^a dx \int_{-\infty}^{\infty} dz \left(H_x^2 + H_z^2\right) \\
&= \frac{\mu_0\Delta_y}{2}\int_{-\infty}^{\infty} dz \sum_{n=0}^{\infty} a\mathcal{H}_n^2\, e^{-2\pi n\frac{|z|}{a}} = \frac{1}{2}\,\mu_0\Delta_y\,\frac{a^2}{\pi}\sum_{n=0}^{\infty}\frac{\mathcal{H}_n^2}{n}\,.
\end{aligned}
\tag{6.2.25}
$$

With this result we may define the low frequency inductance of the system, since $W_{\rm M} = LI^2(t)/2$. Consequently,

$$
\begin{aligned}
L &= \mu_0\Delta_y\frac{a^2}{b^2}\frac{1}{16\pi}\sum_{n=0}^{\infty}\frac{1}{n}g_n^2\left\{\left(1+\frac{b}{a}\right)\,\mathrm{sinc}\left[\frac{\pi}{2}n\left(1+\frac{b}{a}\right)\right]\right. \\
&\qquad\left. -\left(1-\frac{b}{a}\right)\,\mathrm{sinc}\left[\frac{\pi}{2}n\left(1-\frac{b}{a}\right)\right]\right\}^2 .
\end{aligned}
\tag{6.2.26}
$$

The graphical plot of the dimensionless inductance, $\overline{L} = \dfrac{L}{\mu_0\Delta_y/16\pi}$ is illustrated in Figure 6.3. Note that as the current layer becomes wider, the inductance decreases. This result is subject to the quasi-static assumption, that is $\omega a/c \ll 1$ therefore if $a = 1$cm the maximum frequency this expression is valid is $f \ll f_{\rm max} \sim 5$GHz.

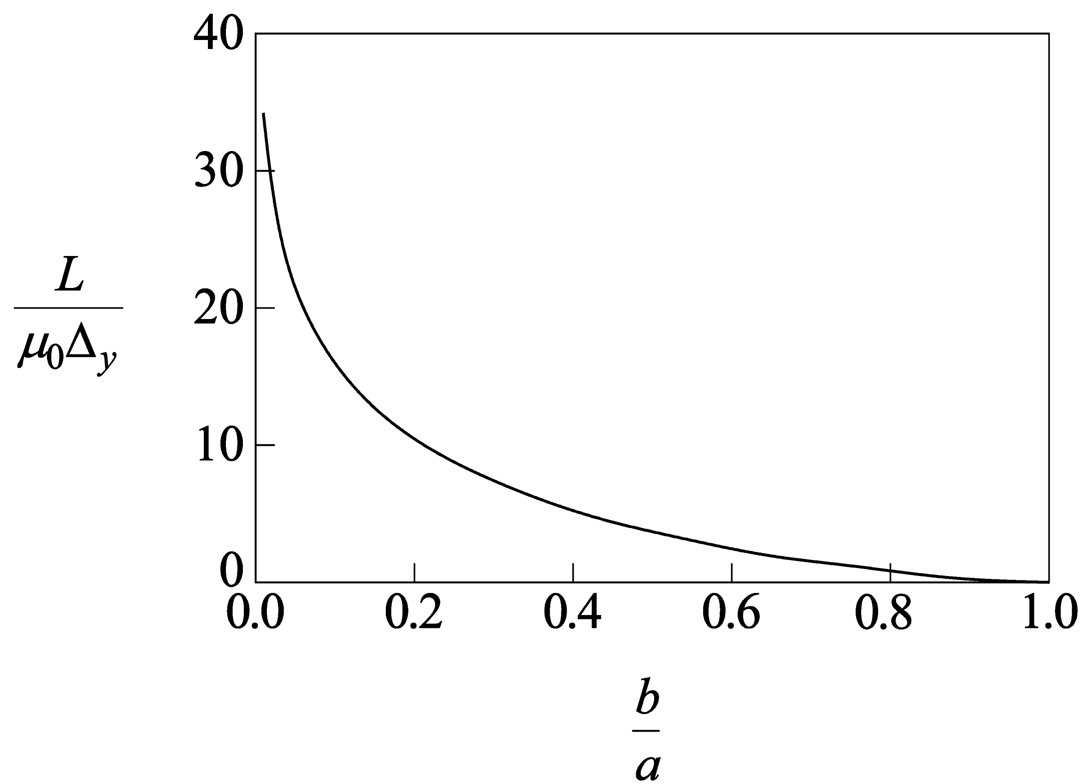

Figure 6.3: Inductance L as a function of the width of the strip (b/a).

6.3. Uniqueness Theorem

So far it was tacitly assumed that once a solution ψ_a of the equation of the type

$$\nabla^2\psi_a = -s \qquad \text{with} \qquad \psi_a = \psi_i \text{ on } \; S_i \tag{6.3.1}$$

was found this is the *only* solution. Let us show that this is a valid assumption. For this purpose let us assume the opposite namely, that another solution ψ_b exists which also satisfies the relations

$$\nabla^2\psi_b = -s \qquad \text{with} \qquad \psi_b = \psi_i \text{ on } S_i\,. \tag{6.3.2}$$

By virtue of the superposition principle we may now define the difference of the two solutions $\psi_d \equiv \psi_a - \psi_b$ as a solution which, evidently, must satisfy the relation

$$\nabla^2\psi_d = 0 \qquad \text{with} \qquad \psi_d = 0 \text{ on } S_i\,. \tag{6.3.3}$$

If ψ_d satisfies Laplace's equation then it cannot reach a maximum or minimum but on the boundaries. This means that "moving" from one boundary to another the function ψ_d must monotonically increase or decrease. But since on the boundaries $\psi_d = 0$ one must conclude that ψ_d is *zero* in the entire volume, including the boundaries. Hence

$$\psi_a = \psi_b\,. \tag{6.3.4}$$

6.4. Two Simple Examples

6.4.1. Example #1

The potential φ_a of a point charge q is given, in free and unbounded space, by the expression

$$\varphi_a(x,y,z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}} \tag{6.4.1}$$

i.e. (6.4.1) is *the* solution of

$$\nabla^2\varphi_a = \frac{\partial^2\varphi_a}{\partial x^2} + \frac{\partial^2\varphi_a}{\partial y^2} + \frac{\partial^2\varphi_a}{\partial z^2} = \frac{-q}{\varepsilon_0}\delta(x)\delta(y)\delta(z)\,. \tag{6.4.2}$$

However, without loss of generality we may write (6.4.2) in a circular cylindrical coordinate system taking advantage of the azimuthal symmetry $\dfrac{\partial}{\partial\phi} \sim 0$, i.e.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_b}{\partial r}\right) + \frac{\partial^2\varphi_b}{\partial z^2} = -\frac{q}{\varepsilon_0}\frac{1}{2\pi r}\,\delta(r)\,\delta(z)\ , \tag{6.4.3}$$

where r is the radial coordinate extending from the location of the source, $r \equiv \sqrt{x^2+y^2}$. The solution of this differential equation may be written as

$$\varphi_b(r,z) = \frac{q}{4\pi\varepsilon_0}\int_0^\infty dk J_0(kr)e^{-k|z|}\ , \tag{6.4.4}$$

where $J_0(\xi)$ is the zero order Bessel function of the first kind, i.e.

$$J_0(\xi) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\psi e^{-j\xi\cos\psi}.$$

The expression in (6.4.4) is a different *representation* of the solution in (6.4.1) so that in fact the identity with the previous definition that

$$\int_0^\infty dk J_0(kr)e^{-k|z|} = \frac{1}{\sqrt{r^2+z^2}} \tag{6.4.5}$$

follows. Let us now utilize this formulation in order to calculate the potential of a point charge located above a grounded (metallic), infinitely extending plane - see Figure 6.4.

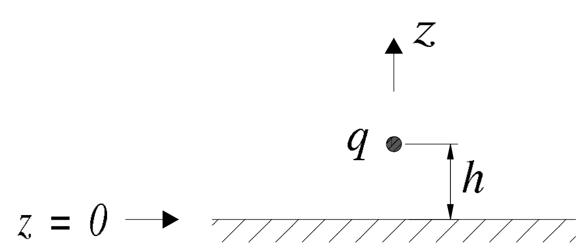

Figure 6.4: Point-charge, q, at a height h above a grounded surface.

The potential of the source (subscript s) excited in the absence of the plane boundary is conveniently written as

$$\varphi_s = \frac{q}{4\pi\varepsilon_0}\int_0^\infty dk J_0(kr)e^{-k|z-h|}. \tag{6.4.6}$$

In the presence of the plane there is an additional contribution to the electrostatic potential. This contribution must be a solution of the homogeneous equation (subscript h)

$$\nabla^2 \varphi_h = 0 \,, \tag{6.4.7}$$

since the source has been already accounted for. Consequently, choosing an expression "coherent" with the exciting one, we assume

$$\varphi_h(r,z) = \frac{q}{4\pi\varepsilon_0} \int_0^\infty dk A(k) J_0(kr) e^{-kz} \,. \tag{6.4.8}$$

In order to establish $A(k)$ we impose on the total solution

$$\varphi_{\text{total}}(r,z) = \varphi_s(r,z) + \varphi_h(r,z) \tag{6.4.9}$$

the boundary condition ($\varphi_{\text{total}} = 0$) at $z = 0$ for any r, i.e.

$$\frac{q}{4\pi\varepsilon_0} \int dk \Big[A(k) + e^{-kh} \Big] J_0(kr) = 0 \,, \tag{6.4.10}$$

thus

$$A(k) = -e^{-kh} \,. \tag{6.4.11}$$

In other words, the presence of the metallic plane has generated an additional contribution to the potential due to the point-charge alone (6.4.6) which has the form

$$\varphi_h(r, z \geq 0) = \frac{-q}{4\pi\varepsilon_0} \int_0^\infty dk J_0(kr) e^{-k(h+z)} \,. \tag{6.4.12}$$

Comparing to (6.4.5) we conclude that we can represent this solution as

$$\varphi_h(r, z \geq 0) = \frac{-q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + (z+h)^2}}\,, \tag{6.4.13}$$

which may be interpreted as an *image charge* $(-q)$ located at $\underline{z = -h}$. Finally, the potential in the upper half-space $(z \geq 0)$ only, is given by

$$\varphi(r, z \geq 0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + (z-h)^2}} + \frac{-q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + (z+h)^2}}\,. \tag{6.4.14}$$

6.4.2. Example #2

The second example to be considered consists of a point charge located between two plates as illustrated in Figure 6.5. We put forward three different representations of the same solution starting from the simple field-solution in integral form, which, by virtue of an adequate expansion is in fact equivalent to an infinite series of charge images; finally, a third representation, which converges quite rapidly, is also presented.

Integral form. As in the previous case, the potential is a superposition of the non-homogeneous

$$\varphi_s(r, z) = \frac{q}{4\pi\varepsilon_0} \int_0^\infty dk J_0(kr)\, e^{-k|z-b|} \tag{6.4.15}$$

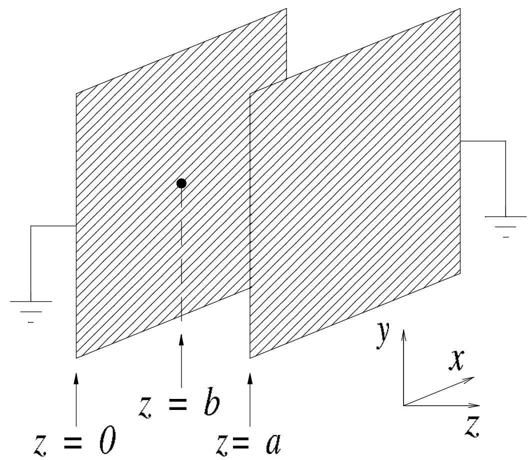

Figure 6.5: A point charge located between two parallel metallic plates.

as well as of the homogeneous solution

$$\varphi_h(r,z) = \frac{q}{4\pi\varepsilon_0} \int_0^\infty dk J_0(kr) \left[A(k)e^{-kz} + B(k)e^{kz}\right]. \tag{6.4.16}$$

The two amplitudes $A(k)$ and $B(k)$ are established by imposing the boundary conditions at $z = 0$ and $z = a$; at both locations the total potential is zero. For $z = 0$ we have

$$\varphi_{\rm total}(r, z = 0) = \frac{q}{4\pi\varepsilon_0} \int_0^\infty dk J_0(kr) \left[e^{-kb} + A + B\right] = 0\,, \tag{6.4.17}$$

which must be satisfied for any k; therefore

$$e^{-kb} + A(k) + B(k) = 0\,. \tag{6.4.18}$$

The potential function is zero also on the second plate and thus

$$\varphi_{\text{total}}(r, z = a) = \frac{q}{4\pi\varepsilon_0} \int_0^\infty dk J_0(kr) \left[e^{-k(a-b)} + Ae^{-ka} + Be^{ka} \right] = 0 \tag{6.4.19}$$

i.e.

$$e^{-k(a-b)} + Ae^{-ka} + Be^{ka} = 0\,. \tag{6.4.20}$$

Using (6.4.18) and (6.4.20) we find that

$$\begin{aligned} A(k) &= -\frac{\sinh\left[k(a-b)\right]}{\sinh(ka)} \\ B(k) &= -e^{-kb} + \frac{\sinh\left[k(a-b)\right]}{\sinh(ka)} = -e^{-ka}\,\frac{\sinh(kb)}{\sinh(ka)}\,. \end{aligned} \tag{6.4.21}$$

The total scalar potential is therefore given by

$$\varphi(r, z) = \frac{q}{4\pi\varepsilon_0} \int_0^\infty dk J_0(kr) \left\{ e^{-k|z-b|} - e^{-kz} \frac{\sinh\left[k(a-b)\right]}{\sinh(ka)} - e^{kz} \left[e^{-ka}\,\frac{\sinh(kb)}{\sinh(ka)} \right] \right\}. \tag{6.4.22}$$

Image Charges Series. From this integral representation we may develop an infinite series of image charges as may intuitively be deduced. In order to obtain this infinite series we use the relation

$$\frac{1}{\sinh(ka)} = \frac{1}{\frac{1}{2}\left[e^{ka} - e^{-ka}\right]} = \frac{2e^{-ka}}{1 - e^{-2ka}} = 2\sum_{n=0}^{\infty} e^{-(2n+1)ka} \tag{6.4.23}$$

with this expression in mind and taking into consideration that

$$\sinh\Big[k(a-b)\Big] = \frac{1}{2}\left[e^{k(a-b)} - e^{-k(a-b)}\right]$$

we find

$$\begin{aligned}\varphi(r,z) &= \frac{q}{4\pi\varepsilon_0}\int_0^\infty dk J_0(kr)\Bigg\{e^{-k|z-b|} \\ & -e^{-kz}\left[e^{k(a-b)} - e^{-k(a-b)}\right]\sum_{n=0}^{\infty} e^{-(2n+1)ka} \\ & -e^{k(z-a)}\left[e^{kb} - e^{-kb}\right]\sum_{n=0}^{\infty} e^{-(2n+1)ka}\Bigg\}\,.\end{aligned} \tag{6.4.24}$$

At this point we introduce the identity in (6.4.5) obtaining

$$\varphi(r,z) = \frac{q}{4\pi\varepsilon_0}\Bigg\{\frac{1}{\sqrt{r^2+(z-b)^2}} - \sum_{n=0}^{\infty}\frac{1}{\sqrt{r^2+\Big[z+b+2na\Big]^2}} + \sum_{n=0}^{\infty}\frac{1}{\sqrt{r^2+\Big[z-b+2(n+1)a\Big]^2}} - \sum_{n=0}^{\infty}\frac{1}{\sqrt{r^2+\Big[z+b-2(n+1)a\Big]^2}} + \sum_{n=0}^{\infty}\frac{1}{\sqrt{r^2+\Big[z-b-2(n+1)a\Big]^2}}\Bigg\}. \quad (6.4.25)$$

The summations correspond to the potential of an infinite series image charges outside the plates $z > a$ or $z < 0$; the first two terms correspond to image charges located at $z < 0$ whereas in the case of the last two, the image charges are located at $z > a$.

Eigen-function Representation. The expressions in (6.4.22) and (6.4.25) are two different representations of the *same* potential. In fact, we are in a position to establish a *third* representation of this potential distribution using the fact that the boundary conditions are satisfied by a set of orthogonal functions. Recall that we have to solve the equation

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right]\varphi=-\frac{q}{\varepsilon_0}\frac{1}{2\pi r}\delta(r)\delta(z-b) \tag{6.4.26}$$

subject to the boundary conditions

$$\varphi(r,z=0)=0 \text{ and } \varphi(r,z=a)=0\,. \tag{6.4.27}$$

Based on the last constraint we have

$$\varphi(r,z)=\sum_{n=1}^{\infty}F_n(r)\sin\left[\frac{\pi n z}{a}\right]\,. \tag{6.4.28}$$

Note that the trigonometric functions form a set of orthogonal functions in the region $0<z\leq d$. In order to determine now the equation for $F_n(r)$ we substitute (6.4.28) in (6.4.26)

$$\sum_{n=1}^{\infty}\sin\left(\pi n\frac{z}{a}\right)\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}F_n(r)\right]-\left(\frac{\pi n}{a}\right)^2F_n(r)\right\}=-\frac{q}{\varepsilon_0}\frac{1}{2\pi r}\delta(r)\,\delta(z-b)\,. \tag{6.4.29}$$

Resorting now to the orthogonality of the trigonometric functions, i.e.

$$\frac{1}{a}\int_0^a dz\sin\left(\frac{\pi n z}{a}\right)\sin\left(\frac{\pi m z}{a}\right)=\frac{1}{2}\delta_{n,m} \tag{6.4.30}$$

we obtain

$$\frac{a}{2}\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}F_n(r)\right]-\left(\frac{\pi n}{a}\right)^2F_n(r)\right\}=-\frac{q}{\varepsilon_0}\frac{1}{2\pi r}\delta(r)\sin\left(\frac{\pi n b}{a}\right)\,. \tag{6.4.31}$$

The solution of the homogeneous equation ($r \neq 0$) has the form

$$F_n(r) = A_n I_0\left(\frac{\pi n r}{a}\right) + B_n K_0\left(\frac{\pi n r}{a}\right) , \tag{6.4.32}$$

where $I_0(\xi)$ is the zero order modified Bessel function of the first kind; $K_0(\xi)$ is the zero order modified Bessel function of the second kind. For $\xi \gg 1$

$$I_0(\xi) \sim \frac{e^{\xi}}{\sqrt{\xi}} \qquad ; \qquad K_0(\xi) \sim \frac{e^{-\xi}}{\sqrt{\xi}} . \tag{6.4.33}$$

Since our charge is at $r = 0$, we conclude that $A_n = 0$ otherwise the solution diverges, implying that $F_n(r) = B_n K_0(\pi n r/a)$. In order to establish B_n we integrate (6.4.31) in the vicinity of $r \to 0$

$$\begin{aligned} \lim_{\Delta\to 0} \frac{a}{2}\int_0^{\Delta} dr\, r\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}F_n(r) &= -\frac{q}{\epsilon_0}\frac{1}{2\pi}\int\limits_0^{\Delta} dr\, r\frac{1}{r}\delta(r)\sin\left(\frac{\pi n b}{a}\right) \\ &= -\frac{q}{\epsilon_0}\frac{1}{2\pi}\sin\left(\frac{\pi n b}{a}\right) \end{aligned} \tag{6.4.34}$$

$$B_n \lim_{\Delta\to 0}\int_0^{\Delta} dr\frac{d}{dr}r\frac{d}{dr}K_0\left(\frac{\pi n r}{a}\right) = -\frac{2}{a}\frac{q}{\varepsilon_0}\frac{1}{2\pi}\sin\left(\frac{\pi n b}{a}\right) \tag{6.4.35}$$

$$B_n \lim_{\Delta\to 0}\left[r\frac{d}{dr}K_0\left(\frac{\pi n r}{a}\right)\right|_0^{\Delta} = -\frac{2}{a}\frac{q}{\varepsilon_0}\frac{1}{2\pi}\sin\left(\frac{\pi n b}{a}\right) . \tag{6.4.36}$$

The modified Bessel function for small arguments ($\xi \ll 1$) varies as

$$K_0(\xi) \simeq -\ln(\xi) \tag{6.4.37}$$

consequently

$$-\xi \frac{d}{d\xi} \ln \xi = -\xi \frac{1}{\xi} = -1 \tag{6.4.38}$$

and assuming that *exactly* at $r = 0$ this value is zero (contemplate a uniform distribution of charge between $r = 0$ and $r = \Delta$) we have

$$B_n \lim_{\Delta \to 0} \left[\int_0^{\Delta} dr\, r \frac{d}{dr} K_0 \left(\frac{\pi n r}{a} \right) \right] = B_n(-1)\,. \tag{6.4.39}$$

Hence

$$B_n = \frac{2}{a} \frac{q}{\varepsilon_0} \frac{1}{2\pi} \sin\left(\frac{\pi n b}{a} \right)\,, \tag{6.4.40}$$

or

$$\phi(r, z) = \frac{q}{4\pi\varepsilon_0} \frac{4}{a} \sum_{n=1}^{\infty} \sin\left(\frac{\pi n b}{a} \right) \sin\left(\frac{\pi n z}{a} \right) K_0 \left(\frac{\pi n r}{a} \right)\,. \tag{6.4.41}$$

For large arguments the modified Bessel function varies as $K_0(\xi) \simeq e^{-\xi}$; consequently, we realize the immediate advantage of this *third* representation of the potential: the sum converges *rapidly*.

The potential in Eq. (6.4.41) contains the singularity expressed by Eq. (6.4.1). In order to prove that it is convenient to express the two trigonometric functions as

$$\sin\left(\pi n\,\frac{b}{a}\right)\sin\left(\pi n\,\frac{z}{a}\right) = \frac{1}{2}\left\{\cos\left[\frac{\pi n}{a}(z-b)\right] - \cos\left[\frac{\pi n}{a}(z+b)\right]\right\} . \tag{6.4.42}$$

For our purpose we shall consider only the first term (why?) and evaluate the limit $a \to \infty$ hence

$$\phi(r,z) = \frac{q}{4\pi\varepsilon_o}\lim_{a\to\infty}\left\{\frac{4}{a}\sum_{n=1}^{\infty}\frac{1}{2}\cos\left[\frac{\pi n}{a}(z-b)\right]\, K_0\left(\frac{\pi n}{a}\,r\right)\right\} . \tag{6.4.43}$$

Using the fact that

$$\int_0^{\infty} d\xi\,\cos(\xi V)K_0(\xi) = \frac{\pi}{2}\frac{1}{\sqrt{1+V^2}} \tag{6.4.44}$$

and that at the limit of $a \to \infty$ the sum may be replaced by integration we obtain

$$
\begin{aligned}
\phi &= \frac{q}{4\pi\varepsilon_0} \lim_{a\to\infty} \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\pi}{a} \cos\left[\frac{\pi}{a} n(z-b)\right] K_0\left(\frac{\pi}{a} nr\right) \\
&= \frac{q}{4\pi\varepsilon_0} \frac{2}{\pi} \int_0^\infty dk \cos\Big[k(z-b)\Big] K_0(kr) \\
&= \frac{q}{4\pi\varepsilon_0} \frac{2}{\pi} \frac{1}{r} \int_0^\infty d\xi \cos\left[\xi\left(\frac{z-b}{r}\right)\right] K_0(\xi) \\
&= \frac{q}{4\pi\varepsilon_0} \frac{2}{\pi} \frac{1}{r} \frac{\pi}{2} \frac{1}{\sqrt{1+\left(\frac{z-b}{r}\right)^2}} \\
&= \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2+(z-b)^2}} \,.
\end{aligned}
\tag{6.4.45}
$$

Exercise #1: Determine the total charge on each one of the plates $\left[\int_0^\infty d\xi \xi K_0(\xi) = 1\right]$.

Exercise #2: Calculate the force acting between two particles located at two arbitrary locations between the grounded plates. Compare your result with the force acting between the same point-charges when in free-space. Is there a difference? Explain!!

Exercise #3: The charge density of a charged loop of radius R located parallel to the electrodes at $z = b$ is given by

$$\rho(r,z) = q \frac{1}{2\pi r} \delta(r-R)\, \delta(z-b)\,. \tag{6.4.46}$$

(a) Show that the potential associated with this charge density is

$$\phi = \frac{q}{4\pi\varepsilon_0} \frac{4}{a} \sum_{n=1}^{\infty} \sin\left(\pi n \frac{b}{a}\right) \sin\left(\pi n \frac{z}{a}\right) \begin{cases} K_0\left(\pi n \frac{r}{a}\right) I_0\left(\pi n \frac{R}{a}\right) & r > R \\ I_0\left(\pi n \frac{r}{R}\right) K_0\left(\pi n \frac{R}{a}\right) & r < R \end{cases} \tag{6.4.47}$$

Note that $\frac{d}{d\xi} K_0(\xi) = -K_1(\xi)$ $\quad \frac{d}{d\xi} I_0(\xi) = I_1(\xi)$ and $I_0(\xi)\, K_1(\xi) + I_1(\xi)\, K_0(\xi) = \frac{1}{\xi}$.

Plot the contour-lines of constant potential.

(b) Show that the force acting on the loop when $\Delta = a/2$ is zero.

(c) Calculate the capacitance of the loop. Make sure that you substract the contribution of the self-field. Hint: consider the limit $a \to \infty$.

(d) What is the potential of an azimuthally symmetric charge distribution $\rho(r, z)$?

(e) What is the potential of an azimuthally symmetric charge distribution $\rho(r, z)$ if on the electrode at $z = a$ we impose a voltage $V_0 \neq 0$?

6.5. Reciprocity Theorem

The Lorentz reciprocity theorem is one of the most useful theorems in the solution of electromagnetic problems, since it may be used to deduce a number of fundamental properties of practical devices. It provides the basis for demonstrating the reciprocal properties of microwave circuits and for showing that the receiving and transmitting characteristics of antennas are the same.

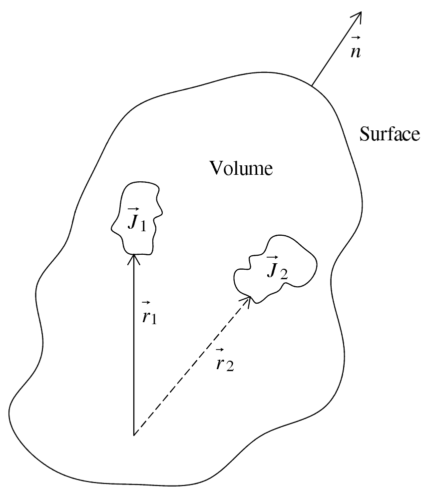

Figure 6.6: Two current densities generate an electromagnetic field $\vec{E}_1, \vec{H}_1)$ and $\vec{E}_2, \vec{H}_2)$ correspondingly.

To derive the theorem, consider a volume V bounded by a closed surface S as in Figure 6.6. Let a current source $\vec{J}_1$ in V produce a field $\vec{E}_1, \vec{H}_1$

$$\nabla \times \vec{E}_1 = -\partial_t \mu_0 \vec{H}_1 \qquad \nabla \times \vec{H}_1 = \partial_t \varepsilon_0 \vec{E}_1 + \vec{J}_1 \tag{6.5.1}$$

while a second source $\vec{J}_2$ produces a field $\vec{E}_2, \vec{H}_2$

$$\nabla \times \vec{E}_2 = -\partial_t \mu_0 \vec{H}_2 \qquad \nabla \times \vec{H}_2 = \partial_t \varepsilon_0 \vec{E}_2 + \vec{J}_2 \,. \tag{6.5.2}$$

Expanding the relation $\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1)$ and using Maxwell's equation show that

$$\begin{aligned} &\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \\ &\quad = (\nabla \times \vec{E}_1) \cdot \vec{H}_2 - (\nabla \times \vec{H}_2) \cdot \vec{E}_1 - (\nabla \times \vec{E}_2) \cdot \vec{H}_1 + (\nabla \times \vec{H}_1) \cdot \vec{E}_2 \\ &\quad = - \vec{J}_2 \cdot \vec{E}_1 + \vec{J}_1 \cdot \vec{E}_2 \,. \end{aligned} \tag{6.5.3}$$

Integrating both sides over the volume V and using Gauss' theorem

$$\begin{aligned} \int\limits_V \nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \, dV &= \oint\!\!\!\oint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot \vec{n} \, dS \\ &= \int\limits_V (\vec{E}_2 \cdot \vec{J}_1 - \vec{E}_1 \cdot \vec{J}_2) \, dV \,, \end{aligned} \tag{6.5.4}$$

where $\vec{n}$ is the unit outward normal to S.

There are at least two important cases where the surface integral vanishes: in case of radiating fields (we shall not discuss here) and in the case of quasi-state fields when $E \propto \dfrac{1}{r^2}$ and $H \propto \dfrac{1}{r^2}$. Since the surface of integration is proportional to r^2 at the limit $r \to \infty$ the surface integral clearly vanishes, therefore Eq. (6.5.4) reduces to

$$\int_V \vec{E}_1 \cdot \vec{J}_2 \, dV = \int_V \vec{E}_2 \cdot \vec{J}_1 \, dV \, . \tag{6.5.5}$$

If $\vec{J}_1$ and $\vec{J}_2$ are *infinitesimal* current elements (Dirac delta functions), then

$$\vec{E}_1(\mathbf{r}_2) \cdot \vec{J}_2(\mathbf{r}_2) = \vec{E}_2(\mathbf{r}_1) \cdot \vec{J}_1(\mathbf{r}_1) \, , \tag{6.5.6}$$

which states that the field $\vec{E}_1$ produced by $\vec{J}_1$ has a component along $\vec{J}_2$ that is equal to the component along $\vec{J}_1$ of the field generated by $\vec{J}_2$ when $\vec{J}_1$ and $\vec{J}_2$ have unit magnitude. The form (6.5.6) is essentially the reciprocity principle used in circuit analysis except that $\vec{E}$ and $\vec{J}$ are replaced by the voltage V and current I.

7. EM FIELD IN THE PRESENCE OF MATTER

In this chapter we consider to a certain extent the effect of matter on the field distribution and also to some degree the effect of the field on the distribution of matter. Although a variety of topics are presented, they constitute only a very small fraction of the array of phenomena engineers may encounter, and, therefore, we mainly emphasize for each case the conceptual approach.

7.1. Polarization Field

So far we have postulated that the relation between $\vec{D}$ and $\vec{E}$ is linear and given by the simple relation $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$ where ε_r stands for the relative dielectric coefficient. We consider in this section a simple model, which provides some basic insight as to the source of this result as well as its limitations.

Step # 1: A point-charge in free space generates a potential

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0 r} \, . \tag{7.1.1}$$

Two such point-charges (q_1, q_2) located at $(\vec{r}_1, \vec{r}_2)$ generate a potential which is a superposition of two single particle potentials namely,

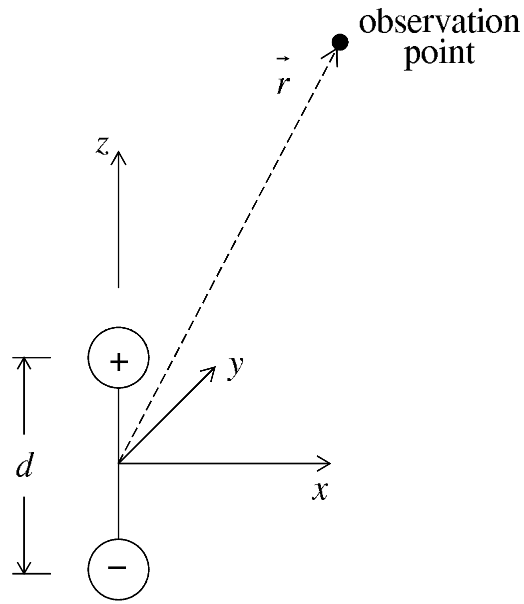

Figure 7.1: Two charges ($\pm q$) separated by a distance d form a dipole moment $p = qd$. Since the dipole moment is a vector in this particular case $\vec{p} = qd\vec{1}_z$.

$$\varphi(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{\sqrt{(x-x_1)^2+(y-y_1)+(z-z_1)^2}}$$

$$+\frac{q_2}{4\pi\varepsilon_0}\frac{1}{\sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2}}\,. \qquad (7.1.2)$$

For the sake of simplicity, we assume that $q_1 = -q_2 = q$ and that their coordinates are $x_1 = x_2 =$

$0\,,\; y_1 = y_2 = 0$ and $z_1 = d/2$, $z_2 = -d/2$ respectively – see Figure 7.1; hence

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+\left(z-\frac{d}{2}\right)^2}} - \frac{1}{\sqrt{x^2+y^2+\left(z+\frac{d}{2}\right)^2}}\right]\,. \tag{7.1.3}$$

Consider now an observer located at a distance $r = \sqrt{x^2+y^2+z^2}$ which is much larger than the distance d between the two charges ($r \gg d$). This observer will determine the scalar potential

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{\underbrace{x^2+y^2+z^2}_{r^2} - zd + \dfrac{d^2}{4}}} - \frac{1}{\sqrt{x^2+y^2+z^2+zd+\dfrac{d^2}{4}}}\right]$$

$$\simeq \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2-zd}} - \frac{1}{\sqrt{r^2+zd}}\right\}\,, \tag{7.1.4}$$

hence

$$\phi \simeq \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\frac{1}{\left(1-\frac{1}{2}\frac{zd}{r^2}\right)} - \frac{1}{r\left(1+\frac{1}{2}\frac{zd}{r^2}\right)}\right]$$

$$\simeq \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left[1+\frac{1}{2}\frac{zd}{r^2} - 1 + \frac{1}{2}\frac{zd}{r^2}\right] \simeq \frac{q}{4\pi\varepsilon_0}\frac{1}{r^3}\, zd\,. \tag{7.1.5}$$

In polar coordinates $z = r\cos\theta$; therefore

$$\phi \simeq \frac{qd}{4\pi\varepsilon_0}\frac{1}{r^2}\cos\theta\,; \tag{7.1.6}$$

the quantity $p \equiv qd$ is called the *dipole moment*. Contrary to the potential of a single charge, that of a dipole decays as r^{-2} (rather than r^{-1}); furthermore it is θ dependent.

Step # 2: In a more general case the potential of a charge distribution is a superposition of all charges

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{7.1.7}$$

as illustrated in Fig. 7.2. Here $\vec{r}$ represents the observer location whereas $\vec{r}\,'$ is the coordinate of the source.

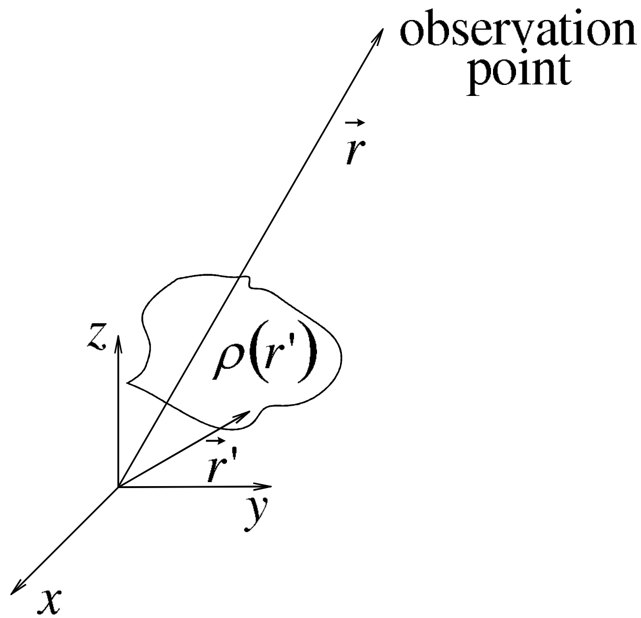

Figure 7.2: Potential of a distribution of charges.

If, as previously, we assume an observer positioned far from the charge distribution we may approximate (7.1.7) by the following expression

$$
\begin{aligned}
\varphi(\vec{r}) \simeq & \frac{1}{4\pi\varepsilon_0} \int dv' \, \frac{\vec{\rho}(r')}{r\left[1 - \dfrac{\vec{r} \cdot \vec{r}\,'}{r^2}\right]} \\
\simeq & \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int dv' \, \rho(\vec{r}\,') \left[1 + \frac{\vec{r} \cdot \vec{r}\,'}{r^2}\right] \\
\simeq & \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \left[\underbrace{\int dv' \rho(r')}_{\equiv q} + \frac{1}{r^2} \vec{r} \cdot \underbrace{\int dv' \, \rho(\vec{r}\,') \vec{r}\,'}_{\equiv \vec{p}} \right] .
\end{aligned} \tag{7.1.8}
$$

According to this result we generalize the concept of electric dipole-moment, and in the case of a general distribution, we define it as

$$
\boxed{\vec{p} \equiv \int dv' \rho(\vec{r}\,') \vec{r}\,'} \tag{7.1.9}
$$

hence

$$
\begin{aligned}
\varphi(\vec{r}) = & \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \left[q + \frac{\vec{p} \cdot \vec{r}}{r^2} \right] \\
= & \frac{1}{4\pi\varepsilon_0} \frac{q}{r} + \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \vec{p} \cdot \hat{r} \, .
\end{aligned} \tag{7.1.10}
$$

In many cases of interest the system is neutral ($q = 0$) (e.g. atom) and the only field we can measure is that of a dipole ignoring higher order terms in the expansion, i.e.

$$\varphi(\vec{r}) \simeq \frac{\vec{p} \cdot \vec{r}}{4\pi\varepsilon_0 r^3} \,. \tag{7.1.11}$$

Step # 3: We now consider an electron at the microscopic level even though we are dealing in our lectures only with macroscopic, phenomenological electrodynamics. Moreover, the analysis is limited to one dimension. In the absence of the external electric field this charge is assumed to be in equilibrium, i.e. no net force acts on it. As the external electric field is applied, the force which acts on it, is given by qE and since the particle moves a distance x from equilibrium (1D model) the equation of motion may be approximated by

$$m \frac{d^2 x(t)}{dt^2} = -kx(t) + qE \,, \tag{7.1.12}$$

where m stands for the mass of the charge, and kx represents the internal reaction force to the deviation (x); therefore the "new equilibrium" occurs when

$$x_0 = \frac{q}{k} E \,. \tag{7.1.13}$$

Based on the previous definition of the dipole moment we may use x_0 to define the equivalent dipole moment as

$$p = q\, x_0 \,. \tag{7.1.14}$$

Clearly, by substituting (7.1.13) in the latter one finds that this dipole is proportional to the local electric field

$$p = q\,x_0 = \varepsilon_0 \alpha E\,, \tag{7.1.15}$$

wherein $\alpha \equiv q^2/\varepsilon_0 k$ is a parameter that has units of volume and it is a measure of the electrostatic force relative to the internal forces represented by the coefficient k. In a macroscopically small volume the contribution of all the dipoles (p_i) determines the polarization field $P = \sum_i n_i p_i$ where n_i is the *volume density* of dipoles of type i'th

$$P = \sum_i n_i p_i = \sum_i n_i \varepsilon_0 \alpha_i E = \varepsilon_0 E \sum_i n_i \alpha_i\,, \tag{7.1.16}$$

wherein it was assumed that each type of dipole p_i is also characterized by a parameter α_i. Now postulating the relation between the polarization field (P) and the electric induction (D)

$$D = \varepsilon_0 E + P = \varepsilon_0 \left[E + \left(\sum_i n_i \alpha_i \right) E \right] = \varepsilon_0 \varepsilon_r\, E \tag{7.1.17}$$

therefore $\varepsilon_r = 1 + \chi_e = 1 + \sum n_i \alpha_i$; χ_e is called the electric susceptibility. It should be pointed out that the above relation has been derived subject to the assumption of a 1D system. Further assuming an isotropic material it is possible to generalize this result to a 3D system namely

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}\,. \tag{7.1.18}$$

Comment #1: There are cases where the dipoles are preferentially aligned in a specific direction and their net contribution to the polarization field is non-zero even in the absence of external electric fields: $P_0 = \sum_i n_i p_{0,i}$. In this case the material is considered to be permanently polarized.

Comment #2: The distribution of the dipoles in *space* is not necessarily *uniform*. In such a case the field equations may differ significantly. Consider for example Maxwell's equations when the dielectric coefficient is space dependent $\varepsilon(r)$

$$\begin{cases} \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\,\mu_0 \vec{H} \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\,\varepsilon_0 \varepsilon(\vec{r})\vec{E} \\ \vec{\nabla} \cdot \left[\varepsilon_0 \varepsilon(\vec{r})\vec{E}\right] = \rho \\ \vec{\nabla} \cdot \mu_0 \vec{H} = 0\,. \end{cases} \tag{7.1.19}$$

Clearly they look similar to the vacuum case however, the wave equation differs. In order to envision this fact let us introduce the magnetic vector potential $\vec{A}$, which defines the magnetic induction $\vec{B} = \vec{\nabla} \times \vec{A}$ and the scalar electric potential Φ, which together with $\vec{A}$ determines the electric field ($\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \nabla\Phi$). Assuming a Lorentz-like gauge

$$\vec{\nabla} \cdot \vec{A} + \frac{\varepsilon(\vec{r})}{c^2}\frac{\partial}{\partial t}\Phi = 0 \tag{7.1.20}$$

it can be shown that $\vec{A}$ and Φ satisfy

$$\left[\nabla^2 - \frac{\varepsilon(\vec{r})}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} - \left(\vec{\nabla}\cdot\vec{A}\right)\nabla\ln\varepsilon(\vec{r}) = -\mu_0\vec{J} \tag{7.1.21}$$

and

$$\left[\nabla^2 - \frac{\varepsilon(\vec{r})}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi - \vec{E}\cdot\nabla\ln\varepsilon(\vec{r}) = -\rho/\varepsilon_0\varepsilon(\vec{r})\,, \tag{7.1.22}$$

which differ from the case of a uniform dielectric medium.

Exercise: Prove (7.1.21)-(7.1.22) based on (7.1.19)-(7.1.20) and the definitions of $\vec{E}$ and $\vec{B}$ in terms of $\vec{A}$ and Φ.

Step # 4: In Eq. (7.1.17) the relation between the electric induction and the polarization was postulated: $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$. In fact, this relation can be concluded from the relation in (7.1.10). Let us assume that in a given region of space there is a free charge distribution of charge $\rho_f(\vec{r}\,')$ and a polarization field $\vec{P}(\vec{r}')$ however, contrary to the expressions in (7.1.10) these are *macroscopic* quantities

hence

$$\text{microscopic: :} \qquad \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3}\right]$$

$$\text{macroscopic:} \qquad \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d\vec{r} \left[\frac{\rho_f(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{\vec{P}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right] . \tag{7.1.23}$$

Based on this last expression, we observe that

$$\frac{\vec{P}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \vec{P}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = \vec{\nabla}'\left(\frac{\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) - \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\vec{\nabla}'\cdot\vec{P}(\vec{r}\,') \tag{7.1.24}$$

implying

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\vec{r}\,'\frac{\rho_1(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi\varepsilon_0}\int d\vec{r}\,'\frac{-\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{7.1.25}$$

since the polarization is assumed to vanish at infinity. From there it is evident that this is a potential generated by an effective charge distribution $\rho_{\text{eff}} = \rho_f - \vec{\nabla}\cdot\vec{P}$ since

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\vec{r}\,'\frac{1}{|\vec{r}-\vec{r}\,'|}\left[\rho_f(\vec{r}\,') - \vec{\nabla}'\cdot\vec{P}(\vec{r}\,')\right] . \tag{7.1.26}$$

Bearing in mind that in a static regime $\vec{E} = -\nabla\phi$ we conclude

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(\rho_f - \vec{\nabla}\cdot\vec{P}) \Rightarrow \nabla\cdot[\varepsilon_0\vec{E} + \vec{P}] = \rho_f \tag{7.1.27}$$

implying

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}\,, \tag{7.1.28}$$

which is exactly the vector form of the relation in a left-hand side of (7.1.17).

7.2. Free and Bound Charge

Gauss' law states that the divergence of the electric displacement equals the free-charge density ρ i.e.

$$\vec{\nabla} \cdot \vec{D} = \rho \; . \tag{7.2.1}$$

In the previous sections it was shown that

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{7.2.2}$$

therefore

$$\vec{\nabla} \cdot \varepsilon_0 \vec{E} = \rho - \vec{\nabla} \cdot \vec{P} \, . \tag{7.2.3}$$

This result indicates that the divergence of the electric field has two sources: the free charge density ρ and the bound (polarization) charge density ρ_p,

$$\vec{\nabla} \cdot \varepsilon_0 \vec{E} = \rho + \rho_{\mathrm{p}} \, , \tag{7.2.4}$$

where

$$\rho_{\mathrm{p}} \equiv -\vec{\nabla} \cdot \vec{P} \, . \tag{7.2.5}$$

This result has implications on the boundary conditions since in the past we have shown that

$$\vec{1}_n \cdot \left(\vec{D}^{(a)} - \vec{D}^{(b)} \right) = \rho_s \, , \tag{7.2.6}$$

therefore we can now define the polarization charge surface density ($\rho_{s,p}$) by means of which we are now able to write

$$\vec{1}_n \cdot \left(\varepsilon_0 \vec{E}^{(a)} - \varepsilon_0 \vec{E}^{(b)} \right) = \rho_s + \rho_{s,p} \,. \tag{7.2.7}$$

Note that in terms of the polarization field in the two regions $\rho_{s,p}$ is being expressed as

$$\rho_{s,p} = -\vec{1}_n \cdot \left(\vec{P}^{(a)} - \vec{P}^{(b)} \right) \;. \tag{7.2.8}$$

7.3. Artificial Dielectrics

We have previously indicated [Section 7.1] that the macroscopic dielectric coefficient may be interpreted as a superposition of *microscopic* dipoles. In principle we can "construct" macroscopic dipoles and build in this way an artificial dielectric material. Let us assume a series of metallic spheres embedded in a dielectric medium ε_r. For simplicity we shall assume that the radius of the spheres is much smaller than the distance between them. In this way it is possible to determine the field in the vicinity of a single sphere and extrapolate for an entire ensemble of spheres.

In order to commence with the analysis, we assume a *grounded* sphere of radius R surrounded by a dielectric material ε_r, residing in a space permeated by an imposed uniform, constant electric field, $E_z = E_0$.

In absence of the sphere the "primary" (superscript p) potential is

$$\phi^{(\mathrm{p})} = -E_0 z = -E_0 r \cos\theta \, . \tag{7.3.1}$$

The presence of the metallic sphere generates a secondary field which is a solution of the partial differential equation

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \, . \tag{7.3.2}$$

Since both the initial field as well as the sphere exhibit azimuthal symmetry, we assume the same for the secondary field (superscript s), i.e. $\dfrac{\partial}{\partial\varphi} \sim 0$. The solution outside the sphere is readily checked to read

$$\phi^{(\mathrm{s})} = A\,\frac{1}{r^2}\,\cos\theta\,. \tag{7.3.3}$$

The coefficient A is determined by the condition that the *total* potential at $r = R$ is zero (grounded sphere), thus

$$\phi(r = R, \theta) = \phi^{(\mathrm{p})}(r = R, \theta) + \phi^{(\mathrm{s})}(r = R, \theta) = 0 \qquad \Rightarrow$$

$$-E_0 R\cos\theta + (A/R^2)\,\cos\theta = 0 \quad \Rightarrow \quad A = E_0 R^3\,, \tag{7.3.4}$$

hence

$$\phi^{(s)} = E_0 R\left(\frac{R}{r}\right)^2 \cos\theta\,. \tag{7.3.5}$$

In (7.1.6) we have shown that the potential of a dipole positioned at the origin of a system of spherical coordinates reads

$$\phi^{(s)} = \frac{p}{4\pi\varepsilon_0\varepsilon_r}\,\frac{\cos\theta}{r^2} \tag{7.3.6}$$

and therefore, we conclude that this sphere "generates" an effective dipole moment

$$p = 4\pi\varepsilon_0\varepsilon_r\,E_0 R^3\,. \tag{7.3.7}$$

In order to determine the polarization field we assume that the average distance between spheres (dipoles) is d, which implies that the average volume density of the dipoles n is of the order of

$$n \simeq \frac{1}{d^3} \,. \tag{7.3.8}$$

Consequently, the intensity of the polarization P is

$$P = np = \frac{1}{d^3}\left(4\pi\varepsilon_0\varepsilon_r\, E_0 R^3\right) = 4\pi\varepsilon_0\varepsilon_r \left(\frac{R}{d}\right)^3 E_0 \tag{7.3.9}$$

and finally we may estimate the contribution to the electric displacement ($\vec{D}$) is given by

$$\begin{aligned} \vec{D} &= \varepsilon_0\varepsilon_r\, \vec{E}_0 + \vec{P}_{\text{spheres}} \\ &= \varepsilon_0\varepsilon_r\, \vec{E}_0 + 4\pi\varepsilon_0\varepsilon_r \left(\frac{R}{d}\right)^3 \vec{E}_0 \\ &= \varepsilon_0\varepsilon_r \left[1 + 4\pi\left(\frac{R}{d}\right)^3\right] \vec{E}_0 = \varepsilon_0\varepsilon_{\text{eff}}\vec{E}_0 \,, \end{aligned} \tag{7.3.10}$$

which implies that we obtain a small artificial increase in the effective dielectric coefficient of the material:

$$\boxed{\varepsilon_{\text{eff}} \equiv \varepsilon_r \left[1 + 4\pi\left(\frac{R}{d}\right)^3\right]}$$

For comparison, if instead of metallic spheres we consider air bubbles, the outer solution has the same form:

$$\phi_{\text{out}} = A\,\frac{1}{r^2}\,\cos\theta \tag{7.3.11}$$

but the inner reads

$$\phi_{\text{in}} = B\,r\cos\theta\,. \tag{7.3.12}$$

Continuity of the E_θ at $r = R$ implies

$$E_\theta = -\frac{1}{r}\,\frac{\partial\phi}{\partial\theta} \Rightarrow -E_0\sin\theta + \frac{A}{R^3}\,\sin\theta = B\sin\theta\,, \tag{7.3.13}$$

hence

$$B = \frac{A}{R^3} - E_0\,. \tag{7.3.14}$$

In a similar way the continuity of the radial component of the displacement is

$$(E_0\cos\theta + \frac{2}{R^3}\,A\cos\theta)\varepsilon_0\varepsilon_r \;= -B\cos\theta\varepsilon_0\,, \tag{7.3.15}$$

hence

$$-B = \left(E_0 + 2\frac{A}{R^3}\right)\,\varepsilon_r\,. \tag{7.3.16}$$

From Eqs. (7.3.14) and (7.3.16) we find

$$A = -R^3E_0\,\frac{\varepsilon_r - 1}{2\varepsilon_r + 1}\,. \tag{7.3.17}$$

The potential is therefore

$$\phi^{(s)}(r > R) = -E_0 R \, \frac{\varepsilon_r - 1}{2\varepsilon_r + 1} \, \frac{R^2}{r^2} \, \cos\theta \,, \tag{7.3.18}$$

implying that the dipole moment of the bubble is

$$p = -4\pi\varepsilon_0\varepsilon_r \; E_0 R \, \frac{\varepsilon_r - 1}{2\varepsilon_r + 1} \, R^2 \,. \tag{7.3.19}$$

Using the same approximations as in (7.3.8) we are able to estimate the intensity of the polarization:

$$P = np = +\frac{1}{d^3} \left[(-4\pi)\varepsilon_0\varepsilon_r \; E_0 R \, \frac{\varepsilon_r - 1}{2\varepsilon_r + 1} \, R^2 \right] \,, \tag{7.3.20}$$

which finally implies that the effective dielectric coefficient is smaller than ε_r

$$\begin{aligned} \vec{D} &= \varepsilon_0\varepsilon_r \; \vec{E}_0 + \vec{P} \\ &= \varepsilon_0\varepsilon_r \; \vec{E}_0 + (-4\pi)\varepsilon_0\varepsilon_r \; \vec{E}_0 \left(\frac{R}{d}\right)^3 \frac{\varepsilon_r - 1}{2\varepsilon_r + 1} \\ &= \varepsilon_0\varepsilon_r \; \left[1 - 4\pi \frac{\varepsilon_r - 1}{2\varepsilon_r + 1} \left(\frac{R}{d}\right)^3 \right] \vec{E}_0 \,, \end{aligned} \tag{7.3.21}$$

hence

$$\boxed{\varepsilon_{\text{eff}} \equiv \varepsilon_r \; \left[1 - 4\pi \, \frac{\varepsilon_r - 1}{2\varepsilon_r + 1} \left(\frac{R}{d}\right)^3 \right].}$$

7.4. Unisotropic Medium

In the first section it was indicated that applying an electric field may "induce" a dipole moment p that in the framework of the one dimensional theory employed, was written as (Eq. (7.1.5)) $p = \varepsilon_0 \alpha E$. Extending this approach to a full three dimensional analysis, we may conceive a situation in which the reaction of a material to an applied field is not identical in all directions. For example, applying a field of the same intensity in one direction (x) will induce a dipole p_x, whereas the same field intensity when applied in the z direction induces p_z and $p_x \neq p_z$. Moreover, a field component in one direction E_x may induce a dipole in the z-direction i.e. p_z. Consequently, in the framework of a linear theory a general constitutive relation describing a *linear unisotropic* (but non-polarized) material is given by

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \varepsilon_0 \underbrace{\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}}_{\underline{\underline{\varepsilon}}} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} . \tag{7.4.1}$$

For envisioning the field distribution in unisotropic materials, let us examine a simple case of a point charge $(-q)$ in a uniform but unisotropic medium. For the sake of simplicity we assume that the dielectric matrix is

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} , \tag{7.4.2}$$

wherein $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{\perp} \quad \varepsilon_{zz} = \varepsilon_{\parallel}$. Bearing in mind that in the static case Faraday's law reads $\vec{\nabla} \times \vec{E} = 0$ we may assume that this field may be derived from a static electric potential

$$\vec{E} = -\nabla\phi \tag{7.4.3}$$

therefore, with Gauss' law

$$\vec{\nabla} \cdot \vec{D} = \rho \Rightarrow \partial_x \varepsilon_{\perp} E_x + \partial_y \varepsilon_{\perp} E_y + \partial_z \varepsilon_{\perp} E_z = \rho/\varepsilon_0 \tag{7.4.4}$$

we conclude that this potential satisfies

$$\varepsilon_{\perp}\left(\frac{\partial^2}{\partial x^2}\,\phi + \frac{\partial^2}{\partial y^2}\,\phi\right) + \varepsilon_{\parallel}\frac{\partial^2}{\partial z^2}\,\phi = -\rho/\varepsilon_0\,. \tag{7.4.5}$$

This equation may be readily brought to the regular form of the Poisson equation by first dividing the last equation by $\varepsilon_{\perp}$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\varepsilon_{\parallel}}{\varepsilon_{\perp}}\,\frac{\partial^2\phi}{\partial z^2} = +\frac{q}{\varepsilon_0\varepsilon_{\perp}}\,\delta(x)\,\delta(y)\,\delta(z) \tag{7.4.6}$$

and further defining $\xi \equiv z\sqrt{\varepsilon_{\perp}/\varepsilon_{\parallel}}$

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} &= \frac{q}{\varepsilon_0\varepsilon_{\perp}}\,\delta(x)\,\delta(y)\,\delta\Big(\xi\sqrt{\varepsilon_{\parallel}/\varepsilon_{\perp}}\,\Big)\\ &= \frac{q}{\varepsilon_0\varepsilon_{\perp}}\,\frac{1}{\sqrt{\dfrac{\varepsilon_{\parallel}}{\varepsilon_{\perp}}}}\,\delta(x)\,\delta(y)\,\delta(\xi)\,. \qquad (7.4.7)\end{aligned}$$

This result indicates that within the framework of the "new" coordinate system (x, y, ξ), we have an effective charge $q_{\text{eff}} = q/\sqrt{\varepsilon_\perp \varepsilon_\parallel}$ with the potential satisfying the Poisson equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \xi^2}\right) \phi = \frac{q_{\text{eff}}}{\varepsilon_0}\, \delta(x)\, \delta(y)\, \delta(\xi)\,. \tag{7.4.8}$$

Consequently, from the solution of this equation in isotropic space

$$\Phi(x, y, \xi) = \frac{-q_{\text{eff}}}{4\pi\varepsilon_0 \sqrt{x^2 + y^2 + \xi^2}}\,, \tag{7.4.9}$$

or

$$\Phi(x, y, z) = -\frac{q}{4\pi\varepsilon_0 \sqrt{\varepsilon_\perp \varepsilon_\parallel}} \frac{1}{\sqrt{x^2 + y^2 + \left(\dfrac{\varepsilon_\perp}{\varepsilon_\parallel}\right) z^2}}\,. \tag{7.4.10}$$

Note that contours of a constant potential are ellipsoids

$$\left(\frac{x}{\sqrt{\varepsilon_\perp}}\right)^2 + \left(\frac{y}{\sqrt{\varepsilon_\perp}}\right)^2 + \left(\frac{z}{\sqrt{\varepsilon_\parallel}}\right)^2 = \left(\frac{q}{4\pi\varepsilon_0 \sqrt{\varepsilon_\perp}} \frac{1}{\phi}\right)^2\,. \tag{7.4.11}$$

Exercise #1: What is the potential if the same charge is located at a height h from a grounded electrode.

Exercise #2: Based on Maxwell equations show that if the matrix $\underline{\underline{\varepsilon}}$ is symmetric i.e. $\varepsilon_{ij} = \varepsilon_{ji}$, then the energy density is given by

$$W_E = \frac{1}{2}\,\varepsilon_0\,\vec{E}\underline{\underline{\varepsilon}}\vec{E}\,. \tag{7.4.12}$$

7.5. Electro-Quasi-Statics in the Presence of Matter

The purpose of this section is to illustrate a quasi-dynamic effect associated with the slow motion of a dielectric layer; the term "slow" implies here that the velocity is much smaller than c.

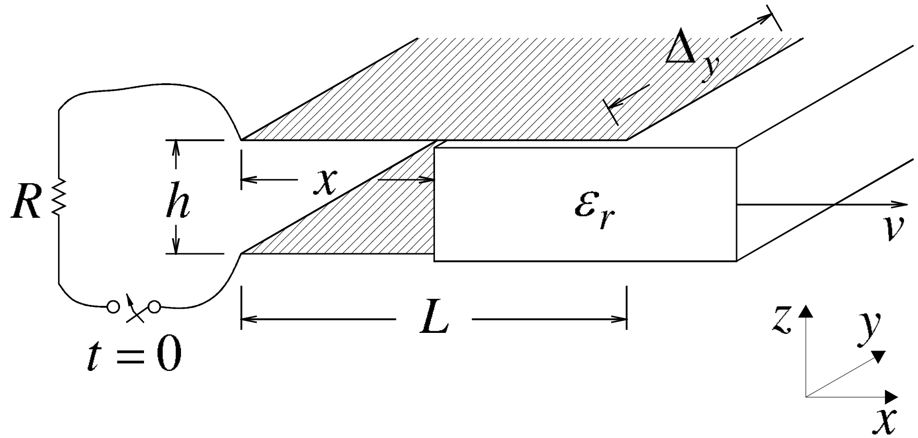

Figure 7.3: A dielectric layer moves between the two plates of a capacitor.

At $t = 0$ the capacitor - see Figure 7.3 – is charged so that $V_c(t = 0) = V_0$. The question posed is: what is the velocity (v) of the dielectric slab (ε_r) in the x-direction which leaves the voltage across the capacitor unchanged.

In order to solve this problem we first have to find an adequate description of its electro-quasi-static characteristics; we start from the calculation of the total electric energy stored between the two metallic plates. Since the z component of the electric field $\left(E_z \sim \dfrac{V_c}{h}\right)$ is continuous, and, further ignoring edge effects, we have

$$W_E = \frac{1}{2}\,\varepsilon_0 \left(\frac{V_c}{h}\right)^2 x\,h\,\Delta_y + \frac{1}{2}\,\varepsilon_0\varepsilon_r \left(\frac{V_c}{h}\right)^2 (L-x)h\Delta_y\,. \tag{7.5.1}$$

This expression determines the capacitance of the system as $W_E = \frac{1}{2}C\,V_c^2$ and therefore

$$C(x) = \varepsilon_0\,\frac{\Delta_y}{h}\,\left[x + \varepsilon_r(L-x)\right]\,. \tag{7.5.2}$$

The equivalent circuit of the system is a capacitor $C(x)$ in series with a resistor R, subject to the initial condition ($x = 0$ at $t = 0$) as well as $V_c(t = 0) = V_0$. The equation describing the voltage dynamics in this circuit is

$$V_c + IR = 0\,, \tag{7.5.3}$$

wherein I is the current linked to the time-variation of charge stored in the capacitor, i.e. $I = dQ/dt$, hence

$$V_c = -R\,\frac{dQ}{dt} = -R\,\frac{d}{dt}\,(C\,V_c)\,. \tag{7.5.4}$$

Here we observe for the first time that capacitance *may vary in time*. Explicitly (7.5.4) reads

$$RV_c\left(\frac{1}{R} + \frac{dC}{dt}\right) + CR\,\frac{dV_c}{dt} = 0\,. \tag{7.5.5}$$

At this point it is interesting to observe two facts:

1. Time variations in the capacitance play a similar role in the circuit as conductance $(1/R)$.

2. Contrary to conductance, which is always positive, $\dfrac{dC}{dt}$ may be negative. This implies that capacitance variations may cause energy to be transferred to the circuit (recall that an ordinary passive resistor always dissipates power!!).

We may now use the explicit expression (7.5.2) for the capacitance and, further bearing in mind that the dielectric slab moves, i.e. x varies in time, we may write

$$\frac{dC}{dt} = \varepsilon_0 \frac{\Delta_y}{h} \left[\frac{dx}{dt} + \varepsilon_r(-)\frac{dx}{dt} \right] = -\varepsilon_0(\varepsilon_r - 1)\frac{\Delta_y}{h} \frac{dx}{dt} \; ; \tag{7.5.6}$$

Further, for simplicity and reality we assume at the outset that the slab moves at a constant velocity (v), being much smaller than the speed of light $(v \ll c)$, hence Eq. (7.5.5) may be explicitly written as

$$\left[1 - R\varepsilon_0(\varepsilon_r - 1) \frac{\Delta_y}{h} v \right] V_c + R \left[C_0 - C_1 \frac{vt}{L} \right] \frac{dV_c}{dt} = 0 \, , \tag{7.5.7}$$

wherein

$$\begin{aligned} C_0 &= \varepsilon_0 \varepsilon_r \, \frac{\Delta_y L}{h} \, , \\ C_1 &= \varepsilon_0(\varepsilon_r - 1)\frac{\Delta_y L}{h} \, . \end{aligned} \tag{7.5.8}$$

The requirement of constant voltage $\left(\frac{dV_c}{dt} = 0\right)$ clearly implies that the first term square brackets on the left hand side of (7.5.7) must vanish, i.e.

$$1 - R\varepsilon_0(\varepsilon_r - 1)\,\frac{\Delta_y}{h}\,v = 0 \tag{7.5.9}$$

so that

$$v = c\,\frac{1}{\varepsilon_r - 1}\,\frac{h}{\Delta_y}\,\frac{\sqrt{\frac{\mu_0}{\varepsilon_0}}}{R}\,. \tag{7.5.10}$$

For $\varepsilon_r \sim 30\,,\quad h/\Delta_y \sim 10^{-3}\,,\quad \frac{1}{R}\sqrt{\frac{\mu_0}{\varepsilon_0}} \sim \frac{120\pi}{4 \times 10^6} \sim 10^{-4}$ we obtain $v \sim 1\text{m/sec}$. When the condition in Eq. (7.5.9) is not satisfied the equation may be rewritten in the form

$$\frac{dV_c}{dt}\,\frac{R\left(C_0 - C_1\,\frac{vt}{L}\right)}{1 - R\varepsilon_0(\varepsilon_r - 1)\,\frac{\Delta_y\,v}{h}} + V_c = 0\,. \tag{7.5.11}$$

We now define a new variable

$$\tau \equiv \frac{t}{RC_0}\left[1 - R\varepsilon_0(\varepsilon_r - 1)\,\frac{\Delta_y\,v}{h}\right]\,, \tag{7.5.12}$$

as well as the coefficient

$$\xi \equiv \frac{C_1}{C_0}\,\frac{v}{L}\,\left\{\frac{1}{RC_0}\left[1 - R\varepsilon_0(\varepsilon_r - 1)\,\frac{\Delta_y}{h}\,v\right]\right\}^{-1}\,. \tag{7.5.13}$$

With these definitions (7.5.11) reads

$$\frac{dV_c}{d\tau}\,(1 - \xi\tau) + V_c = 0$$

yielding therefore the solution

$$\begin{aligned} \frac{dV_c}{V_c} &= -\frac{d\tau}{1 - \xi\tau} \\ \rightsquigarrow \ln\, V_c &= \frac{1}{\xi}\,\ln\,(1 - \xi\tau) \\ V_c(\tau) &= V_c(0)\,\left[1 - \xi\tau\right]^{1/\xi} \end{aligned}$$

$$\boxed{V_c(t) = V_c(0)\,\left[1 - \frac{C_1}{C_0}\frac{vt}{L}\right]^{1/\xi}}\,. \tag{7.5.14}$$

Note that according to the sign of ξ i.e. $1 - \dfrac{R}{\eta_0}\,(\varepsilon_r - 1)\,\dfrac{\Delta_y}{h}\,\dfrac{v}{c}$ the voltage across the capacitor may increase or decrease. In case the velocity is below its critical value determined by (7.5.10), i.e. $\xi > 0$, the voltage decreases whereas above this value, it actually increases. The former case implies dissipated power whereas the latter implies that there is net energy conversion from kinetic energy (the slab) into electric energy. A similar approach may be resorted to in the synthesis and in the analysis of micro-mechanical detectors.

Exercise: Calculate the electric energy stored in the capacitor, the power dissipated in the resistor and the power provided by the moving dielectric slab.

7.6. Magnetization Field

7.6.1. Dipole Moment

For the magnetic field we follow an approach similar to that introduced in Section 7.1. The main difference in this case is the experimental lack of a magnetic monopole. As a result, the magnetic field depends on the electric current density whose divergence in quasi-statics is zero namely, $\vec{\nabla} \cdot \vec{J} = 0$. In a boundless space the magnetic vector potential, in Cartesian coordinates, is given by

$$\vec{A}(x, y, z) = \frac{\mu_0}{4\pi} \int dx'dy'dz' \, \frac{\vec{J}(x', y', z')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \,. \tag{7.6.1}$$

Let us consider for simplicity a rectangular loop in the z-plane, its dimensions being $a_x \times a_y$. Explicitly the current density is given by

$$\begin{aligned} J_x(x, y, z) &= I\delta(z) \left[\delta\Big(y + \frac{a_y}{2}\Big) - \delta\Big(y - \frac{a_y}{2}\Big)\right] \\ J_y(x, y, z) &= I\delta(z) \left[\delta\Big(x - \frac{a_x}{2}\Big) - \delta\Big(x + \frac{a_x}{2}\Big)\right] . \end{aligned} \tag{7.6.2}$$

Based on the previous relation, the relevant magnetic vector potential reads

$$
\begin{aligned}
A_x(x,y,z) &= \frac{\mu_0 I}{4\pi}\int_{-a_x/2}^{a_x/2} dx' \left[\frac{1}{\sqrt{(x-x')^2+\left(y+\frac{a_y}{2}\right)^2+z^2}} - \frac{1}{\sqrt{(x-x')^2+\left(y-\frac{a_y}{2}\right)^2+z^2}}\right] \\
&\simeq \frac{\mu_0 I}{4\pi r}\int_{-a_x/2}^{a_x/2} dx' \left[\left(1+\frac{xx'}{r^2}-\frac{1}{2}\frac{y a_y}{r^2}\right)-\left(1+\frac{xx'}{r^2}+\frac{1}{2}\frac{y a_y}{r^2}\right)\right] \\
&\simeq \frac{\mu_0 I}{4\pi r}\left(-\frac{y a_y}{r^2}\right) a_x = -\frac{\mu_0 m_z y}{4\pi r^3}\,,
\end{aligned}
\tag{7.6.3}
$$

where we have defined here the *magnetic dipole moment* as

$$
m_z \equiv I a_x a_y\,. \tag{7.6.4}
$$

In a similar way

$$
\begin{aligned}
A_y(x,y,z) &= \frac{\mu_0 I}{4\pi}\int_{-a_y/2}^{a_y/2} dy' \left[\frac{1}{\sqrt{\left(x-\frac{a_x}{2}\right)^2+\left(y-y'\right)^2+z^2}} - \frac{1}{\sqrt{\left(x+\frac{a_x}{2}\right)^2+\left(y-y'\right)^2+z^2}}\right] \\
&\simeq \frac{\mu_0 I}{4\pi r}\int_{-a_y/2}^{a_y/2} dy' \left[\left(1+\frac{1}{2}\,\frac{x a_x}{r^2}+\frac{yy'}{r^2}\right)-\left(1-\frac{1}{2}\,\frac{x a_x}{r^2}+\frac{yy'}{r^2}\right)\right] \\
&\simeq \frac{\mu_0 I}{4\pi r}\,\frac{a_x x}{r^2}\,a_y = \frac{\mu_0 m_z x}{4\pi r^3}
\end{aligned}
\tag{7.6.5}
$$

and finally

$$A_z(x,y,z) = 0\,. \tag{7.6.6}$$

Based upon these results we find that

$$\vec{A} = \frac{\mu_0}{4\pi r^3}\,\vec{m}\times\vec{r}\,, \tag{7.6.7}$$

and the definition of the $\vec{m}$, the *magnetic dipole moment* can be generalized to its vector form

$$\vec{m} = \frac{1}{2}\int dx'dy'dz'\,\vec{r'}\times\vec{J}(\vec{r'})\,. \tag{7.6.8}$$

Relying on (7.6.7) the magnetic induction

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left[\frac{\mu_0}{4\pi r^3} \vec{m} \times \vec{r} \right]$$
$$= \frac{\mu_0}{4\pi} \frac{3\vec{1}_r (\vec{m} \cdot \vec{1}_r) - \vec{m}}{r^3} . \tag{7.6.9}$$

In our particular case $\vec{m} = m_z \hat{z}$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3\vec{1}_r - \vec{1}_z}{r^3} m_z \Rightarrow \begin{cases} B_x = & \dfrac{\mu_0 m_z}{4\pi} \dfrac{3xz}{r^5} \\ B_y = & \dfrac{\mu_0 m_z}{4\pi} \dfrac{3yz}{r^5} \\ B_z = & \dfrac{\mu_0 m_z}{4\pi} \dfrac{2z^2 - x^2 - y^2}{r^5} \end{cases} , \tag{7.6.10}$$

consequently, the magnetic induction components form an ellipsoid

$$\left(\frac{B_x}{3} \right)^2 + \left(\frac{B_y}{3} \right)^2 + \left(\frac{B_z}{2} \right)^2 = \left(\frac{1}{2} \frac{\mu_0 m_z}{4\pi} \frac{1}{r^3} \right)^2 . \tag{7.6.11}$$

Exercise #1: Plot the contours of constant magnetic field in the various planes.

Exercise #2: In order to avoid a breakdown, the secondary parts of a transformer are 30 cm apart. Assuming that the wires form a square loop carrying 500 Amp of current, show that the magnetic induction 10 meters away is of the order 8×10^{-8} [T], or 0.08 [mG].

7.6.2. Force on a Magnetic Dipole

The Lorentz force acting on a negative electric charge moving in an *external* field (B) is given by

$$\vec{F} = -q\vec{v} \times \vec{B}\,. \tag{7.6.12}$$

For a distribution of particles of density n, the force density is given by

$$\vec{f} = -(qn\vec{v}) \times \vec{B}\,. \tag{7.6.13}$$

Bearing in mind that the current density linked to a flow of charge is defined by $\vec{J} = -qn\vec{v}$, we have for the force on the entire ensemble

$$\vec{F} = \int dx'dy'dz'\,\vec{J}(x',y',z') \times \vec{B}(x',y',z')\,. \tag{7.6.14}$$

Assuming that the magnetic field varies slowly in the region of the current distribution we may expand, obtaining

$$\begin{aligned}\vec{B}(x',y',z') = \vec{B}(0) &+ \vec{r}' \cdot (\nabla B_x)_{\vec{r}'=0}\,\vec{1}_{x'}\\ &+ \vec{r}' \cdot (\nabla B_y)_{\vec{r}'=0}\,\vec{1}_{y'}\\ &+ \vec{r}' \cdot (\nabla B_z)_{\vec{r}'=0}\,\vec{1}_{z'}\,.\end{aligned} \tag{7.6.15}$$

For dc or quasi-static currents (flowing in closed loops) the integral over the current density is zero i.e. $\int dx'dy'dz'\vec{J}(x',y',z') = 0$ therefore,

$$\vec{F} = \int dx'dy'dz'\vec{J}(x',y',z') \times \Big\{ \vec{r}' \cdot (\nabla B_x)_{\vec{r}'=0} \vec{1}_{x'} + \vec{r}' \cdot (\nabla B_y)_{\vec{r}'=0} \vec{1}_{y'} + \vec{r}' \cdot (\nabla B_z)_{\vec{r}'=0} \vec{1}_{z'} \,. \qquad (7.6.16)$$

Following each component separately we find (using the definition of $\vec{m}$ and the fact that $\vec{\nabla} \cdot \vec{B} = 0$)

$$\begin{aligned} \vec{F} &= \left(\vec{m} \times \vec{\nabla}\right) \times \vec{B} = \vec{\nabla}\left(\vec{m} \cdot \vec{B}\right) - \vec{m}\left(\nabla \cdot \vec{B}\right) \\ &= \nabla\left(\vec{m} \cdot \vec{B}\right) . \end{aligned} \qquad (7.6.17)$$

This directly implies that the mechanical energy associated with the presence of a magnetic dipole moment in the magnetic field:

$$\mathcal{E} = -\vec{m} \cdot \vec{B} \,, \qquad (7.6.18)$$

since

$$\vec{F} = -\nabla \mathcal{E} \,. \qquad (7.6.19)$$

Exercise #1: Prove the steps that lead to (7.6.17).

Exercise #2: Show in a similar way that for an electrostatic dipole moment in an electric field the energy is $\mathcal{E} = -\vec{p} \cdot \vec{E}$.

7.6.3. Bound Magnetic Charge

If we assume that at the *microscopic* level a magnetic dipole moment is represented by $\vec{m}_i$, then the magnetization

$$\vec{M} = \sum_i n_i \, \vec{m}_i \tag{7.6.20}$$

with this definition, and similar to Gauss' law and the electric polarization, we may define the magnetic induction as

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \, , \tag{7.6.21}$$

where $\vec{M}$ is the magnetization and we recall that

$$\vec{\nabla} \cdot \vec{B} = 0 \; . \tag{7.6.22}$$

Consequently

$$\vec{\nabla} \cdot \mu_0 \vec{H} = -\vec{\nabla} \cdot \mu_0 \vec{M} \tag{7.6.23}$$

and it is therefore convenient to define the magnetization charge density (ρ_{m}):

$$\rho_{\mathrm{m}} = -\vec{\nabla} \cdot \mu_0 \vec{M} \tag{7.6.24}$$

and, correspondingly, the magnetization charge surface density $\rho_{s,m}$ is

$$\rho_{s,m} = -\vec{1}_n \cdot \mu_0 \left(\vec{M}^{(a)} - \vec{M}^{(b)} \right) \; . \tag{7.6.25}$$

7.6.4. Magneto-Quasi-Statics in the Presence of Matter

Magnetization fields are widely used for data storage. In this section we consider a simple model for the process of signal retrieval from a pre-magnetized tape as well as the electromagnetic process associated with it. A schematic drawing of the system is illustrated in Figure 7.4:

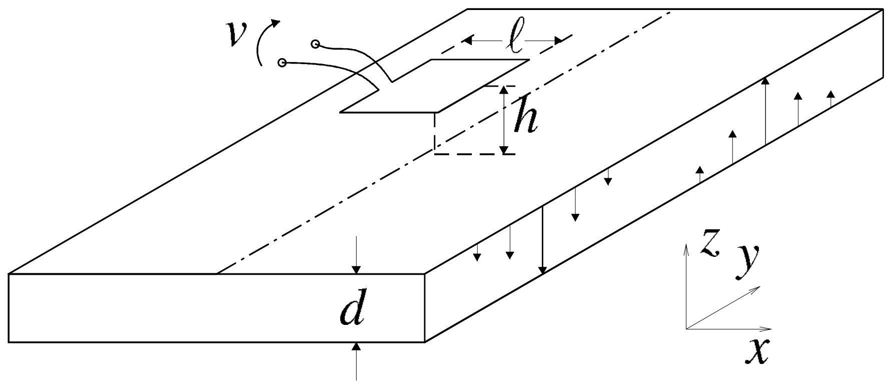

Figure 7.4: The magnetic flux imprinted in a moving tape is measured as voltage by a suspended loop.

The magnetization of a *motionless* thin tape (thickness d) is in the z-direction and is taken to be periodic along the y-axis:

$$\vec{M} = M_0 \cos(ky)\, \vec{1}_z\,; \tag{7.6.26}$$

the tape is infinite in both y and x directions.

At an elevation h above the tape a rectangular (ℓ) loop is positioned. The tape moves in the y-direction with a velocity v and the question is what voltage V is induced by the moving tape in this loop.

The first step is to "translate" the moving magnetization into the (rest) frame of the loop. By virtue of the non-relativistic transformation i.e.,

$$y \to y' = y + vt \tag{7.6.27}$$

we conclude that in the laboratory frame of reference

$$\vec{M} = M_0 \cos\left[k(y + vt)\right] \vec{1}_z \,. \tag{7.6.28}$$

Explicitly this means that in the laboratory frame of reference the magnetization varies periodically in time with characteristic angular frequency

$$\omega = kv \,. \tag{7.6.29}$$

Since for practical purposes we have assumed

$$v \ll c \,, \tag{7.6.30}$$

we are able to calculate the magnetic field in the entire space assuming $v = 0$. After calculating the magnetic field in the entire space we shall make the adequate transformation to account for the motion of the tape.

The magnetic scalar potential ψ is a solution of the Laplace equation therefore,

$$\psi(y,z) = \begin{cases} A\, e^{-k\left(z-\frac{d}{2}\right)} \cos(ky) & z \geq \frac{d}{2} \\ B \sinh(kz) \cos(ky) & |z| < \frac{d}{2} \\ -A\, e^{+k\left(z+\frac{d}{2}\right)} \cos(ky) & z \leq -\frac{d}{2}\,, \end{cases} \tag{7.6.31}$$

where use has been made of the fact that H_z is an even function of z which implies that ψ has to be an odd function of z.

Let us now examine the boundary conditions associated with this problem:

(a) The condition that the tangential magnetic field is continuous at $z = \pm d/2$ implies

$$A = B \sinh(kd/2)\,. \tag{7.6.32}$$

(b) The divergence of M is zero throughout the space therefore there exists at the discontinuity a magnetic surface charge density

$$\rho_{s,m} = \mu_0 M_0 \cos(ky)\;. \tag{7.6.33}$$

As indicated earlier, $\rho_{s,m}$ determines the discontinuity in the normal component of the magnetic field

(H_z) hence

$$\mu_0 H_z \left(z = \frac{d}{2} + 0 \right) - \mu_0 H_z \left(z = \frac{d}{2} - 0 \right) = \mu_0 M_0 \cos(ky) \tag{7.6.34}$$

therefore, bearing in mind that $H_z = -\dfrac{\partial \psi}{\partial z}$, we obtain

$$+kA\mu_0 + kB\mu_0 \cosh(kd/2) = \mu_0 M_0 \tag{7.6.35}$$

and the solution of (7.6.32,35) reads

$$B = \frac{M_0}{k}\, e^{-kd/2}\,, \qquad A = \frac{M_0}{k} \sinh(kd/2)\, e^{-kd/2}\,. \tag{7.6.36}$$

The scalar magnetic potential in the upper half-space for a *motionless* tape is given by

$$\psi(y,z) = \frac{M_0}{k}\, e^{-k\left(z - \frac{d}{2}\right)} \left[\sinh(kd/2)\, e^{-kd/2}\right] \cos[ky]\,. \tag{7.6.37}$$

When the tape *moves*

$$\psi(y,z;t) = \frac{M_0}{k}\, e^{-k(z - d/2)} \left[\sinh(kd/2) e^{-kd/2}\right] \cos[k(y + vt)]\,. \tag{7.6.38}$$

The magnetic *flux* through the loop is

$$
\begin{aligned}
\Phi(t) \quad &= \quad \mu_0 \int_{-l/2}^{l/2} dy \int_{-l/2}^{l/2} dx\, H_z \Big|_{z=h+d/2} = -\mu_0\, l \int_{-l/2}^{l/2} dy\, \frac{\partial \psi}{\partial z} \\
&= - \quad \mu_0\, l\, \frac{M_0}{k}(-k)e^{-kh} \left[\sinh(kd/2)e^{-kd/2}\right] \int_{-l/2}^{l/2} dy \cos k(y+vt) \\
&= + \quad \mu_0\, l\, M_0 e^{-kh} \left[\sinh(kd/2)e^{-kd/2}\right] \frac{1}{k} \left\{ \sin\left[k\left(\frac{l}{2}+vt\right)\right] \right. \\
&\qquad\qquad \left. - \sin\left[k\left(-\frac{l}{2}+vt\right)\right] \right\} \\
&= \quad \frac{\mu_0 l}{k} M_0 \left[\sinh(kd/2)\, e^{-kd/2}\, e^{-kh}\right] 2 \sin\left(k\frac{l}{2}\right) \cos(kvt)\,. \qquad (7.6.39)
\end{aligned}
$$

By virtue of Faraday's law (in its integral form)

$$
\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int d\vec{a} \cdot \vec{B} \equiv -\frac{d}{dt}\, \Phi \qquad (7.6.40)
$$

we observe that the external magnetic flux serves as an external voltage source given by $V = \dfrac{d\Phi}{dt}$ consequently,

$$\boxed{V(t) = -(2\mu_0 l M_0) v \left[\sinh(kd/2)\, e^{-kd/2}\, e^{-kh}\right] \sin(kl/2)\sin(kvt)}. \qquad (7.6.41)$$

Comments:

1. The amplitude of this voltage is a linear function of the velocity (v) and of the magnetization of the tape.

2. Storing more information along a given length implies a larger value of k. This, in turn, implies that h has to be reduced $\left[M_0\, e^{-kh}\right]$ or that M_0 must increase.

3. If $k\ell/2 \ll 1$ (no significant field variation across the magnetic tape) then

$$V = -(kv)(\mu_0 M_0 l^2)\left[\sinh(kd/2)\, e^{-kd/2}\, e^{-kh}\right] \sin(kvt)$$

   the first term is the angular frequency as defined in (7.6.4) and $\mu_0 M_0 \ell^2$ is the maximum flux crossing the rectangular loop.

4. The term $\sinh(kd/2)\, e^{-kd/2}$ for large $kd/2$ tends asymptotically to $1/2$, whereas for $kd/2 \ll 1$, it equals $kd/2$. Consequently, the smaller the value of d, the lower the induced voltage.

7.7. Field in the Vicinity of an Edge

In the chapter on quasi-statics we have discussed extensively the transition from statics to dynamics with regard to the characteristics of devices. It has been pointed out that one criterion of deciding between quasi-statics and dynamics is the ratio between a typical geometric parameter and the exciting wavelength. So far we have considered primarily low frequency devices, but the concepts of quasi-statics are valid even if we consider optical wavelengths (1μm), provided the geometric parameter is smaller than 100 nm.

7.7.1. Metallic Edge

Consider for example the case when the wavelength is much longer than the radius of curvature of an edge ($\lambda \gg R$) as illustrated in Figure 10.

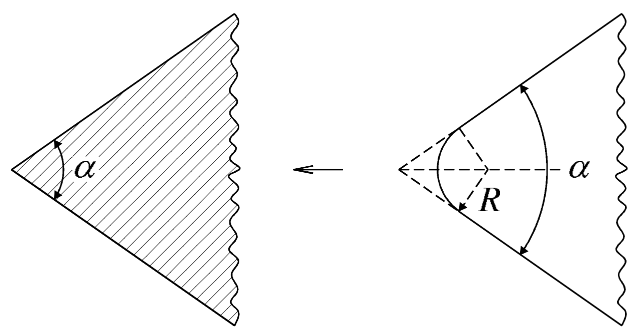

Figure 7.5: Two dimensional edge.

For example, one may conceive a laser beam ($\lambda \sim 1\mu m$) illuminating a nano-meter size probe ($R \sim 10\,\mathrm{nm}$). Based on this assumption the electric field in the vicinity of the edge is a solution of the Laplace equation since

$$\left[\nabla^2 - \cancel{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\right]\Phi = 0 \tag{7.7.1}$$

or explicitly

$$\nabla^2 \Phi = 0\,. \tag{7.7.2}$$

Assuming that the system is infinite in the z-direction and so is the excitation, then

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]\Phi = 0 \tag{7.7.3}$$

is the equation to be solved subject to the zero potential on the walls

$$\Phi(r, \phi = \alpha/2) = 0 \quad \text{and} \quad \Phi\left(r, \phi = 2\pi - \frac{\alpha}{2}\right) = 0\,. \tag{7.7.4}$$

The solution has the form

$$\Phi \sim \left(A\, e^{j\nu\phi} + B\, e^{-j\nu\phi}\right) r^{\nu}\,,$$

thus imposing the boundary conditions

$$
\begin{aligned}
&\underline{\phi = \alpha/2}: \qquad && A\,e^{j\nu\alpha/2} + B\,e^{-j\nu\alpha/2} = 0 \\
&\underline{\phi = 2\pi - \alpha/2}: \qquad && A\,e^{j\nu(2\pi - \alpha/2)} + B\,e^{-j\nu(2\pi - \alpha/2)} = 0\,,
\end{aligned}
\tag{7.7.5}
$$

or in a matrix form

$$
\begin{bmatrix} e^{j\nu\alpha/2} & e^{-j\nu\alpha/2} \\ e^{j\nu(2\pi - \alpha/2)} & e^{-j\nu(2\pi - \alpha/2)} \end{bmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0\,.
$$

A non-trivial solution is possible provided the *determinant of the matrix is zero* implying

$$
\sin\Big[\nu(2\pi - \alpha)\Big] = 0\,,
\tag{7.7.6}
$$

hence

$$
\nu(2\pi - \alpha) = \pi n \qquad\qquad n = 1, 2, 3\,\ldots
$$

or

$$
\boxed{\nu = \frac{\pi n}{2\pi - \alpha}}\,.
\tag{7.7.7}
$$

According to the boundary conditions

$$
A\,e^{j\nu\alpha/2} + B\,e^{-j\nu\alpha/2} = 0 \Rightarrow B = -A\,e^{j\nu\alpha}\,,
\tag{7.7.8}
$$

thus

$$\Phi(r,\phi) = \left[A\, e^{j\nu\phi} - A\, e^{j\nu\alpha}\, e^{-j\nu\phi}\right] r^{\nu}$$

$$= A\, e^{j\nu\alpha/2}\, 2j \sin\left[\nu(\phi - \alpha/2)\right] r^{\nu} \qquad (7.7.9)$$

and finally the general solution may be written as

$$\boxed{\Phi\left(r,\, \frac{\alpha}{2} < \phi < 2\pi - \frac{\alpha}{2}\right) = \sum_{n=1}^{\infty} A'_n \sin\left[\pi n\, \frac{\phi - \dfrac{\alpha}{2}}{2\pi - \alpha}\right]\, r^{\dfrac{\pi n}{2\pi - \alpha}}}. \qquad (7.7.10)$$

It is important to point out already at this stage that the angle of the edge determines the curvature of the potential. Figure 7.6 illustrates the contour of constant potential for $\alpha = \pi/6$.

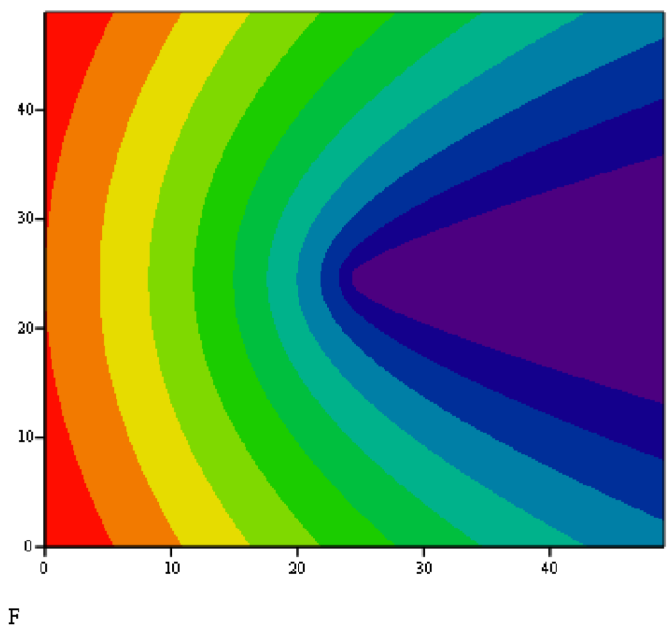

Figure 7.6: Contours of constant potentials in the vicinity of an edge of an angle 30°.

For a thorough analysis let us examine the field components linked to the first harmonic

$$\Phi_1 = A_1 \sin\left[\pi \frac{\phi - \frac{\alpha}{2}}{2\pi - \alpha}\right] r^{\frac{\pi}{2\pi - \alpha}} \tag{7.7.11}$$

$$E_r = -\frac{\partial}{\partial r}\Phi_1 = -\frac{\pi}{2\pi - \alpha} A_1 \sin\left[\pi \frac{\phi - \frac{\alpha}{2}}{2\pi - \alpha}\right] r^{\frac{\pi}{2\pi - \alpha} - 1} \tag{7.7.12}$$

$$E_\phi = -\frac{1}{r}\frac{\partial \Phi}{\partial \phi} = -\frac{\pi}{2\pi - \alpha} A_1 \cos\left[\pi \frac{\phi - \frac{\alpha}{2}}{2\pi - \alpha}\right] r^{\frac{\pi}{2\pi - \alpha} - 1}. \tag{7.7.13}$$

One of the most intriguing properties of this field is that at the limit $r \to 0$, if $\alpha < \pi$, then the electric field *diverges* nevertheless, the energy is *finite*. The first part of this statement is evident from Eqs. (7.7.12)-(7.7.13) whereas for the second part, one needs to examine the energy in a finite volume around the edge. Explicitly this is given by

$$W = \Delta_z \int_{\alpha/2}^{2\pi - \alpha/2} d\phi \int_0^R dr r \left[\frac{1}{2}\varepsilon_0 E_r^2 + \frac{1}{2}\varepsilon_0 E_\phi^2\right]. \tag{7.7.14}$$

$$W = \frac{1}{2}\,\varepsilon_0 \Delta_z A_1^2 \left(\frac{\pi}{2\pi-\alpha}\right)^2 \int\limits_{\alpha/2}^{2\pi-\alpha/2} d\phi \int\limits_0^R dr r\, r^{2\left[\frac{\pi}{2\pi-\alpha}-1\right]} \underbrace{\left[\cos^2(\ \ ) + \sin^2(\ \ )\right]}_{\equiv 1}$$

$$= \frac{1}{2}\,\varepsilon_0 \Delta_z A_1^2 \left(\frac{\pi}{2\pi-\alpha}\right)^2 (2\pi-\alpha) \int\limits_0^R dr\, r^{\frac{2\pi}{2\pi-\alpha}-1}$$

$$= \frac{1}{2}\,\varepsilon_0 \Delta_z A_1^2 \left(\frac{\pi}{2\pi-\alpha}\right)^2 (2\pi-\alpha)\,\frac{2\pi-\alpha}{2\pi}\, R^{\frac{2\pi}{2\pi-\alpha}}$$

$$W = \frac{1}{2}\,\varepsilon_0 \Delta_z A_1^2\, \frac{\pi}{2}\, R^{\frac{2\pi}{2\pi-\alpha}} \,. \tag{7.7.15}$$

Consequently, regardless the value of α, the power of R is *positive*

$$\frac{2\pi}{2\pi-\alpha} > 0\,, \tag{7.7.16}$$

therefore, at the limit of $R \to 0$ the energy is finite. Clearly, assuming that the voltage is known, and it is proportional to A, it is possible to determine the capacitance in the system.

7.7.2. Dielectric Edge

A similar approach may be followed to investigate the field distribution in the vicinity of a dielectric edge. Based on the knowledge of the field in the vicinity of a metallic edge, the character of a typical solution may be anticipated to be symmetric relative to the x-axis. Specifically,

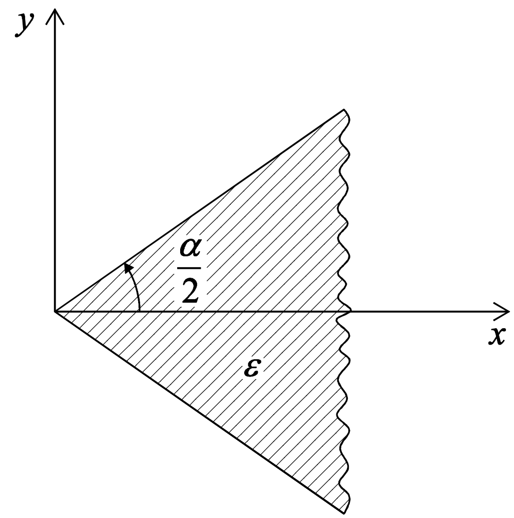

Figure 7.7: A dielectric edge.

$$E_\phi(\phi = 0) = 0 \Rightarrow \Phi_\varepsilon(r,\phi) = A\cos(\nu\phi)\, r^{\nu} \tag{7.7.17}$$

$$E_\phi(\phi = \pi) = 0 \Rightarrow \Phi_v(r,\phi) = B\cos\Big[\nu(\pi-\phi)\Big]\, r^{\nu}\,. \tag{7.7.18}$$

Since we have taken care of the symmetry, we impose the boundary conditions only at $\phi = \alpha/2$.

Continuity of Φ entails

$$A\cos(\nu\alpha/2) = B\cos\Big[\nu(\pi-\alpha/2)\Big]\,, \tag{7.7.19}$$

whereas continuity of D_ϕ

$$A\varepsilon \sin(\nu\alpha/2) = -B \sin\Big[\nu(\pi - \alpha/2)\Big] \,. \tag{7.7.20}$$

For a non trivial solution, this equation system formulated next in a matrix form

$$\begin{pmatrix} \cos(\nu\alpha/2) & -\cos\Big[\nu(\pi-\alpha/2)\Big] \\ \varepsilon\sin(\nu\alpha/2) & +\sin\Big[\nu(\pi-\alpha/2)\Big] \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{7.7.21}$$

entailing the determinant ought to vanish i.e.

$$\cos\left(\nu\frac{\alpha}{2}\right)\sin\left[\nu\left(\pi-\frac{\alpha}{2}\right)\right] + \varepsilon\sin\left(\nu\frac{\alpha}{2}\right)\cos\left[\nu\left(\pi-\frac{\alpha}{2}\right)\right] = 0$$

$$\varepsilon = -\frac{\tan\left[\nu\left(\pi-\frac{\alpha}{2}\right)\right]}{\tan\left(\nu\frac{\alpha}{2}\right)} \,. \tag{7.7.22}$$

Clearly, the curvature parameter depends on the dielectric coefficient of the medium. Note that at the limit $\varepsilon \to \infty$ the first solution is

$$\nu\left(\pi-\frac{\alpha}{2}\right) = \frac{\pi}{2} + 0 \,,$$

or

$$\nu = \frac{\pi}{2\pi-\alpha}$$

similar to the result in (7.7.7).

Figure 7.8 illustrates the dependence of curvature parameter on the dielectric coefficient.

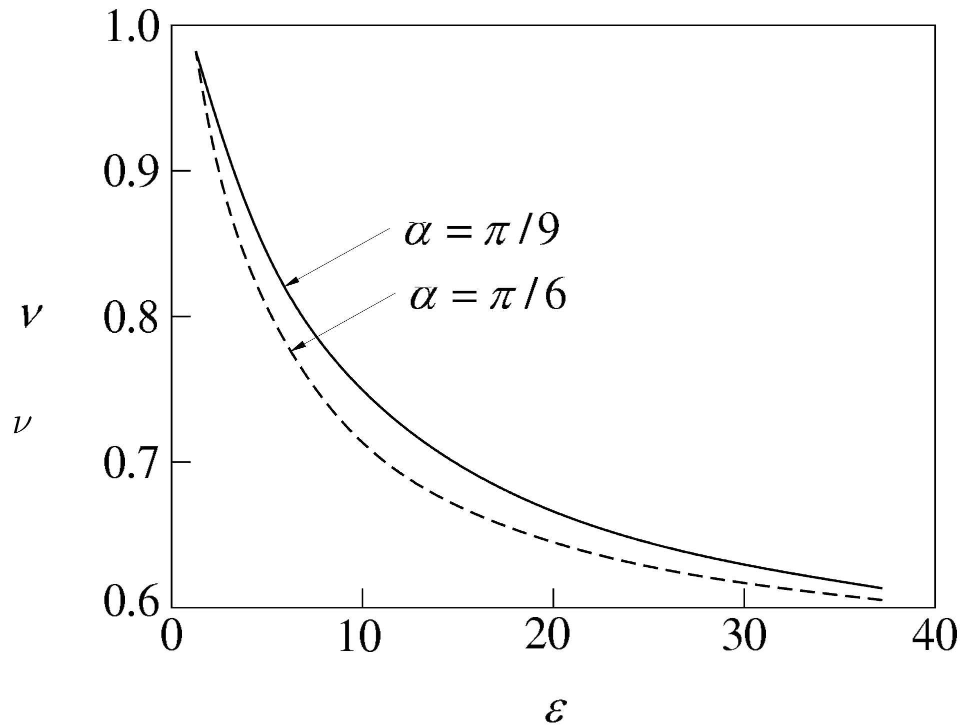

Figure 7.8: Curvature parameter ν as a function of the dielectric coefficient (ε).

7.7.3. Capacitance in the Vicinity of an Edge

Consider an electrode that is infinite in the z-direction, infinitesimally thin and it has an angular spread $|\phi - \gamma| < \dfrac{\beta}{2}$. A voltage V_0 is applied to this electrode, all the other surfaces being grounded – including the edge (α) and the surrounding surface see Figure 7.14. Consequently, the potential is

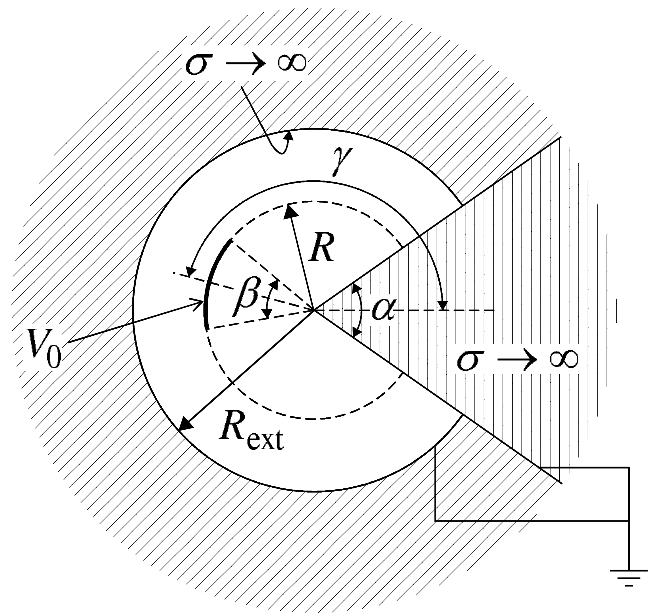

Figure 7.9: Capacitance associated with an edge.

given by

$$\Phi(r,\phi) = \sum_{n=1}^{\infty} \sin\left[n\nu\left(\phi - \frac{\alpha}{2}\right)\right] \begin{cases} A_n\left[\left(\frac{r}{R_{\text{ext}}}\right)^{\nu n} - \left(\frac{R_{\text{ext}}}{r}\right)^{\nu n}\right] & R < r < R_{\text{ext}} \\ B_n\left(\frac{r}{R}\right)^{\nu n} & 0 \le r \le R\,, \end{cases} \tag{7.7.23}$$

wherein as before, ν is given by $\nu = \dfrac{\pi}{2\pi - \alpha}$. Continuity of Φ at $r = R$ implies

$$A_n\left[\left(\frac{R}{R_{\text{ext}}}\right)^{\nu n} - \left(\frac{R_{\text{ext}}}{R}\right)^{\nu n}\right] = B_n\,. \tag{7.7.24}$$

Therefore, we may rewrite the potential

$$\Phi(r,\phi) = \sum_{n=1}^{\infty} U_n\,\mathbb{R}_n(r)\sin\left[n\nu\left(\phi - \frac{\alpha}{2}\right)\right]\,, \tag{7.7.25}$$

wherein

$$\mathbb{R}_n(r) = \begin{cases} \dfrac{\left(\frac{r}{R_{\text{ext}}}\right)^{\nu n} - \left(\frac{r}{R_{\text{ext}}}\right)^{-\nu n}}{\overline{R}^{\nu n} - \overline{R}^{-\nu n}} & R < r < R_{\text{ext}} \\ \left(\frac{r}{R}\right)^{\nu n} & 0 \le r \le R \end{cases} \tag{7.7.26}$$

and $\overline{R} \equiv \dfrac{R}{R_{\text{ext}}}$. In order to determine the amplitude it is assumed that the electrode is infinitesimally thin, thus the charge-density reads

$$\rho(r,\phi) = \rho_s(\phi)\,\delta(r-R) \tag{7.7.27}$$

entailing that the discontinuity in the electric induction D_r is

$$\int\limits_{R-0}^{R+0} dr\, r\,\frac{1}{r}\frac{\partial}{\partial r} r\,\frac{\partial}{\partial r}\,\Phi(r,\phi) = -\frac{1}{\varepsilon_0}\int\limits_{R-0}^{R+0} dr\, r\rho(r,\phi)\,, \tag{7.7.28}$$

thus

$$\begin{aligned}\left[r\,\frac{\partial\Phi}{\partial r}\right|_{r\,=\,R+0} - \left[r\,\frac{\partial\Phi}{\partial r}\right|_{r\,=\,R-0} &= -\frac{1}{\varepsilon_0}\int\limits_{R-0}^{R+0} dr\, r\rho(r,\phi)\\ &= -\frac{1}{\varepsilon_0}\,R\rho_s(\phi)\,. \end{aligned}\tag{7.7.29}$$

Substituting (7.7.25) we get

$$R\sum_{n=1}^{\infty} U_n \sin\left[n\nu\left(\phi-\frac{\alpha}{2}\right)\right]\,\left[\left.\frac{d\mathbb{R}_n(r)}{dr}\right|_{r\,=\,R+0} - \left.\frac{d\mathbb{R}_n(r)}{dr}\right|_{r\,=\,R-0}\right] = \frac{-1}{\varepsilon_0}\,R\rho_s(\phi)\,, \tag{7.7.30}$$

or explicitly based on the definition of $\mathbb{R}_n(r)$ in Eq. (7.7.26)

$$\left(\frac{d}{dr}\mathbb{R}_n(r)\right|_{r\,=\,R+0} - \left(\frac{d}{dr}\mathbb{R}_n(r)\right|_{r\,=\,R-0} = \frac{-\nu n}{R}\,\frac{2}{1-\overline{R}^{2\nu n}} \tag{7.7.31}$$

and using the orthogonality of the trigonometric function we find

$$U_n = \frac{1}{2\varepsilon_0}\,\frac{R}{\nu n}\left(1-\overline{R}^{2\nu n}\right)\frac{2}{2\pi-\alpha}\int_{\frac{\alpha}{2}}^{2\pi-\frac{\alpha}{2}} d\phi\,\rho_s(\phi)\sin\left[\nu n\left(\phi-\frac{\alpha}{2}\right)\right] \tag{7.7.32}$$

implying that if the surface charge-density, $\rho_s(\phi)$, on the electrode is known, the potential may be established. However, in practice the opposite is the case; we know the voltage on the electrode

$$\Phi\left(r=R\,,\quad |\phi-\gamma|<\frac{\beta}{2}\right) = V_0 \tag{7.7.33}$$

based on which the surface-charge distribution should be determined. The *exact* solution requires solving an integral equation that is beyond the scope of this lecture. Nevertheless, we may proceed to an approximate solution by replacing the continuous electrode with N line charges located at certain angles ϕ_i i.e.

$$\rho_s(\phi) = \sum_{i=1}^{N}\frac{1}{\Delta_z R\beta}\,Q_i\delta(\phi-\phi_i)\,. \tag{7.7.34}$$

In this way the unknown charges may be readily established by imposing the potential in N points on the electrode

$$\Phi_j = \Phi(r = R\,,\ \phi = \phi_j) = \sum_{n=1}^{\infty} \sin\left[n\nu\left(\phi_j - \frac{\alpha}{2}\right)\right]$$
$$\times \frac{1}{2\varepsilon_0}\,\frac{R}{\nu n}\left(1 - \overline{R}^{2\nu n}\right)\frac{2}{2\pi - \alpha}\,\frac{1}{\Delta_z R\beta}\sum_{i=1}^{N} Q_i \sin\left[n\nu\left(\phi_j - \frac{\alpha}{2}\right)\right]\,. \quad (7.7.35)$$

By defining the matrix

$$\chi_{ji} \equiv \frac{1}{2\pi\varepsilon_0\Delta_z\beta}\,\frac{2\pi}{2\pi - \alpha}\sum_{n=1}^{\infty}\frac{1}{\nu n}\left(1 - \overline{R}^{2\nu n}\right)\sin\left[n\nu\left(\phi_j - \frac{\alpha}{2}\right)\right]\,\sin\left[n\nu\left(\phi_i - \frac{\alpha}{2}\right)\right]\,, \quad (7.7.36)$$

we realize that

$$\Phi_j = \sum_{i=1}^{N} \chi_{ji}\,Q_i\,, \quad (7.7.37)$$

or

$$Q_i = \sum_{j=1}^{N}\left[\chi^{-1}\right]_{ij}\Phi_j\,. \quad (7.7.38)$$

Bearing in mind that for any j, $\Phi_j = V_0$ implies that the potential is known in the entire space. Moreover, it is possible to calculate the capacitance since the total change on the electrode is $Q_{\text{total}} = \sum_{i=1}^{N} Q_i$, then

$$Q_{\text{total}} = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[\chi^{-1}\right]_{ij} V_0 \,,$$

hence

$$C \equiv \frac{Q_{\text{total}}}{V_0} = \sum_{i,j=1}^{N} \left[\chi^{-1}\right]_{ij} . \tag{7.7.39}$$

A simpler approach to the solution of the potential in Eq. (7.7.25) subject to the charge distribution that determines U_n in Eq. (7.7.32) is possible if we assume that the charge is *uniformly distributed* on the electrode. Explicitly, the surface charge density ρ_s is assumed to be given by

$$\rho_s(\phi) = \frac{Q}{\Delta_z R\beta} \left[h\left(\phi - \gamma + \frac{\beta}{2}\right) - h\left(\phi - \gamma - \frac{\beta}{2}\right)\right] \tag{7.7.40}$$

implying

$$U_n = \frac{Q}{2\pi\varepsilon_0\Delta_z} \frac{1}{\nu n} \left(1 - \overline{R}^{2\nu n}\right) \frac{2\pi}{2\pi - \alpha} \operatorname{sinc}\left(\frac{1}{2}\,\nu n\beta\right) \sin\left[n\nu\left(\gamma - \frac{\alpha}{2}\right)\right] . \tag{7.7.41}$$

For the evaluation of the total charge Q we again impose the condition in Eq. (7.7.33), or if to be more precise, since we assumed that the surface charge density is uniform, the potential on the electrode is expected to vary. Therefore, we shall assume that the *average* potential on the electrode equals the voltage V_0 i.e.

$$V_0 = \frac{1}{\beta} \int\limits_{\gamma - \frac{\beta}{2}}^{\gamma + \frac{\beta}{2}} d\phi \, \Phi(r = R, \phi) = \sum_{n=1}^{\infty} U_n \frac{1}{\beta} \int\limits_{\gamma - \frac{\beta}{2}}^{\gamma + \frac{\beta}{2}} d\phi \sin\left[n\nu\left(\phi - \frac{\alpha}{2}\right)\right]$$

$$= \sum_{n=1}^{\infty} U_n \, \mathrm{sinc}\left(\frac{1}{2}n\nu\beta\right) \sin\left[n\nu\left(\gamma - \frac{\alpha}{2}\right)\right] , \tag{7.7.42}$$

consequently, the charge

$$Q = \frac{V_0}{\frac{1}{2\pi\varepsilon_0\Delta_z}\frac{2\pi}{2\pi - \alpha}\sum_{n=1}^{\infty}\frac{1}{\nu n}\left(1 - \overline{R}^{2\nu n}\right)\mathrm{sinc}^2\left(\frac{1}{2}n\nu\beta\right)\sin^2\left[n\nu\left(\gamma - \frac{\alpha}{2}\right)\right]} , \tag{7.7.43}$$

whereas the normalized capacitance

$$\overline{C} \equiv \frac{Q}{2\pi\varepsilon_0\Delta_z V_0} = \frac{1}{\frac{2\pi}{2\pi - \alpha}\sum_{n=1}^{\infty}\frac{1}{\nu n}\left(1 - \overline{R}^{2\nu n}\right)\mathrm{sinc}^2\left(\frac{1}{2}n\nu\beta\right)\sin^2\left[n\nu\left(\gamma - \frac{\alpha}{2}\right)\right]} . \tag{7.7.44}$$

The potential distribution associated with this electrode, in the framework of this last approximation, is illustrated in Figure 7.10 and the dependence of the capacitance on the various parameters is shown in Figure 7.11.

With the capacitance established, it is possible to determine the energy stored in the system.

$$W\left(\alpha, \beta, \gamma, R, R_{\text{ext}}, \Delta_z\right) = \frac{1}{2} C\left(\alpha, \beta, \gamma, R, R_{\text{ext}}, \Delta_z\right) V_0^2 \,. \tag{7.7.45}$$

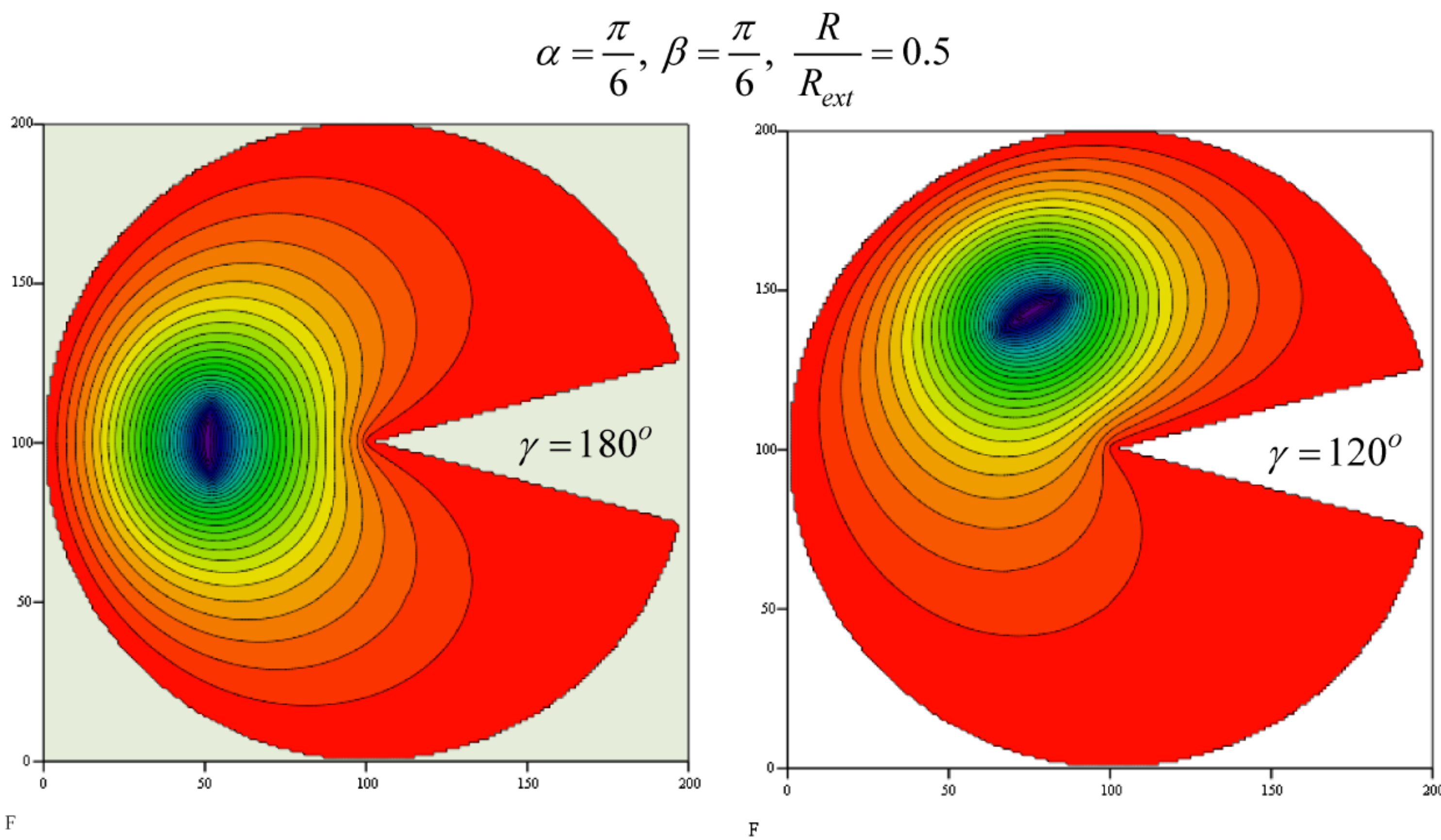

Figure 7.10: Contours of constant potential.

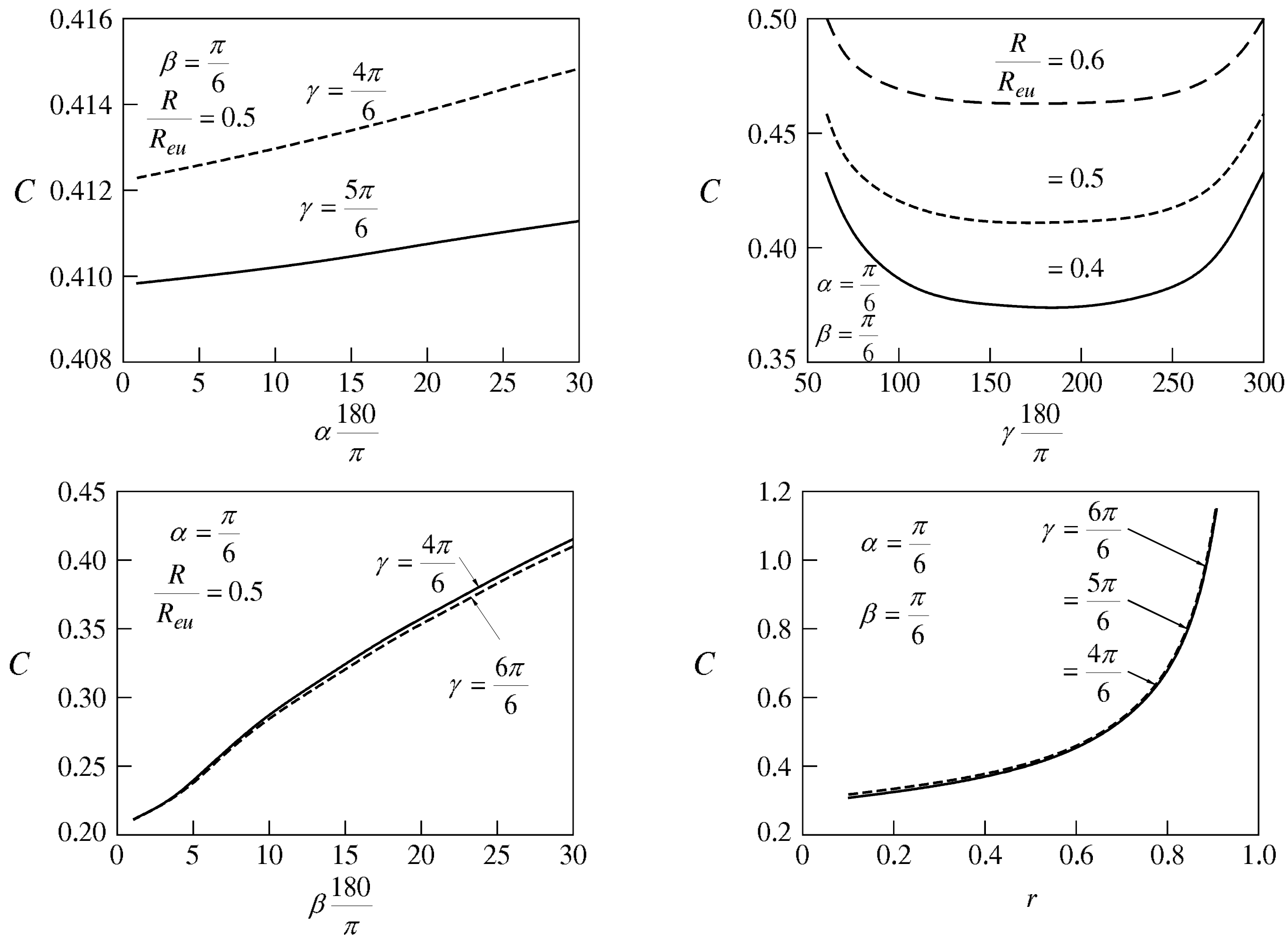

Figure 7.11: Capacitance as a function of the various parameters.

Assuming that the voltage V_0 is set, we may determine the *force on the electrode*. For example, the radial force on the electrode is

$$F_r = -\frac{\partial W}{\partial R}\,, \tag{7.7.46}$$

whereas the azimuthal force is

$$F_\phi = -\frac{1}{R}\frac{\partial W}{\partial \gamma}\,. \tag{7.7.47}$$

Exercise #2: Calculate the resistance per unit length between the edge and the electrode – see Figure 7.12. Analyze the effect of the various parameters on this quantity.

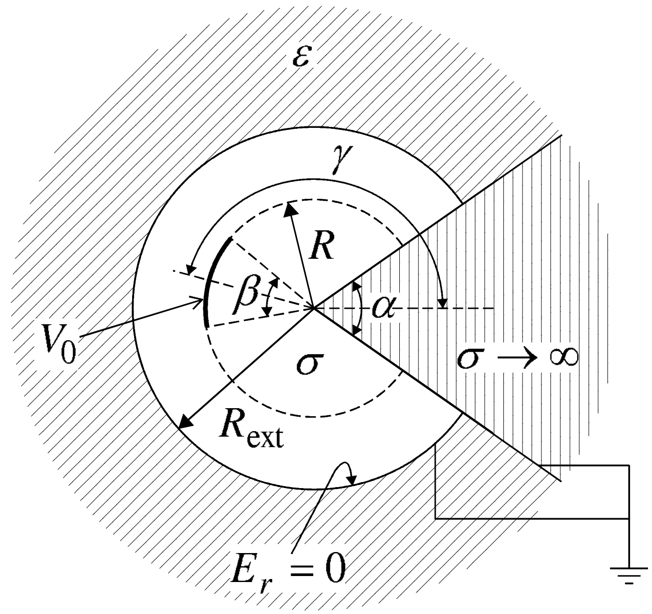

Figure 7.12: Resistor with an "edgy" electrode.

7.8. EM Field in Superconducting Materials

Superconducting materials play an important role in *low noise* devices or *intense magnetic field* devices. In order to understand the basic process occurring in a superconductor, let us examine a simplified model which describes the conduction in regular metals. In *vacuum* the motion of an electron subject to an electric field E is given by

$$\frac{d}{dt}v = -\frac{e}{m}E\,. \tag{7.8.1}$$

In a *metal*, the electron encounters collisions therefore, if we denote by τ the typical time between two collisions, then the dynamics of the particle is determined by

$$\frac{d}{dt}v = -\frac{v}{\tau} - \frac{e}{m}E\;. \tag{7.8.2}$$

From this relation we may deduce that the average velocity of the electron is given by

$$v = -\frac{e\tau}{m}E \tag{7.8.3}$$

hence, the current density

$$J = -env = -en\left(-\frac{e\tau}{m}\right)E = \frac{e^2 n\tau}{m}E = \sigma\,E\;, \tag{7.8.4}$$

wherein

$$\sigma = \frac{e^2 n\tau}{m} \tag{7.8.5}$$

represents the conductivity.

For *superconductors* $\tau \to \infty$ therefore we end up with a situation similar to vacuum therefore

$$\frac{dv}{dt} = -\frac{e}{m}\,E = -\frac{e}{m}\left(-\frac{\partial A}{\partial t}\right) , \tag{7.8.6}$$

where A is the magnetic vector potential; we have ignored the electric scalar potential and consequently we may assume a typical velocity

$$v = \frac{e}{m}\,A\,. \tag{7.8.7}$$

Based on this relation we may define the current density associated with the superconducting electrons as

$$J_{\rm sc} = -en_{sc}v = -en_{\rm sc}\frac{e}{m}\,A\,, \tag{7.8.8}$$

where $n_{\rm sc}$ is the density of the "super-conducting" electrons. Note that as in semiconductors, the mass is not necessarily the mass of a free electron.

An important observation is that the relation in (7.8.8) dictates in the static case the gauge since charge conservation entails $\vec{\nabla} \cdot \vec{J} = 0$ hence $\vec{\nabla} \cdot \vec{A} = 0$ also dictating zero electric scalar potential as it reflects from (7.8.6). Equation (7.8.8) dictates the so called London's equations for the microscopic electric and magnetic field

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\left(-\frac{m}{e^2 n_{sc}}\,\vec{J}_{\rm sc}\right) = \frac{m}{e^2 n_{sc}}\,\frac{\partial \vec{J}_{\rm sc}}{\partial t} \tag{7.8.9}$$

and

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{1}{\mu_0}\vec{\nabla}\times\left(-\frac{m}{e^2 n_{sc}}\vec{J}_{\rm sc}\right) = \frac{-m}{\mu_0 e^2 n_{\rm sc}}\vec{\nabla}\times\vec{J}_{\rm sc}\,. \tag{7.8.10}$$

Combining the last equation with Ampere's law ($\vec{\nabla}\times\vec{H} = \vec{J}$) and neglecting the contribution of the electric induction $\vec{D}$ we find

$$\begin{aligned} \nabla\times(\nabla\times\vec{H}) &= \nabla\times\vec{J} \\ \nabla\left(\underbrace{\vec{\nabla}\cdot\vec{H}}_{=0}\right) - \nabla^2\vec{H} &= -\mu_0\frac{e^2 n_{sc}}{m}\vec{H} \end{aligned} \tag{7.8.11}$$

hence

$$\nabla^2\vec{H} = \frac{e^2 n_{sc}}{m\varepsilon_0}\frac{1}{c^2}\vec{H} = \frac{1}{\lambda^2}\vec{H}\,. \tag{7.8.12}$$

λ — is called the Meisner coefficient and it describes the penetration depth of the magnetic field in a superconducting material. In order to illustrate the effect consider a 1D problem, see Figure 7.11 where we set $H_z(x=0) = H_0$, all other components being identically zero:

$$\left[\frac{d^2}{dx^2} - \frac{1}{\lambda^2}\right] H_z = 0\,. \tag{7.8.13}$$

Consequently $H_z = U\,e^{-x/\lambda} + W\,e^{x/\lambda}$ and since the solution needs to converge at $x\to\infty$ we conclude that $W = 0$ therefore

$$H_z = H_o\,e^{-x/\lambda}\,. \tag{7.8.14}$$

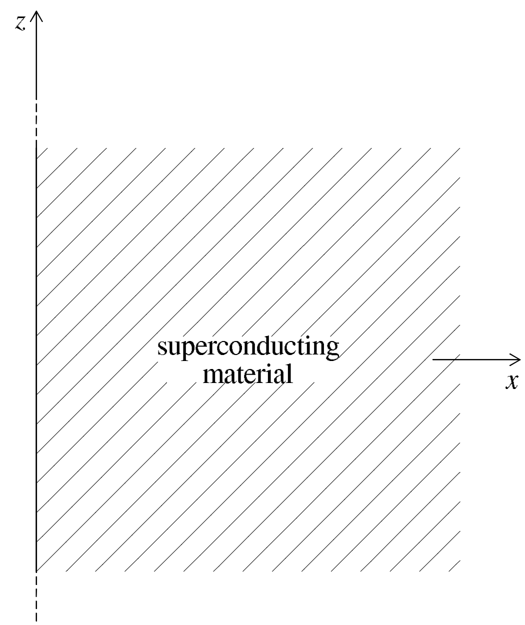

Figure 7.13: Schematic of a superconducting half-space where the magnetic field penetrates.

In order to have an idea of the penetration depth consider $n_{sc} \sim 10^{27} m^{-3}$ hence

$$\frac{1}{\lambda^2} = \frac{1.6 \times 10^{-19} \times 10^{27}}{0.51 \times 10^6 \times 8.85 \times 10^{12}} \sim \frac{1}{2.7} \frac{1}{10^{-14}}$$

$$\lambda \sim 10^{-7}\, m \sim 100nm\,. \tag{7.8.15}$$

Note that although the electric field may be zero (static case) the current density is nonzero, and in spite of the static condition the magnetic field does not penetrate the material.

7.9. EM Field in the Presence of Moving Metal

When a metallic film moves in the vicinity of a magnet, currents are induced in the film. These currents have two effects firstly they dissipate power and secondly, they generate a decelerating force $\vec{J} \times \vec{B}$; the two are related and in this section we shall consider this relation. The example to be examined in this section relies on the observation based on the Coulomb force for a motionless charge

$$\vec{F} = q\vec{E} \tag{7.9.1}$$

and its extension to a moving charge

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{7.9.2}$$

we can extend the constitutive relation in a motionless metal

$$\vec{J} = \sigma \vec{E} \tag{7.9.3}$$

to

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \tag{7.9.4}$$

when the metal moves at a velocity v, Figure 7.14 illustrates the simple example to be considered consisting of a periodic magnet and a metallic layer.

In the absence of the moving metallic layer the magnetic field generated by the periodic magnets may

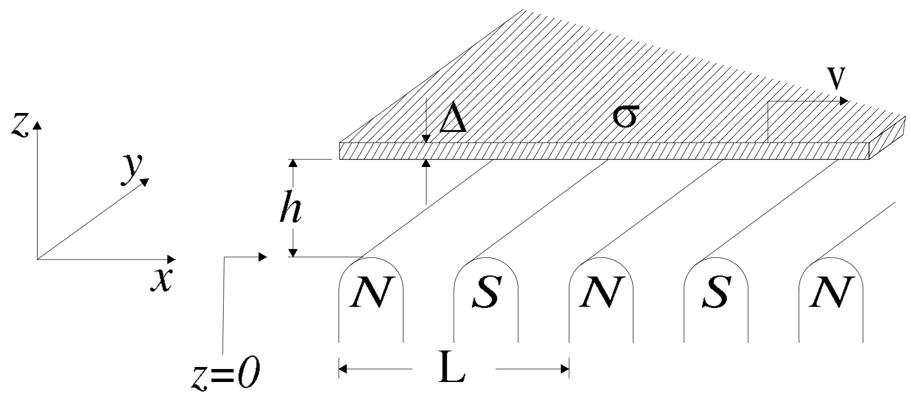

Figure 7.14: Metallic plate moving above an array of periodic magnets.

be roughly approximated by

$$A_y^{(p)} = B_0 \frac{L}{2\pi} \cos\left(2\pi \frac{x}{L}\right) e^{-2\pi \frac{z}{L}} \tag{7.9.5}$$

therefore the magnetic field components read

$$B_x^{(p)} = -\frac{\partial A_y^{(p)}}{\partial z} = B_0 \cos\left(\frac{2\pi x}{L}\right) e^{-2\pi \frac{z}{L}} \tag{7.9.6}$$

$$B_z^{(p)} = \frac{\partial A_y^{(p)}}{\partial x} = -B_0 \sin\left(\frac{2\pi x}{L}\right) e^{-2\pi \frac{z}{L}} ; \tag{7.9.7}$$

the superscript p indicates that this is a *primary* field. The presence of the moving metallic layer generates a *secondary* magnetic vector potential (superscript p) which is a solution of the Laplace equation $\left(\frac{2\pi}{L} \gg \frac{\omega}{c}\right)$.

Between the magnets and the metallic layer $A_y^{(sec)}$ is

$$A_y^{(sec)}(x,\, 0 < z < h) = \left[A_1\, e^{+2\pi\frac{z}{L}} + A_2\, e^{-2\pi\frac{z}{L}}\right] \sin\left(\frac{2\pi x}{L}\right) \tag{7.9.8}$$

and

$$A_y^{(sec)}(x,\, z > h) = A_3\, e^{-2\pi\frac{z-h}{L}} \sin\left(\frac{2\pi x}{L}\right) \,. \tag{7.9.9}$$

Since the magnets which impose the primary magnetic vector potential may be represented by a surface current density and assuming that the secondary field is weak enough such that it does not affect this current,

$$H_x^{(sec)}(z = 0) = 0 \qquad \Rightarrow \qquad A_1 - A_2 = 0 \tag{7.9.10}$$

and consequently

$$A_y^{(sec)}(x,\, 0 < z < h) = 2A_1\, \cosh\left(2\pi\frac{z}{L}\right) \sin\left(2\pi\frac{x}{L}\right) \,. \tag{7.9.11}$$

The amplitudes A_1 and A_3 are next determined by imposing the boundary conditions at $z = h$. First the $A_y^{(sec)}$ has to be continuous $[\vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0]$ at $z = h$:

$$2A_1 \cosh\left(2\pi\frac{h}{L}\right) = A_3 \tag{7.9.12}$$

and second the x-component of the secondary magnetic field is discontinuous. The discontinuity is determined by the current which flows in the metallic layer

$$H_x(z = h + 0) - H_x(z = h - 0) = J_{s,y}\,. \tag{7.9.13}$$

Based on (7.9.4) and bearing in mind that $E = 0$, we conclude that

$$J_y = -\sigma v B_z^{(p)} \Rightarrow J_{s,y} = -\Delta v \sigma B_z^{(p)} \tag{7.9.14}$$

hence

$$-\frac{1}{\mu_0}\left.\frac{\partial A_y}{\partial z}\right|_{z=h+0} + \frac{1}{\mu_0}\left.\frac{\partial A_y}{\partial z}\right|_{z=h-0} = -\Delta v \sigma \frac{\partial A_z^{(p)}}{\partial x}$$

$$\frac{2\pi}{L} A_3 \left[e^{-2\pi\frac{z-h}{L}} \sin\left(2\pi\frac{x}{L}\right)\right|_{z=h} + \frac{2\pi}{L}\cdot 2A_1 \left[\sinh\left(2\pi\frac{z}{L}\right)\sin\left(2\pi\frac{x}{L}\right)\right|_{z=h}$$

$$= +\mu_0 \Delta\sigma v \frac{2\pi}{L}\left[B_0 \cdot \frac{L}{2\pi}\sin\frac{2\pi x}{L}\, e^{-2\pi\frac{z}{L}}\right|_{z=h}\,. \tag{7.9.15}$$

Explicitly this reads

$$\frac{2\pi}{L} A_3 + \frac{2\pi}{L} 2A_1 \sinh\left(2\pi\frac{h}{L}\right) = \mu_0 \Delta\sigma v\, B_0\, e^{-2\pi\frac{h}{L}} \tag{7.9.16}$$

and together with (7.9.12) we can determine the two amplitudes:

$$A_1 = \frac{1}{2}\left(B_0\frac{L}{2\pi}\right)(\mu_0 \Delta v \sigma)\ e^{-4\pi\frac{h}{L}} \tag{7.9.17}$$

and

$$A_3 = \left(B_0\frac{L}{2\pi}\right)(\mu_0 \Delta v \sigma)\ e^{-4\pi\frac{h}{L}}\cosh\left(2\pi\frac{h}{L}\right) . \tag{7.9.18}$$

Consequently, the secondary magnetic field reads

$$B_x^{(sec)} = -\frac{\partial}{\partial z}A_y^{(\mathrm{sec})} = B_0(\mu_0\Delta v\sigma)\sin\left(2\pi\frac{x}{L}\right)\begin{cases} -\sinh\left(2\pi\frac{z}{L}\right)\ e^{-4\pi\frac{h}{L}} & 0 \le z < h \\ e^{-2\pi\frac{z-h}{L}}\ e^{-4\pi\frac{h}{L}}\cosh\left(2\pi\frac{h}{L}\right) & z > h \end{cases} \tag{7.9.19}$$

$$B_z^{(sec)} = \frac{\partial A_y^{(sec)}}{\partial x} = B_0(\mu_0\Delta v\sigma)\cos\left(2\pi\frac{x}{L}\right)\begin{cases} \cosh\left(2\pi\frac{z}{L}\right)\ e^{-4\pi\frac{h}{L}} & 0 < z < h \\ e^{-2\pi\frac{z-h}{L}}\ e^{-4\pi\frac{h}{L}}\cosh\left(2\pi\frac{h}{L}\right) & z > h\,. \end{cases} \tag{7.9.20}$$

7.9.1. Power and Force

The power dissipated in the metallic layer is given by

$$P = \Delta_y \int_0^L dx \cdot \Delta\, J_y^2 \frac{1}{\sigma} \simeq \Delta_y \Delta \frac{1}{\sigma} \left[\sigma v B_0\, e^{-2\pi \frac{h}{L}} \right]^2 \int_0^L dx\, \sin^2\left(2\pi \frac{x}{L}\right) \tag{7.9.22}$$

hence

$$P = \Delta_y \Delta \frac{1}{\sigma} \left[\sigma v B_0\, e^{-2\pi \frac{h}{L}} \right]^2 \frac{L}{2}\,. \tag{7.9.23}$$

Recall that the power is the product of the force and velocity, consequently the decelerating force which acts on the metallic layer is

$$F_x = \frac{1}{\Delta} \left[\frac{1}{2\mu_0} B_0^2 (\Delta_y \Delta L) \right] \left[\mu_0 \sigma v \Delta\, e^{-4\pi \frac{h}{L}} \right]\,. \tag{7.9.24}$$

Note that this force is proportional to the magnetic field stored in the metal, it is proportional to the velocity and the conductivity and it decays exponentially with the height (h).

7.A. Appendix: Non-linear Constitutive Relation – Hysteresis

The constitutive relations considered so far were always linear. In general these relations may be non-linear, time-dependent and given in the form of a matrix. Electromagnetic characterization of a material requires examination of its microscopic features. In the present section we shall examine a more complex constitutive relation. Our goal being determination of the relation between $\vec{H}$ and $\vec{M}$ in a spin-1/2 system, namely a system having two distinct energetic states. It will be shown that rather than a simple linear relation form, the constitutive relation is actually a differential equation.

Let us assume a 1D model in which the individual magnetic dipole of the atom/molecule is denoted by $\vec{m}$. In its vicinity we assume the existence of an effective magnetic field $\vec{H}_{\text{eff}}$ which is a superposition of the external magnetic field $\vec{H}$ and the contribution of the material (magnetization) denoted by $\vec{M}$ hence

$$\vec{H}_{\text{eff}} = \vec{H} + \vec{M}\,. \tag{7.A.1}$$

We further assume that the dipole may be positioned either parallel or anti-parallel to $\vec{H}_{\text{eff}}$ and as such, it may be either in a high energy state

$$\mathcal{E}_{+} = -\vec{m}\cdot(\mu_0\vec{H}_{\text{eff}})\,, \tag{7.A.2}$$

or in a low energy state

$$\mathcal{E}_{-} = +\vec{m}\cdot(\mu_0\vec{H}_{\text{eff}})\,. \tag{7.A.3}$$

Within the framework of this 1D model we denote by N_+ the number of dipoles per unit volume in the higher energy state and by N_- the density of dipoles in the lower energy state. Using this notation the magnetization (M) is given by

$$\vec{M} = \vec{m}\,(N_+ - N_-)\ . \tag{7.A.4}$$

We have to bear in mind that the total number of dipoles per unit volume is conserved:

$$N_+ + N_- = N = \text{const}\ . \tag{7.A.5}$$

Based upon (7.A.4) we observe that the variation in time of the magnetization is determined by the time variation of N_+ and N_- namely,

$$\frac{d}{dt}\,\vec{M} = \vec{m}\,\left(\frac{dN_+}{dt} - \frac{dN_-}{dt}\right)\ , \tag{7.A.6}$$

where it has been assumed that $\vec{m}$ does not vary in time. We now denote by $T_{+\to-}$ the *transition rate* from the upper to lower state and by $T_{-\to+}$ the transition from the lower to the upper state. Consequently, within the framework of linear transition rate theory we are able to determine the following density dynamics:

$$\begin{aligned} \frac{dN_+}{dt} &= -\ \ N_+T_{+\to-} + N_-T_{-\to+} \\ \frac{dN_-}{dt} &= -\ \ N_-T_{-\to+} + N_+T_{+\to-} \end{aligned} \tag{7.A.7}$$

obviously, consistent with the conservation law in (7.A.5).

The transition rate is determined by the deviation from "equilibrium" as generated by the effective magnetic field which acts on the individual dipole:

$$\begin{aligned} T_{+\to-} &= \frac{1}{\tau}\exp\left[-\frac{\vec{m}\cdot(\mu_0\vec{H}_{\text{eff}})}{k_B T}\right], \\ T_{-\to+} &= \frac{1}{\tau}\exp\left[+\frac{\vec{m}\cdot(\mu_0\vec{H}_{\text{eff}}}{k_B T}\right], \end{aligned} \tag{7.A.8}$$

where k_B is the Boltzman constant, T is the temperature of the ensemble and τ is a characteristic time of the material. Substituting (7.A.7-8) in (7.A.6) we obtain

$$\boxed{\frac{d}{dt}\vec{M} + \frac{2}{\tau}\vec{M}\cosh\left[\frac{\vec{m}\cdot(\mu_0\vec{H}+\mu_0\vec{M})}{k_B T}\right] = \frac{2}{\tau}\vec{M}_0\sinh\left[\frac{\vec{m}\cdot(\mu_0\vec{H}+\mu_0\vec{M})}{k_B T}\right],} \tag{7.A.9}$$

which is the differential equation describing a constitutive relation and $\vec{M}_0 \equiv \vec{m}N$ represents the maximum magnetization possible.

7.A.1 Permanent Magnetization

In the absence of any time variations and of an applied field (7.A.9) reads

$$\vec{M} = \vec{M}_0 \tanh \left[\frac{\vec{m} \cdot \mu_0 \vec{M}}{k_B T} \right] . \tag{7.A.10}$$

Figure 7.5 illustrates schematically the two sides of this equation

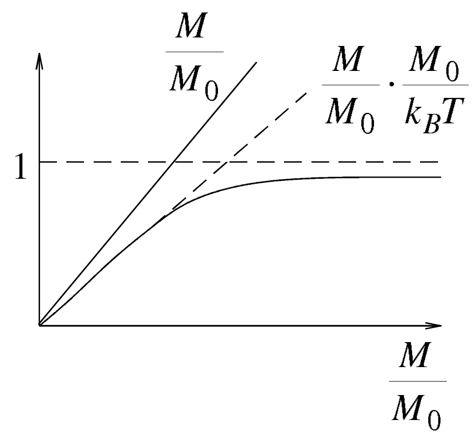

Figure 7.A.1: Schematic representation of the two sides of Eq. (7.A.10) are illustrated.

In case $\frac{\mu_0 M_0 m}{k_B T} < 1$ the only possible solution is $M = 0$. However, if $\frac{\mu_0 M_0 m}{k_B T} > 1$ then the two curves intersect as shown in Figure 7.6. The quantity $T_c = \frac{\mu_0 M_0 m}{k_B}$ is called the *Curie temperature* and it represents the critical temperature below which permanent magnetization is possible, whereas above this temperature the permanent magnetization vanishes. Dependence of the magnetization on temperature is shown in Figure 7.7.

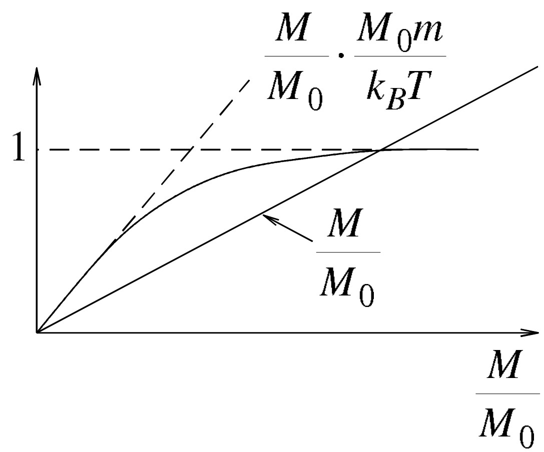

Figure 7.A.2: Condition of permanent magnetization.

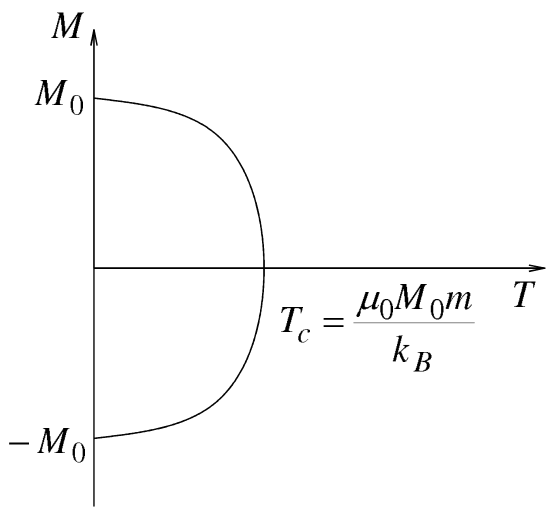

Figure 7.A.3: Magnetization field as a function of temperature.

Exercise #1: Explain why the magnetization decreases as a function of the temperature.

7.A.2 Permanent Magnetization & Applied Magnetic Field

When an external field is applied (7.7.9) simplifies to

$$\vec{M} = \vec{M}_0 \tanh\left[\frac{\vec{m}\cdot(\mu_0\vec{H} + \mu_0\vec{M})}{k_B T}\right] . \tag{7.A.11}$$

We observe that if $\vec{H}$ is parallel to $\vec{M}$ then the magnetization increase whereas in the anti-parallel case it is being reduced. Let us now assume a sphere of radius R and initial magnetization M_1 (in the z-direction). Our goal is to calculate the magnetic field in the whole space. Inside the sphere the magnetic scalar potential is given by

$$\psi = A_1 r\cos\theta \tag{7.A.12}$$

whereas outside

$$\psi = A_2 \frac{1}{r^2}\cos\theta\,. \tag{7.A.13}$$

Continuity of the tangential magnetic field at $r = R$ implies

$$A_1 = \frac{A_2}{R^3} \tag{7.A.14}$$

and the continuity of the radial magnetic induction imposes the condition

$$\begin{aligned} &\mu_0 M_1\cos\theta - \mu_0 A_1\cos\theta = \mu_0\frac{2A_2}{R^3}\cos\theta \\ &\Rightarrow \quad M_1 - A_1 = \frac{2A_2}{R^3}\,. \end{aligned} \tag{7.A.15}$$

From these two equations we conclude that

$$A_1 = \frac{1}{3}\, M_1 \tag{7.A.16}$$

implying that inside the sphere the magnetic field is given by

$$\begin{aligned} H_r &= -\ \frac{1}{3}\, M_1 \cos\theta \\ H_\theta &= \ \ \ \frac{1}{3}\, M_1 \sin\theta\ , \end{aligned} \tag{7.A.17}$$

or

$$H_z = -\frac{1}{3}\, M_1\,. \tag{7.A.18}$$

It is important to point out here that the *finite* boundary of the body is now responsible to the existence of a magnetic field $\vec{H}$. From the perspective of the material this is an applied magnetic field therefore if we now return to (7.A.11) we find that

$$\begin{aligned} M_1 &= M_0 \tanh\left[\frac{m\mu_0}{k_B T}\left(-\frac{1}{3}\, M_1 + M_1\right)\right] \\ &= M_0 \tanh\left[\frac{m\mu_0}{k_B T}\,\frac{2}{3}\, M_1\right] = M_0 \tanh\left[\frac{m\mu_0 M_1}{k_B\left(T\dfrac{3}{2}\right)}\right]\,. \end{aligned} \tag{7.A.19}$$

Comments:

1. The effect of the finite geometry is to reduce the magnetization as if the temperature was 50% higher ($T \to \frac{3}{2}T$)!! See Figure 7.8.

2. The magnetic field inside the material is directed anti-parallel to $\vec{M}_1$.

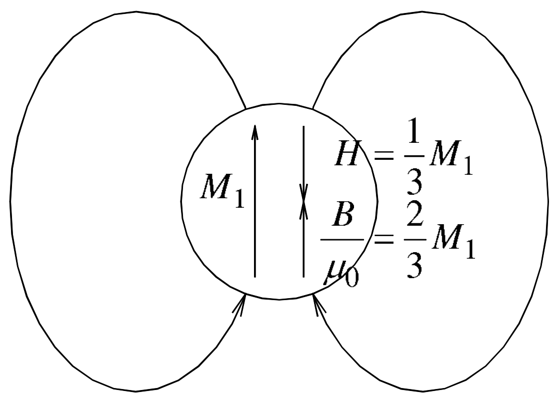

Figure 7.A.4: The magnetic field in the material is anti-parallel to the magnetization field.

7.A.3 Dynamics of Magnetization

The equation in (7.A.9) may be solved numerically and some typical solutions are plotted in Figure 7.A.5. Here we wish to point out some of the main features:

1. The system preserves the information concerning the *history* of the circuit, $\Rightarrow$ in other words it possesses memory characteristics. Specifically, if for a given $\vec{H}$ the system in the fourth quadrant we know that sometimes in the past the system was in the third quadrant in the state $M = -M_0$.
2. There are two regions where the energy is *negative* in the second and fourth quadrant implying that when in this region the system may supply energy.
3. The initial state of polarization does not affect the long-term behavior of the system.

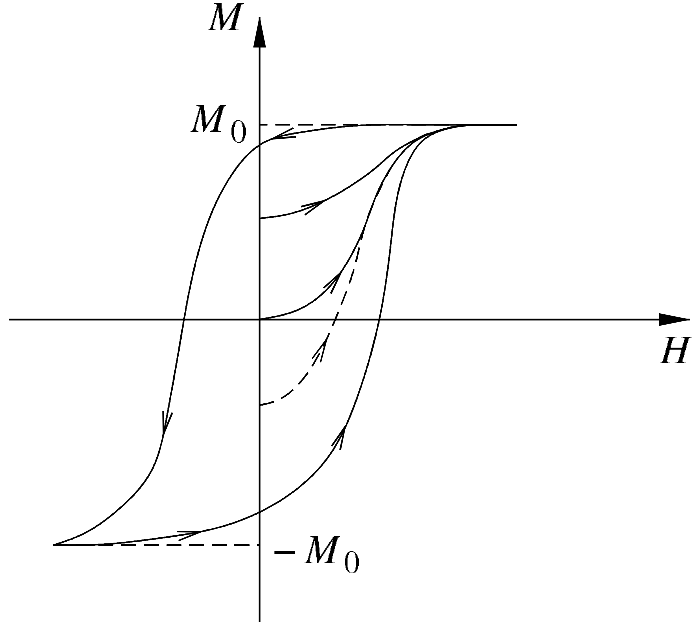

Figure 7.A.5: Hysteresis curve.

7.B. Appendix: Child-Langmuir Limiting Current

Electron emission and electron flow/control are important factors in electronic devices. In many cases it is important to avoid electron emission in order to reduce the probability of breakdown. In other cases it is necessary to control the emission of electrons and to utilize them for monitors, radiation sources and many practical applications. In the present section we discuss a few topics associated with electron emission. There are three main ways to generate free electrons:

Photo-emission: where a photon releases an electron from from a metallic surface provided its energy is larger than the work function (ϕ_W) of the metal.

Thermionic Emission: In this case the temperature (T) of the metal is sufficiently high such that electrons have enough energy to overcome the potential barrier associated with the so-called work-function (ϕ_W). The current density achievable via thermionic emission is given by

$$J[\mathrm{kA/cm^2}] \simeq 0.12 T^2 \, e^{-e\phi_W/k_B T} \, . \tag{7.B.1}$$

Field Emission: The third way to generate free electrons is to "pull" them out from the metal using an intense electric field. The current density in this case is related to the field by

$$J \propto E^2 e^{-E_{\mathrm{cr}}/E} \, ; \tag{7.B.2}$$

This is called Fowler-Nordheim relation. It indicates that the current is quadratic with the electric field provided the latter is significantly larger than a critical value E_{cr}, which is a characteristic of the material and the surface.

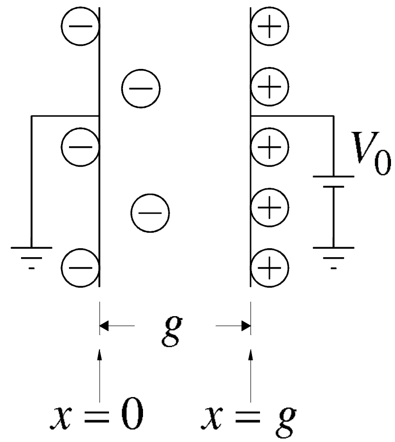

Figure 7.B.1: Vacuum diode.

In either case the total current density which can be extracted, is limited by the charge extracted from the cathode. In what follows this limit will be considered.

For a visualization of this process consider two uniform electrodes separated by a distance g - see Figure 7.20. One is grounded and on the other a positive voltage is applied. If no electrons are extracted from the cathode (left electrode) then the electric field is given by $-V_0/g$. As electrons are extracted they *screen* the cathode from the anode: maximum current is attained when the cathode is completely screened by the electrons, since there is no more field to pull electrons from the surface.

Let us now calculate this limiting current within the framework of a 1D model. The electron density in the diode gap is denoted by $n(x)$ implying a charge density $\rho = -en(x)$ and consequently the Poisson equation reads

$$\frac{d^2}{dx^2}\,\phi = -\frac{\rho}{\varepsilon_0} = \frac{e}{\varepsilon_0}\,n(x)\ . \tag{7.B.3}$$

Since the problem is assumed to be stationary the continuity equation reads

$$\vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \frac{d}{dx}\,J_x = 0 \Rightarrow J_x = -J\ . \tag{7.B.4}$$

Given the density $n(x)$, we can determine the current density as

$$J_x = -en(x)v(x) \tag{7.B.5}$$

where $v(x)$ is the velocity of the electrons at x. Hence

$$n(x) = \frac{J}{ev(x)}\,. \tag{7.B.6}$$

Energy conservation implies

$$\frac{1}{2}\,mv^2(x) - e\phi(x) = \text{const.} \tag{7.B.7}$$

At the cathode the velocity of the electrons is zero and so is the potential, therefore

$$\frac{1}{2}\,mv^2(x) - e\phi(x) = 0 \rightsquigarrow v(x) = \sqrt{\frac{2e\phi(x)}{m}}\,. \tag{7.B.8}$$

We now return to statics: using (7.B.6) and (7.B.8) we may write the Poisson equation in (7.B.3) as

$$\frac{d^2}{dx^2}\phi(x) = \frac{e}{\varepsilon_0}\,\frac{J}{ev(x)} = \frac{e}{\varepsilon_0}\,\frac{J}{e}\,\frac{1}{\sqrt{\dfrac{2e\phi(x)}{m}}}\,. \tag{7.B.9}$$

In an attempt to solve this equation we multiply both sides by $\dfrac{d\phi}{dx}$ thus

$$\left(\frac{d\phi}{dx}\right)\frac{d^2\phi}{dx^2} = \frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}\,\frac{1}{\sqrt{\phi}}\,\frac{d\phi}{dx}\,, \tag{7.B.10}$$

which may also be written as

$$\frac{1}{2}\,\frac{d}{dx}\left[\left(\frac{d\phi}{dx}\right)^2\right] = 2\,\frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}\,\frac{d}{dx}\sqrt{\phi} \tag{7.B.11}$$

or

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - 2\,\frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}\sqrt{\phi}\right] = 0\,. \tag{7.B.12}$$

This implies that

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - 2\,\frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}\sqrt{\phi} = C\,. \tag{7.B.13}$$

The first term on the left is just the electric field square and since we are interested in the *maximum* current density we follow the previous arguments namely, that the cathode is screened. Consequently, we impose $E_x(x = 0) = 0$ therefore $C = 0$ [recall that the potential is zero on the cathode]. This conclusion enables us to write (7.B.13) as

$$\frac{d\phi}{dx} = \sqrt{\frac{4J_{\rm max}}{\varepsilon_0}\sqrt{\frac{m}{2e}}}\,\phi^{1/4}\,. \tag{7.B.14}$$

The integration of this equation is straightforward,

$$\frac{d\phi}{\phi^{1/4}} = \sqrt{\frac{4J_{\rm max}}{\varepsilon_0}\sqrt{\frac{m}{2e}}}\,dx\,, \tag{7.B.15}$$

hence

$$\frac{4}{3}\,V_o^{3/4} = \sqrt{\frac{4J_{\rm max}}{\varepsilon_0}\sqrt{\frac{m}{2e}}}\,g \tag{7.B.16}$$

and finally

$$\begin{aligned} J_{\rm max} &= \frac{16}{9}\,\frac{\varepsilon_0}{4}\,\sqrt{\frac{2e}{m}}\,\frac{V_o^{3/2}}{g^2} \\ J_{\rm max}\left[\frac{\rm A}{\rm cm^2}\right] &\simeq 2.33\times 10^{-6}\frac{(V[{\rm V}])^{3/2}}{(g[{\rm m}])^2}\,. \end{aligned} \tag{7.B.17}$$

This is the Child-Langmuir limiting current. It is proportional to $V_o^{3/2}$ and inverse proportional to g^2. It is important to point out that here we have assumed an ideal vacuum and further, it has been tacitly assumed that any existing ions do not contribute to the net current density.

8. SOME BASIC ELECTRODYNAMIC CONCEPTS

8.1. Plane Waves in Non-Conducting Medium

So far we have seen that energy can be transferred by an electromagnetic field when the latter is guided by wires or plates. One of the most important features of Maxwell's equations is that they support solutions which carry energy in free space without guidance. The simplest manifestation of such a solution is the plane wave. It deserves its name from the fact that the locus of its constant phase is a plane; the solution has the form

$$e^{j\omega t - j\vec{k}\cdot\vec{r}}; \tag{8.1.1}$$

$\vec{k}$ being the wave number and at this stage we shall assume that all its three components are real. Substituting in Maxwell's equation we find immediately several features:

1. Faraday's law

$$\begin{aligned} \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\mu_0\mu_r\vec{H} \;&\rightarrow\; -j\vec{k}\times\underline{\vec{E}} = -j\omega\mu_0\mu_r\underline{\vec{H}} \\ &\rightarrow\; \underline{\vec{H}} = \vec{k}\times\underline{\vec{E}}\frac{1}{\omega\mu_0\mu_r}\,, \end{aligned} \tag{8.1.2}$$

i.e. $\underline{\tilde{\mathrm{E}}}$ & $\underline{\tilde{\mathrm{H}}}$ are perpendicular.

2. Gauss' law

$$\begin{aligned}\vec{\nabla}\cdot\varepsilon_0\varepsilon_r\vec{E}=0 \;&\rightarrow\; -j\vec{k}\cdot(\varepsilon_0\varepsilon_r\underline{\vec{E}})=0\\ &\rightarrow\; \vec{k}\cdot\underline{\vec{E}}=0\end{aligned} \tag{8.1.3}$$

implying that $\vec{k}$ and $\vec{E}$ are perpendicular.

3. Conservation of the magnetic induction

$$\begin{aligned}\vec{\nabla}\cdot\rho_0\mu_r\vec{H}=0 \;&\rightarrow\; -j\vec{k}\cdot(\mu_0\mu_r\underline{\vec{H}})=0\\ &\rightarrow\; \vec{k}\cdot\underline{\vec{H}}=0\;,\end{aligned} \tag{8.1.4}$$

thus $\vec{k}$ is perpendicular to $\vec{H}$ too. If we go back to (8.1.2) we observe that $\vec{k}, \underline{\vec{E}}$ and $\underline{\vec{H}}$ are three orthogonal (in space) vectors.

4. The complex vector Poynting

$$\begin{aligned}\underline{\vec{S}}\equiv\frac{1}{2}\underline{\vec{E}}\times\underline{\vec{H}}^* \;&=\; \frac{1}{2}\left[\underline{\vec{E}}\times(\vec{k}\times\underline{\vec{E}})^*\frac{1}{\omega\mu_0\mu_r}\right]\\ \underline{\vec{S}} \;&=\; \frac{1}{2\omega\mu_0\mu_r}\left[(\underline{\vec{E}}\cdot\underline{\vec{E}}^*)\vec{k}-\underbrace{(\underline{\vec{E}}\cdot\vec{k})}_{=0}\underline{\vec{E}}^*\right]=\frac{1}{2\omega\mu_0\mu_r}|\underline{\vec{E}}|^2\vec{k}\,,\end{aligned} \tag{8.1.5}$$

where we used the fact $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.

Therefore the complex vector Poynting is *real* and parallel to $\vec{k}$. Note that $\vec{\nabla}\cdot\underline{\vec{S}}=0$.

5. Magnetic and electric energy density

$$w_E = \frac{1}{4}\,\varepsilon_0\varepsilon_r \underline{\vec{E}} \cdot \underline{\vec{E}}^* \tag{8.1.6}$$

$$w_M = \frac{1}{4}\,\mu_0\mu_r \underline{\vec{H}} \cdot \underline{\vec{H}}^* = \frac{1}{4}\,\mu_0\mu_r \left[(\vec{k} \times \underline{\vec{E}}) \cdot (\vec{k} \times \underline{\vec{E}}^*)\right] \frac{1}{(\omega\mu_0\mu_r)^2}\,, \tag{8.1.7}$$

where we have substituted (8.1.2).

Now we can take advantage of the fact that

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) \tag{8.1.8}$$

and obtain

$$\begin{aligned} w_M &= \frac{1}{4}\frac{1}{\omega^2\mu_0\mu_r}\left[k^2\,\underline{\vec{E}}\cdot\underline{\vec{E}}^* - (\vec{k}\cdot\underline{\vec{E}}^*)\,(\underbrace{\vec{k}\cdot\ \underline{\vec{E}}}_{=0})\right] \\ &= \frac{1}{4}\,\varepsilon_0\varepsilon_r\ \underline{\vec{E}}\cdot\underline{\vec{E}}^*\frac{k^2c^2}{\omega^2}\,\frac{1}{\mu_r\varepsilon_r}\,. \end{aligned} \tag{8.1.9}$$

Since we have shown that $\vec{\nabla}\cdot\underline{\vec{S}} = 0$ we anticipate that $W_E - W_M = 0$ which implies

$$\vec{k}\cdot\vec{k} = \frac{\omega^2}{c^2}\,\varepsilon_r\mu_r\,. \tag{8.1.10}$$

6. Wave equation: Using Ampere's law

$$\begin{aligned} \vec{\nabla}\times\vec{H} = \partial_t\varepsilon_0\varepsilon_r\vec{E} \ &\rightsquigarrow\ -j\vec{k}\times\underline{\vec{H}} = j\omega\varepsilon_0\varepsilon_r\,\underline{\vec{E}} \\ &\rightarrow\ \underline{\vec{E}} = -\frac{1}{\omega\varepsilon_0\varepsilon_r}\,\vec{k}\times\underline{\vec{H}}\,. \end{aligned} \tag{8.1.11}$$

We may now multiply (8.1.2) by $\vec{k}$:

$$-\varepsilon_0\varepsilon_r\,\omega\underline{\vec{E}} = \vec{k}\times\underline{\vec{H}} = \vec{k}\times\frac{(\vec{k}\times\underline{\vec{E}})}{\omega\mu_0\mu_r} = \left[(\underbrace{\vec{k}\cdot\underline{\vec{E}}}_{=0})\vec{k} - (\vec{k}\cdot\vec{k})\underline{\vec{E}}\right]\ \frac{1}{\omega\mu_0\mu_r} \tag{8.1.12}$$

hence

$$\left[\vec{k}\cdot\vec{k} - \frac{\omega^2}{c^2}\,\varepsilon_r\mu_r\right]\,\underline{\vec{E}} = 0\,, \tag{8.1.13}$$

which is the wave equation and for a non zero solution $\vec{E} = 0$ it implies

$$\vec{k}\cdot\vec{k} = \frac{\omega^2}{c^2}\,\varepsilon_r\mu_r\,. \tag{8.1.14}$$

This result is identical to that in (8.1.11) obtained from energy considerations.

7. Phase velocity. In case of a plane wave the temporal and spatial variations of the field are according to $\cos(\omega t - kz + \psi)$. Clearly, the region in space of constant phase is determined by a plane (from here the name plane wave) $\omega t - kz = \text{const}$. For an external observer to measure this *constant phase* it needs to be stationary with this plane i.e. $\omega\Delta t - k\Delta z = 0$ thus its velocity ought to be

$$V_{\text{ph}} \equiv \frac{\Delta z}{\Delta t} = \frac{\omega}{k}\,; \tag{8.1.15}$$

this is the *phase-velocity*.

8. Group velocity. Consider **two** plane waves

$$\psi(z,t) = \cos(\omega_+ t - k_+ z) + \cos(\omega_- t - k_- z) \tag{8.1.16}$$

of two different frequencies $\omega_\pm = \omega \pm \Delta\omega$ and their corresponding wave numbers $k_\pm = k \pm \Delta k$. Using trigonometric identities we have

$$\psi(z,t) = 2\cos(\Delta\omega t - \Delta k z)\cos(\omega t - kz)\,. \tag{8.1.17}$$

Now, assuming that $\omega \gg |\Delta\omega|$ and $k \gg |\Delta k|$ we may consider $\cos(\Delta\omega t - \Delta k z)$ as a slowly varying amplitude. Therefore, we may ask what is the speed an observer needs to move in order to measure a *constant amplitude*. Similar to the previous case, $\Delta\omega t - \Delta k z = \text{const}$. This is to say that, $\Delta\omega\delta t - \Delta k\delta z = 0$, or at the limit $\delta t \to 0$ we may define the so-called group velocity

$$V_{\rm gr} = \frac{\partial\omega}{\partial k}\,. \tag{8.1.18}$$

In the simple configurations discussed here both $V_{\rm ph}$ and $V_{\rm gr}$ are constant satisfying

$$V_{\rm gr}\,V_{\rm ph} = c^2/\varepsilon_r\mu_r\,. \tag{8.1.19}$$

9. Dispersion. Consider an electromagnetic pulse that when monitored at $z = 0$ has a known form, for example a pulse (though the next steps are not dependent on this specific form):

$$E_x(z=0,t) = E_0\begin{cases}1 & |t| < \tau_p/2\\ 0 & |t| > \tau_p/2\end{cases} \equiv E_0\mathbb{P}(t,\tau_p)\,, \tag{8.1.20}$$

where τ_p is the pulse duration. The question is what will measure a different observer located down the road at $z = d$ equipped with an identical probe.

In order to address this question it is convenient to define the Fourier transform of the signal at $z = 0$ i.e.

$$E_x(z=0,t) = \int_{-\infty}^{\infty} d\omega\, \mathcal{E}(\omega)\, e^{j\omega t} \Leftrightarrow \mathcal{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt'\, e^{-j\omega t'}\, E_x(z=0,t')\,. \tag{8.1.21}$$

With the spectrum established we may proceed and determine the functional dependence of this pulse in space, assuming it propagates along the positive direction of the z-axis

$$E_x(z \neq 0,t) = \int_{-\infty}^{\infty} d\omega\, \mathcal{E}(\omega)\, e^{j\omega t - jk(\omega)z}\,. \tag{8.1.22}$$

Substituting Eq. (8.1.21) we obtain

$$\begin{aligned} E_x(z \geq 0,t) &= \int_{-\infty}^{\infty} d\omega \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dt'\, e^{-j\omega t'}\, E_x(z=0,t')\right] e^{j\omega t - jk(\omega)z} \\ &= \int_{-\infty}^{\infty} dt'\, E_x(z=0,t')\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{j\omega(t-t') - jk(\omega)z}\,. \end{aligned} \tag{8.1.23}$$

The expression

$$G(t,z|t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{j\omega(t-t') - jk(\omega)z} \tag{8.1.24}$$

may be conceived as Green's function i.e.,

$$E_x(z \geq 0, t) = \int\limits_{-\infty}^{\infty} dt'\, G(t, z|t')\, E_x(z = 0, t')\,. \tag{8.1.25}$$

Evidently, this is a convolution integral therefore the shape of the pulse at $z > 0$ will be preserved only if Green's function becomes a Dirac delta function and the only case where it happens is, if the phase velocity of *all* frequencies is the same. In order to prove this statement consider Eq. (8.1.10) which entails

$$G(t, z|t') = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\omega\, e^{j\omega\left(t - t' - \frac{z}{c}\sqrt{\varepsilon_r \mu_r}\right)}$$

$$= \delta\left(t - t' - \frac{z}{c}\sqrt{\varepsilon_r \mu_r}\right)$$

or

$$E_x(z \geq 0, t) = E_x\left(0, t' = t - \frac{z}{c}\sqrt{\varepsilon_r \mu_r}\right)\,. \tag{8.1.26}$$

This is to say that the pulse propagates at a velocity $c/\sqrt{\varepsilon_r \mu_r}$ which in case of vacuum is c. And the pulse reaches the point $z = d$ after $\Delta t = \dfrac{d}{c}\sqrt{\varepsilon_r \mu_r}$.

In order to understand how important the fact is that all the waves have the same phase velocity, let us assume $\mu_r = 1$ and a dielectric coefficient which is frequency dependent. For the sake of simplicity, we shall assume that it equals ε_r for frequencies below a critical value (ω_{cr}) and unity otherwise

$$\varepsilon(\omega) = \begin{cases} \varepsilon_r & |\omega| < \omega_{cr} \\ 1 & |\omega| > \omega_{cr} \,. \end{cases} \tag{8.1.27}$$

This is to say that waves of low frequency have a phase velocity $c/\sqrt{\varepsilon_r}$, whereas high frequency waves have the same speed as if in vacuum.

We may now evaluate $G(t, z|t')$ subject to the assumption above i.e.

$$G(t, z|t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{j\omega(t - t') - jk(\omega)z}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{j\omega(t - t') - j \, \frac{\omega}{c} \, \sqrt{\varepsilon(\omega)} z}$$

$$= \frac{1}{2\pi} \left\{ \int_{-\infty}^{-\omega_{cr}} d\omega \, e^{j\omega(t - t') - j \, \frac{\omega}{c} \, z} + \int_{-\omega_{cr}}^{\omega_{cr}} d\omega \, e^{j\omega(t - t') - j \, \frac{\omega}{c} \, \sqrt{\varepsilon_r} z} \right.$$

$$\left. + \int_{\omega_{cr}}^{\infty} d\omega \, e^{j\omega(t - t') - j \, \frac{\omega}{c} \, z} \right\}. \tag{8.1.28}$$

Without loss of generality we may add and subtract $\frac{1}{2\pi}\int\limits_{-\omega_{cr}}^{\omega_{cr}} d\omega\, e^{j\omega(t-t') - j\frac{\omega}{c}z}$, in which case

$$G(t,z|t') = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} d\omega\, e^{j\omega\left(t-t'-\frac{z}{c}\right)} + \frac{1}{2\pi}\int\limits_{-\omega_{cr}}^{\omega_{cr}} d\omega\, e^{j\omega(t-t') - \frac{z}{c}\sqrt{\varepsilon_r}} - \frac{1}{2\pi}\int\limits_{-\omega_{cr}}^{\omega_{cr}} d\omega\, e^{j\omega\left(t-t'-\frac{z}{c}\right)}$$

$$= \delta\left(t'-t+\frac{z}{c}\right) + \frac{\omega_{cr}}{\pi}\,\mathrm{sinc}\left[\omega_{cr}\left(t'-t+\frac{z}{c}\sqrt{\varepsilon_r}\right)\right] - \frac{\omega_{cr}}{\pi}\,\mathrm{sinc}\left[\omega_{cr}\left(t'-t+\frac{z}{c}\right)\right], \qquad (8.1.29)$$

wherein $\mathrm{sinc}\,(\xi) \equiv \sin(\xi)/\xi$. Substituting in Eq. (8.1.25) we get

$$E_x(z,t) = \int\limits_{-\infty}^{\infty} dt'\, G(t,z|t')\, E_x(0,t')$$

$$= E_x\left(0, t-\frac{z}{c}\right) + \frac{\omega_{cr}}{\pi}\int\limits_{-\infty}^{\infty} dt'\, E_x(0,z')$$

$$\times\left\{\mathrm{sinc}\left[\omega_{cr}\left(t'-t+\frac{z}{c}\sqrt{\varepsilon_r}\right)\right] - \mathrm{sinc}\left[\omega_{cr}\left(t'-t+\frac{z}{c}\right)\right]\right\} \qquad (8.1.30)$$

clearly indicating that the pulse does not preserve its shape. This phenomenon when a pulse changes its shape because waves at different frequencies have different velocities is called *dispersion*. The equation that determines the relation between the wave number k and the angular frequency ω is called the *dispersion relation*. Equation (8.1.14) or $k = \frac{\omega}{c}\sqrt{\varepsilon_r \mu_r}$ is the simplest manifestation of a dispersion relation.

Comment: It is important to point out that the dispersion is not only a result of frequency dependence of the material but also the geometry of the structure where the wave propagates.

8.2. White Noise and Cerenkov Radiation

The source of electromagnetic waves is always charged particles. Let us now investigate the waves attached to a point-charge

$$J_z(r,z,t) = -qv\delta\left(z - vt\right)\frac{1}{2\pi r}\,\delta(r) \tag{8.2.1}$$

that moves in a dielectric medium (e.g. glass). It is convenient to determine the Fourier transform of the current density

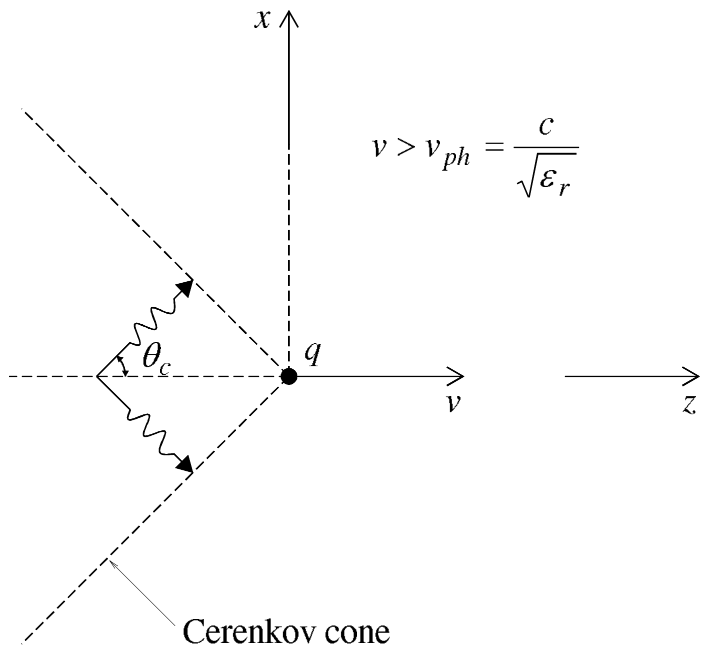

Figure 8.1: Charge that moves in a dielectric medium faster than the speed of light, $c/\sqrt{\varepsilon_r}$, in that medium, generates Cerenkov radiation.

$$\begin{aligned}\underline{J}_z(r,z;\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-j\omega t}\, J_z(r,z,t) \\ &= \frac{1}{2\pi}\,(-qv)\frac{1}{2\pi r}\,\delta(r)\int_{-\infty}^{\infty} dt\, e^{-j\omega t}\,\delta(z-vt) \\ &= \frac{1}{2\pi}\,(-qv)\frac{1}{2\pi r}\,\delta(r)\, e^{-j\frac{\omega}{v}z}\,\frac{1}{v} \\ &= \frac{-q}{2\pi}\,\frac{\delta(r)}{2\pi r}\, e^{-j\frac{\omega}{v}z}\,. \end{aligned} \tag{8.2.2}$$

Here we observe that the spectrum of a moving point charge is constant (frequency independent) and the phase is linear in frequency. This is one possible interpretation of *white noise*. This particle moves in a dielectric medium $\varepsilon_r \geq 1$ and $\mu_r = 1$. The z-component of J_z excites the z-component of the magnetic vector potential $\underline{A}_z(r,z;\omega)$ which is a solution of:

$$\left[\nabla^2 + \varepsilon_r\,\frac{\omega^2}{c^2}\right]\underline{A}_z(r,z;\omega) \;=-\;\mu_0\underline{J}_z(r,z;\omega) = \frac{q\mu_0}{2\pi}\,\frac{\delta(r)}{2\pi r}\,e^{-j\frac{\omega}{v}z}\,. \tag{8.2.3}$$

The term $e^{-j\frac{\omega}{v}z}$ dictates that $\underline{A}_z(r,z;\omega)$ is given by

$$\underline{A}_z(r,z;\omega) = a_z(r)e^{-j\frac{\omega}{v}z} \tag{8.2.4}$$

hence

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{\omega^2}{c^2}\left(\frac{1}{\beta^2}-\varepsilon_r\right)\right]a_z(r)=\frac{q\mu_0}{2\pi}\frac{\delta(r)}{2\pi r}, \tag{8.2.5}$$

where $\beta = v/c$.

The general solution of this equation is

$$a_z(r)=C_1K_o\left[r\frac{\omega}{c}\sqrt{\frac{1}{\beta^2}-\varepsilon_r}\right]+C_2I_o\left(r\frac{\omega}{c}\sqrt{\frac{1}{\beta^2}-\varepsilon_r}\right) \tag{8.2.6}$$

but $K_o(\xi)\underset{\xi\gg 1}{\longrightarrow}\dfrac{e^{-\xi}}{\sqrt{\xi}}$ and $I_o(\xi)\underset{\xi\gg 1}{\longrightarrow}\dfrac{e^{\xi}}{\sqrt{\xi}}$. Consequently $C_2 = 0$ otherwise the solution diverges for $r\to\infty$. In order to determine C_1 we can integrate (8.2.5) in the interval $r: 0\to\delta$

$$C_1\int_0^\delta dr\,r\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}K_o\left[\frac{\omega}{c}\sqrt{\frac{1}{\beta^2}-\varepsilon_r}\,r\right]=\frac{q\mu_0}{(2\pi)^2}, \tag{8.2.7}$$

hence

$$C_1\left[r\frac{d}{dr}K_o\left(\frac{\omega}{c}\sqrt{\frac{1}{\beta^2}-\varepsilon_r}\,r\right)\right|_{r=\delta}=\frac{q\mu_0}{(2\pi)^2}.$$

It was previously indicated that for small argument $K_o(\xi)\simeq -\ln(\xi)$ therefore

$$C_1\left[\xi\frac{d}{d\xi}K_o(\xi)\right]=C_1\left[\xi(-)\frac{1}{\xi}\right]=-C_1=\frac{q\mu_0}{(2\pi)^2}. \tag{8.2.8}$$

The magnetic vector potential reads

$$\underline{A}_z(r,z;\omega) = -\frac{q\mu_0}{(2\pi)^2}\, K_o\left(\left|\frac{\omega}{c}\right|\, r\,\sqrt{\frac{1}{\beta^2}-\varepsilon_r}\right)\, e^{-j\frac{\omega}{v}z} \tag{8.2.9}$$

and for large arguments

$$A_z(r,z;\omega) \simeq -\frac{q\mu_0}{(2\pi)^2}\,\frac{e^{-\frac{\omega}{c}r\sqrt{\frac{1}{\beta^2}-\varepsilon_r}}}{\sqrt{\frac{2}{\pi}\,\frac{\omega}{c}r}\,\sqrt{\frac{1}{\beta^2}-\varepsilon_r}}\, e^{-j\frac{\omega}{v}z}\,. \tag{8.2.10}$$

1. In vacuum ($\varepsilon_r = 1$) the field decays exponentially in the radial direction.

2. If $\frac{1}{\beta^2} - \varepsilon_r > 0$ i.e. $v < V_{\rm ph} = c/\sqrt{\epsilon_r}$ then the behavior is similar to that in vacuum.

3. If $\frac{1}{\beta^2} - \varepsilon_r < 0$ i.e. $v < V_{\rm ph} = c/\sqrt{\epsilon_r}$ then the particle emits a wave which propagates at an angle

$$\tan\theta = \frac{\frac{\omega}{c}\sqrt{\varepsilon_r - \frac{1}{\beta^2}}}{\frac{\omega}{c}\,\frac{1}{\beta}} = \sqrt{\varepsilon_r\beta^2 - 1} \Rightarrow \boxed{\cos\theta_c = \frac{1}{\beta\sqrt{\varepsilon_r}}}.$$

This is called Cerenkov radiation and θ_c is the Cerenkov angle.

8.3. Radiation from an Oscillating Dipole

Among the simplest and at the same time widespread manifestation of radiation emission is the one occurring when two charges oscillate. In Chapter 7 we examined the potential of two stationary charges. In this section we allow them to oscillate. Here are the main assumptions of the problem we analyze:

1. The charges oscillate at an angular frequency ω. The motion of the charges is in the z-direction only generating a current density $J_z = j\omega \frac{d}{2} q \frac{1}{2\pi r}\delta(r)\,\delta(z)$. In free space (no boundary conditions) this current density generates magnetic vector potential A_z.
2. The system has azimuthal symmetry $\Rightarrow \partial_\phi \simeq 0$.
3. By virtue of a Lorentz gauge $\Phi \neq 0$.
4. $(A_z, \Phi) \rightsquigarrow \Pi_z$

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right] \underline{\Pi}_z = -\underline{\mathcal{P}}_z/\varepsilon_0\,. \tag{8.3.1}$$

Away from the source

$$\frac{1}{r}\,\frac{d^2}{dr^2}\,(r\underline{\Pi}_z) + \frac{\omega^2}{c^2}\,\underline{\Pi}_z = 0 \tag{8.3.2}$$

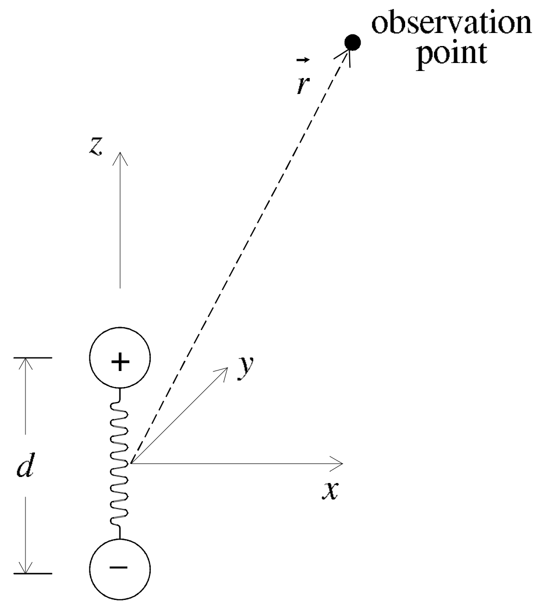

Figure 8.2: Oscillating dipole.

and the solution is

$$\underline{\Pi}_z = \underline{A}\,\frac{e^{-j\frac{\omega}{c}r}}{r}\,. \tag{8.3.3}$$

In order to determine the amplitude A recall that the potential of a static dipole is [see Eq. (7.1.6)]

$$\underline{\Phi} = \frac{p}{4\pi\varepsilon_0}\,\frac{1}{r^2}\,\cos\theta\,, \tag{8.3.4}$$

where $p \equiv qd$.

According to the definition in (2.3.8)

$$\underline{\Phi} = -\frac{\partial \underline{\Pi}_z}{\partial z} \tag{8.3.5}$$

which for $\omega = 0$ reads

$$\Phi = -\frac{\partial}{\partial z}\left(\frac{A}{r}\right) = A\,\frac{1}{r^2}\,\cos\theta\,. \tag{8.3.6}$$

Comparing with (8.3.4) we obtain

$$A = \frac{1}{4\pi\varepsilon_0}\,p$$

hence

$$\underline{\Pi}_z = \frac{p}{4\pi\varepsilon_0}\,\frac{1}{r}\,e^{-j\dfrac{\omega}{c}r}\,. \tag{8.3.7}$$

The next step is to determine the emitted power. For this purpose we have to calculate the Poynting vector

$$\underline{S}_r = \frac{1}{2}\,\left[\underline{E}_\theta \underline{H}_\varphi^* - \underline{E}_\varphi \underline{H}_\theta^*\right]\,. \tag{8.3.8}$$

The only terms which contribute to the total power are those which decay as $\dfrac{1}{r}$ hence

$$\begin{aligned}
\underline{A}_z = j\frac{\omega}{c^2}\,\underline{\Pi}_z \quad &= j\frac{\omega}{c^2}\frac{p}{4\pi\varepsilon_0}\frac{e^{-j\frac{\omega}{c}r}}{r} \\
\underline{\Phi} = -\frac{\partial\underline{\Pi}_z}{\partial z} \quad &= -\frac{p}{4\pi\varepsilon_0}\left[\frac{-j\frac{\omega}{c}r\,e^{-j\frac{\omega}{c}r} - e^{-j\frac{\omega}{c}r}}{r^2}\right]\cos\theta \\
&\simeq j\,\frac{\omega}{c}\,\frac{p}{4\pi\varepsilon_0}\,\frac{e^{-j\frac{\omega}{c}r}}{r}\cos\theta\,.
\end{aligned} \tag{8.3.9}$$

Since electric scalar potential $\underline{\Phi}$ is φ independent and the magnetic vector potential $\underline{\vec{A}}$ has no component in the azimuthal direction we conclude that $\underline{E}_\varphi = 0$ which implies that

$$\underline{S}_r = \frac{1}{2}\,\underline{E}_\theta\,\underline{H}_\varphi^* \,. \tag{8.3.10}$$

In cylindrical coordinates

$$\mu_0 \underline{H}_\varphi = -\frac{\partial \underline{A}_z}{\partial \rho}\,, \tag{8.3.11}$$

where $r = \sqrt{\rho^2 + z^2}$, therefore the azimuthal component of the magnetic field is given by

$$\mu_0 \underline{H}_\varphi = -j\frac{\omega}{c^2}\,\frac{p}{4\pi\varepsilon_0}\,\frac{\partial}{\partial\rho}\,\left(\frac{e^{-j\frac{\omega}{c}r}}{r}\right)$$

$$\simeq -j\frac{\omega}{c^2}\,\frac{p}{4\pi\varepsilon_0}\,\frac{e^{-j\frac{\omega}{c}r}}{r}\,\left(-j\frac{\omega}{c}\right)\sin\theta$$

$$\underline{H}_\varphi \simeq \frac{-1}{\mu_0}\,\frac{\omega^2}{c^3}\,\frac{p}{4\pi\varepsilon_0}\,\frac{e^{-j\frac{\omega}{c}r}}{r}\sin\theta\,. \tag{8.3.12}$$

Next we calculate E_θ. In principle it is given by $\underline{E}_\theta = -(1/r)\partial\underline{\Phi}/\partial\theta - j\omega\underline{A}_\theta$ but since the first term is proportional to $1/r^2$ we shall consider only the second one. Thus

$$\underline{A}_\theta \simeq \underline{A}_z \sin\theta \tag{8.3.13}$$

and consequently

$$\begin{aligned}
\underline{E}_\theta &= - \; j\omega\, \underline{A}_\theta = +j\omega\, \underline{A}_z \sin\theta \\
&\simeq - \; j\frac{\omega}{c} \frac{p}{4\pi\varepsilon_0} \frac{e^{-j\frac{\omega}{c}r}}{r} \left(-j\frac{\omega}{c}\right) \sin\theta \\
&\simeq \; - \frac{\omega^2}{c^2} \frac{p}{4\pi\varepsilon_0} \frac{e^{-j\frac{\omega}{c}r}}{r} \sin\theta \,.
\end{aligned} \tag{8.3.14}$$

According to (8.3.10) we now obtain

$$\begin{aligned}
\underline{S}_r &= \frac{1}{2} \left[-\frac{\omega^2}{c^2} \frac{p}{4\pi\varepsilon_0} \frac{e^{-j\frac{\omega}{c}r}}{r} \sin\theta \right] \left[\frac{-1}{\mu_0} \frac{\omega^2}{c^2} \frac{p}{4\pi\varepsilon_0} \frac{e^{+j\frac{\omega}{c}r}}{r} \sin\theta \right] \\
&= \frac{1}{2\mu_0 c} \left(\frac{\omega}{c}\right)^4 \left(\frac{p}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^2} \sin^2\theta
\end{aligned} \tag{8.3.15}$$

and the total average power reads

$$P = r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \left[\frac{1}{2\mu_0} \left(\frac{\omega}{c} \frac{p}{4\pi\varepsilon_0} \right)^2 \frac{1}{r^2} \sin^2\theta \right]$$

$$= 2\pi \frac{1}{2\mu_0 c} \left(\frac{p\frac{\omega^2}{c^2}}{4\pi\varepsilon_0} \right)^2 \int_0^{\pi} d\theta \, \sin^3\theta \, . \tag{8.3.16}$$

The integral can be evaluated analytically

$$\int_0^{\pi} d\theta \sin^3\theta = \int_0^{\pi} d\theta [\sin\theta - \sin\theta \cos^2\theta] = 2 + \left(\frac{1}{3} \cos^3\theta \Big|_0^{\pi} \right) = 2 - \frac{2}{3} = \frac{4}{3}$$

hence

$$P = 2\pi \frac{1}{2\mu_0 c} \left(\frac{p\frac{\omega^2}{c^2}}{4\pi\varepsilon_0} \right)^2 \frac{4}{3} = \frac{1}{2\eta_o} \left[\frac{p\frac{\omega^2}{c^2}}{4\pi\varepsilon_0} \right]^2 \frac{8\pi}{3} \, . \tag{8.3.17}$$

Denoting the current associated with the oscillation of the dipole by $I \equiv q\omega$ we obtain

$$\boxed{P = \frac{1}{2} \eta_o I^2 \left[\left(\frac{\omega}{c} d \right)^2 \frac{1}{6\pi} \right]} . \tag{8.3.18}$$

Comment #1: Based on this observation we may define the impedance of this radiating dipole

$$\mathbb{R} = \frac{P}{\frac{1}{2}I^2} = \eta_0 \frac{1}{6\pi}\left(\frac{\omega}{c}d\right)^2 . \tag{8.3.19}$$

Comment #2: Another interesting observation relies on the fact that the average power emitted per period of the radiation field

$$W = PT = 2\pi P/\omega \,,$$

hence

$$W = \frac{1}{2}\eta_0 \, p^2 \frac{\omega^3}{c^2} . \tag{8.3.20}$$

Bearing in mind that the energy of a single photon is expressed in terms of Plank's constant is given by ($\hbar = 1.054 \times 10^{-34}\,\mathrm{Joule} \cdot \mathrm{sec}$)

$$W_{\mathrm{ph}} = \hbar\omega \,,$$

then the number of photons by the dipole is given by

$$
\begin{aligned}
N_{\rm ph} = & \frac{W}{W_{\rm ph}} = \left[\frac{p\,\dfrac{\omega}{c}}{e}\right]^2 \frac{2\pi}{3}\,\frac{e^2}{4\pi\varepsilon_0}\,\frac{1}{\hbar c} \\
= & \frac{2\pi}{3}\,\alpha\left(\frac{p\,\dfrac{\omega}{c}}{e}\right)^2 ,
\end{aligned}
\tag{8.3.21}
$$

wherein $\alpha \equiv \dfrac{e^2}{4\pi\varepsilon_0}\dfrac{1}{\hbar c} \simeq \dfrac{1}{137}$ is the so-called fine structure constant and $e \equiv 1.6 \times 10^{-19}[C]$ is the electron's charge. Examining this relation (8.3.21) for a single dipole reveals that the number of photons emitted per dipole is very small bearing in mind that typically the size of the dipole is much smaller than the wavelength.

Comment #3: We may now re-examine one of the basic assumptions of the analysis in Section (8.3). Namely, the fact that the oscillating dipole has a constant complitude d. It is possible to conceive this dipole as a harmonic oscillator which when its radiation is ignored satisfies

$$
m\,\frac{d^2}{dt^2}\,Z + KZ = 0\,,
\tag{8.3.22}
$$

wherein m is the mass and K is the "Hook coefficient" representing the binding potential – whatever its characteristics are. Multiplying by $\dfrac{dZ}{dt}$ we obtain

$$\frac{d}{dt}\left[\underbrace{\frac{m}{2}\left(\frac{dZ}{dt}\right)^2}_{W_{\rm KIN}}+\underbrace{\frac{K}{2}Z^2}_{W_{\rm POT}}\right]=0\,, \tag{8.3.23}$$

which is the energy conservation when the radiation emitted is ignored. Note that based on Eq. (8.3.22) the oscillating frequency is $\omega=\sqrt{\dfrac{K}{m}}$. It is therefore evident that according to Figure 8.2 the oscillatory motion is

$$Z(t)=\frac{d}{2}\cos(\omega t+\psi) \tag{8.3.24}$$

implying, according to Eq. (8.3.23), that the total energy

$$W_{\rm TOTAL}=W_{\rm KIN}+W_{\rm POT}=\frac{m}{2}\left(\frac{d}{2}\omega\right)^2 \tag{8.3.25}$$

is constant. This is justified as long as we ignore the electromagnetic energy emitted by this dipole. When accounting for this energy namely, assuming that the change in the total energy ($W_{\rm TOTAL}$) equals the emitted (em) energy then

$$\frac{d}{dt} W_{\text{TOTAL}} = -P = -\left[\frac{2}{3\pi}\frac{\eta_0 I^2}{mc^2}\right] W_{\text{TOTAL}} \,. \tag{8.3.26}$$

Since in the past we denoted $I = e\omega$ and assuming now that e is the charge of an electron, then

$$\frac{d}{dt} W_{\text{TOTAL}} = -\left[\frac{2}{3}\frac{r_e}{c}(2\omega)^2\right] W_{\text{TOTAL}} \,, \tag{8.3.27}$$

wherein $r_e \simeq 2.8 \times 10^{-15}$ [m] is the so-called classical radius of the electron

$$r_e \equiv \frac{e^2}{4\pi\varepsilon_0}\frac{1}{mc^2} \,. \tag{8.3.28}$$

According to Eq. (8.3.27) the energy of the dipole is not constant but decays exponentially with a time-constant

$$\tau = \frac{3}{8}\frac{1}{\omega}\frac{1}{\frac{\omega}{c} r_e} \,. \tag{8.3.29}$$

At 1[GHz] this time-constant is 45[min] whereas at 100[THz] it is less than 1[μsec]. Consequently, for most practical purposes the decay process is negligible on the scale of a few periods of the oscillation.

Comment #4: Subject to the far-field approximation we have concluded that the amplitude of the oscillating dipole is not constant. From the perspective of the oscillating charge this implies that in addition to the "binding" force there is another force component associated with the radiation field.

Evaluation of this field is facilitated by three straightforward observations: first we know the total average emitted power

$$P = \frac{1}{2}\eta_0 \, |I|^2 \left[\left(\frac{\omega}{c} d \right)^2 \frac{1}{6\pi} \right] \tag{8.3.30}$$

as evaluated in the framework of the far-field approximation. Second, this radiation is emitted in a loss-less medium implying that the far-field power equals the power generated by the dipole or according to the Poynting theorem

$$P = -\,\mathrm{Re}\left\{ \frac{1}{2} \int dv \vec{J}^{*} \cdot \vec{E} \right\} . \tag{8.3.31}$$

Third, the current density which generates this power is

$$\begin{aligned} J_z &\simeq -q \left(j\omega \frac{d}{2} \right) \frac{1}{2\pi\rho} \, \delta(\rho) \, \delta(z) \\ &\simeq -j\omega p \frac{1}{2} \frac{1}{2\pi\rho} \, \delta(\rho) \, d(z) \end{aligned} \tag{8.3.32}$$

implying that the power in (8.3.31) is

$$P = -\frac{1}{2} \, \mathrm{Re} \left\{ j\omega p \frac{1}{2} \, E_z^{(\mathrm{rad})} \right\} . \tag{8.3.33}$$

Using the definition of the dipole-moment ($p = qd$) and comparing (8.3.33) with (8.3.30) we find that the *radiation reaction* is proportional to the dipole moment and cubic with the frequency

$$\boxed{E_z^{(\mathrm{rad})} = j\,\frac{2}{3}\,(2p)\,\left(\frac{\omega}{c}\right)^3\,\frac{1}{4\pi\varepsilon_0}}. \tag{8.3.34}$$