To appear in IEEE Trans. on Automatic Control

# Competitive routing in networks with polynomial cost

**Eitan Altman**
INRIA B.P. 93
06902 Sophia Antipolis Cedex
France
Tel: (33) 4 92 38 77 86;
Fax: (33) 4 92 38 77 65;
Email: altman@sophia.inria.fr

**Tamer Başar**[*] [1]
Coordinated Science Laboratory
University of Illinois, 1308 West Main Str.
Urbana, IL 61801-2307, USA
Tel: (1) 217 333-3607;
Fax: (1) 217 244-1653;
Email: tbasar@decision.csl.uiuc.edu

**Tania Jiménez**
C.E.S.I.M.O., Universidad de los Andes
Mérida, Venezuela, and INRIA
B.P. 93, 06902 Sophia Antipolis Cedex, France
Tel: (33) 4 92 38 79 74;
Fax: (33) 4 92 38 71 93;
Email: tjimenez@sophia.inria.fr

**Nahum Shimkin**
Electrical Engineering Department
Technion, Haifa 32000
Israel
Tel: (+972) 4 829 4734;
Fax: +972 4 906990;
Email: shimkin@ee.technion.ac.il

May 2000

**Abstract**

We study a class of noncooperative general topology networks shared by $N$ users. Each user has a given flow which it has to ship from a source to a destination. We consider a class of polynomial link cost functions, adopted originally in the context of road traffic modeling, and show that these costs have appealing properties that lead to predictable and efficient network flows. In particular, we show that the Nash equilibrium is unique, and is moreover efficient, i.e. it coincides with the solution of a corresponding global optimization problem with a single user. These properties make the cost structure attractive for traffic regulation and link pricing in telecommunication networks. We finally discuss the computation of the equilibrium in the special case of the affine cost structure for a topology of parallel links.

**Keywords.** Networks; routing; nonzero-sum games; noncooperative equilibria.

# 1 Introduction

We consider in this paper a routing problem in networks, in which paths from a source to a destination are to be established by non-cooperating agents, to whom we refer to as *players* or *users*. In the context of telecommunication networks, players correspond to different traffic sources that need to determine the routing of their traffic over a shared network. A similar setting has also been studied in the context of road traffic networks [8], where a player can be viewed as a transportation company which is to ship a flow of vehicles.

A natural framework to analyze this class of problems is that of noncooperative game theory, and an appropriate solution concept is that of Nash equilibrium [2]: a composite routing policy for the users constitutes a Nash equilibrium if no user can gain by unilaterally deviating from his own policy.

There exists rich literature [6, 11, 14, 16] on the analysis of equilibria in networks, particularly in the context of the Wardrop equilibrium [20] which pertains to the case of infinitesimal users (e.g. a single car in a road traffic context). In such a framework the impact of a single user on the congestion experienced by other users is negligible. Our focus here, however, is *competitive routing*, where the network is shared by several users with each one having a non-negligible amount of flow. This concept has attracted considerable attention in recent years, such as the work reported in [1, 8, 9, 12, 15], which is our starting point here. In these cited papers, conditions have been obtained for the existence and uniqueness of an equilibrium. This has allowed, in particular, the design of network management policies that induce efficient equilibria [12]. This framework has also been extended to the context of repeated games in [13], in which cooperation can be induced by using policies that include threats of punishments for deviating users.

A desired property that an equilibrium would possess is efficiency, i.e. social optimality. It is well known that Nash equilibria in routing games are generally non-efficient. In particular, this non-efficiency can lead to paradoxical phenomena (e.g. the Braess's paradox [2]) where adding a link to the network could result in an increase of cost to all users. For examples of non-efficient behavior and paradoxes, we could cite [6, 21] for road-traffic examples, [5, 12] for examples in a telecommunications setting, and [9, 10] for examples in distributed computing.

In this paper we study equilibria that arise in networks of general-topology under some polynomial cost functions introduced originally in the context of road traffic by the US Bureau of Public Roads. We obtain conditions for the uniqueness of the equilibrium. The uniqueness result is important since to date, there is little known about the uniqueness in the case of general topologies; only for a few special cases the uniqueness has been established, see [1, 9, 15]. We further present conditions under which the Nash equilibrium is efficient.

The fact that a class of nonlinear costs can be used for which the equilibrium turns out to be unique and efficient for general-topology networks may be especially useful for pricing purposes; indeed, if the network operator chooses prices that correspond to a unique and efficient equilibrium, it can enforce a globally optimal use of the network.

In the next section we introduce the general model considered in this paper. In Section 3 we establish conditions for the uniqueness of the Nash equilibrium. In Section 4 we establish the existence of an *efficient* Nash equilibrium under appropriate conditions. We then focus in Section 5 on the computation of the equilibrium in the special case of affine cost and parallel links.

# 2 The model

The topology is given by a directed graph $\{V, L, f\}$ where $V$ is the set of nodes, $L$ is the set of directed arcs and $f = (f_l, l \in L)$ where $f_l : \mathbb{R} \to \mathbb{R}$ is the cost function of link $l$, which gives the cost per unit traffic as a function of the load $\lambda_l$ on that link. We consider in this paper

$$f_l(\lambda) = a(l)\lambda^{p(l)} + b_l, \qquad p(l) > 0.$$

The link cost in the case where $p(l)$ does not depend on $l$ is widely used as confirmed in [4] and [17] in the context of road traffic. It is the cost adopted by the US Bureau of Public Roads [19], and we thus call it the BPR cost. The extra term $b_l$ could be interpreted as an additional fixed toll per packet (or unit traffic) for the use of link $l$.

There are $N$ users. User $i$ has a fixed amount $\Lambda_i$ of flow which he wants to ship from a source $s(i)$ to a destination $d(i)$. Each user has to decide on which part of its traffic to send on each given link. Introduce the quantities:
$\lambda_l^i$:= the rate of traffic that user $i$ sends over link $l$,
$\lambda_l$:= the total rate of traffic sent over link $l$,
$\lambda_l^{-i}$:= the total rate of traffic sent over link $l$ excluding the traffic from source $i$. In other words,

$$\lambda_l = \sum_i \lambda_l^i, \quad \lambda_l^{-i} := \lambda_l - \lambda_l^i.$$

We formulate this problem as a noncooperative network, i.e., a network in which the rate at which traffic is sent is determined by selfish users or players (corresponding to the different users) each having its own utility (or cost).

The cost for player $i$ is given by

$$J^i(\lambda_l^i, \lambda_l, l \in L) = \sum_{l \in L} J_l^i(\lambda_l^i, \lambda_l)$$

where $J_l^i(\lambda_l^i, \lambda_l) = \lambda_l^i f_l(\lambda_l)$. It should be noted that $J^i$ is a convex function of the flow vector of user $i$. We shall make frequent use of this property below without further note.

The flows of each user $i$ have to satisfy the feasibility conditions:

$$r_i(x) + \sum_{l \in In(x)} \lambda_l^i = \sum_{l \in Out(x)} \lambda_l^i, \tag{1}$$

$\forall x \in V, \lambda_l \geq 0$, where

$$r_i(x) = \begin{cases} \Lambda^i & \text{if } x = s(i), \\ -\Lambda^i & \text{if } x = d(i), \\ 0 & \text{otherwise.} \end{cases}$$

$In(x)$ (respectively, $Out(x)$) are the set of links which are input (respectively, output) to node $x$.

The solution concept adopted is that of Nash equilibrium, i.e., we seek a feasible multi-policy $\lambda = \{\lambda_l^i\}_{l,i}$ such that no player (user) $i$ can benefit by deviating unilaterally from his policy $\lambda^i = \{\lambda_l^i\}_l$ to another feasible multi-policy. In other words, for every player $i$,

$$\begin{aligned} J^i(\lambda) &= J^i(\lambda^1, \ldots, \lambda^{i-1}, \lambda^i, \lambda^{i+1}, \ldots, \lambda^N) \\ &= \min_{\tilde{\lambda}^i} J^i(\lambda^1, \ldots, \lambda^{i-1}, \tilde{\lambda}^i, \lambda^{i+1}, \ldots, \lambda^N) \end{aligned}$$

where the minimum is taken over all policies $\tilde{\lambda}^i$ which lead to a feasible multi-policy.

Under the assumptions made above on the topology and the cost structure, we know from [15] that a Nash equilibrium indeed exists.

# 3 Uniqueness of the Nash equilibrium

We consider $p(l) > 0$ for all links. The existence of a Nash equilibrium follows from [15]. Below we establish the uniqueness of the equilibrium under appropriate conditions. The uniqueness result is important since to date, there is little known about the uniqueness in the case of general topologies; only for a few special cases the uniqueness has been established, see [1, 9, 15]. Define

$$p^* := \frac{3N-1}{N-1}.$$

**Theorem 1** *Consider the BPR cost with* $0 < p(l) < p^*$ *for all links, and a general topology. Then the Nash equilibrium is unique.*

**Proof:** Let $\rho$ be a positive N-dimensional vector. Define the pseudo-gradient vector

$$g_l(\lambda, \rho) = \begin{bmatrix} \rho_1 \nabla_1 J_l^1(\lambda) \\ \vdots \\ \rho_N \nabla_N J_l^N(\lambda) \end{bmatrix} \tag{2}$$

A sufficient condition for the uniqueness of the Nash equilibrium is that for any two different vectors $\lambda$ and $\lambda'$ satisfying the flow constraints, we have (see Theorem 2 in [18])

$$\sum_{l \in L} (\lambda_l - \lambda'_l)(g_l(\lambda, \rho) - g_l(\lambda', \rho)) > 0,$$

where $g_l$ is as defined in (2). We will show that in fact, for any $l$ for which $\lambda_l \neq \lambda'_l$,

$$(\lambda_l - \lambda'_l)(g_l(\lambda, \rho) - g_l(\lambda', \rho)) > 0. \tag{3}$$

We first consider the case $p(l) = 1$, and introduce the Jacobian of $g_l$ with respect to $\lambda_l^j$:

$$G_l(\lambda_l, \rho) \stackrel{\text{def}}{=} \left\{ \rho_i \frac{\partial^2 J_l^i}{\partial \lambda_l^i \partial \lambda_l^j}(\lambda_l) \right\}_{i,j}$$

$$= \begin{bmatrix} 2a_l\rho_1 & a_l\rho_1 & \cdots & a_l\rho_1 \\ a_l\rho_2 & 2a_l\rho_2 & \cdots & a_l\rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_l\rho_N & a_l\rho_N & \cdots & 2a_l\rho_N \end{bmatrix}$$

If the matrix $[G_l + G_l^T]$ is positive definite for all possible values of $\lambda_l^i, i = 1, ..., N$, then (3) indeed holds, which implies that the Nash equilibrium is unique, see [15] and [18, Theorem 6]. But in fact, it is easy to check in the proof given in those references that (3) holds if $[G_l + G_l^T]$ is positive definite for all possible values of $\lambda_l^i, i = 1, ..., N$, for which $\lambda_l > 0$.

If we choose $\rho_1 = \rho_2 = \cdots = \rho_N = 1$, the matrix above has the structure:

$$a_l \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$$

which is clearly positive definite for $a_l > 0$ (being the sum of the identity matrix and a nonnegative definite matrix).

Next, we consider the case of a general $p(l)$, $0 < p(l) < p^*$. We use again $\rho_i = 1, i = 1, ..., N$. Then

$$G_l(\lambda_l, \rho) = \{\frac{\partial^2 J_l^i}{\partial \lambda_l^i \partial \lambda_l^j}(\lambda_l)\}_{i,j}$$

$$= a_l p(l) \lambda_l^{p(l)-2} (\mathcal{G}_l + \lambda_l \mathcal{I}),$$

where

$$\mathcal{G}_l = \begin{bmatrix} q_l^1 & q_l^1 & \cdots & q_l^1 \\ q_l^2 & q_l^2 & \cdots & q_l^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_l^N & q_l^N & \cdots & q_l^N \end{bmatrix}$$

where $\mathcal{I}$ is the identity matrix, and $q_l^i \stackrel{\text{def}}{=} \lambda_l + (p(l) - 1)\lambda_l^i$. We will show that the eigenvalues of $(\mathcal{G}_l + \lambda_l \mathcal{I}) + (\mathcal{G}_l + \lambda_l \mathcal{I})^T$ are positive if $\lambda_l > 0$. They are clearly zero if $\lambda_l = 0$, and hence we shall restrict the discussion below to the case $\lambda_l > 0$.

Let

$$z \stackrel{\text{def}}{=} \sum_{j=1}^{N} q_l^j + 2\lambda_l = (N+1+p(l))\lambda_l,$$
$$r \stackrel{\text{def}}{=} \sqrt{N\sum_{j=1}^{N}(q_l^j)^2}. \tag{4}$$

Denote by $x_i$, $i = 1, ..., N$ the eigenvalues of $\mathcal{G}_l+\mathcal{G}_l^T$. We first show that there are $N-2$ eigenvalues which equal 0, which we denote by $x_3 = \ldots = x_N = 0$. (This will imply that there are $N-2$ eigenvalues of $(\mathcal{G}_l + \lambda_l\mathcal{I}) + (\mathcal{G}_l + \lambda_l\mathcal{I})^T$ which equal $2\lambda_l$ each.) Indeed, a vector whose elements are all zero except for some arbitrary two entries $i, j$ $(i \neq j)$ whose values are 1 and -1 respectively is an eigenvector of $\mathcal{G}_l$ corresponding to an eigenvalue of $0$. It then follows that any one of the vectors

$$(q_l^2 - q_l^i, q_l^i - q_l^1, 0, ...0, q_l^1 - q_l^2, 0, ..., 0)^T,$$

$i = 3, ..., N$ (where the third nonzero entry is the $i$th) is also an eigenvector of $\mathcal{G}_l$ and is in fact an eigenvector of $\mathcal{G}_l + \mathcal{G}_l^T$ (corresponding to the eigenvalue $0$).

As shown in the appendix, the two remaining eigenvalues $x_1$ and $x_2$ of $\mathcal{G}_l + \mathcal{G}_l^T$ are $z + r - 2\lambda_l$ and $z - r - 2\lambda_l$.

To complete the proof, we need to show that $x_1$ and $x_2$ are both greater than $-2\lambda_l$. Clearly the eigenvalue $x_1 = z + r - 2\lambda_l > -2\lambda_l$ is positive as $z$ and $r$ are positive. That $z - r - 2\lambda_l > -2\lambda$ (or simply $z - r > 0$) follows from:

$$\begin{aligned}
\frac{r^2}{N} &= \sum_{i=1}^{N}(\lambda_l + (p(l)-1)\lambda_l^i)^2 \\
&= (N-1)(\lambda_l)^2 + (\lambda_l)^2 \\
&\quad +(p(l)-1)^2\sum_{j=1}^{N}(\lambda_l^j)^2 + 2(\lambda_l)^2(p(l)-1) \\
&\leq (N-1)(\lambda_l)^2 + (\lambda_l)^2 \\
&\quad +(p(l)-1)^2(\lambda_l)^2 + 2(\lambda_l)^2(p(l)-1) \\
&= (N-1)(\lambda_l)^2 + (\lambda_l + (p(l)-1)\lambda_l)^2 \\
&= (N-1+p(l)^2)(\lambda_l)^2.
\end{aligned} \tag{5}$$

It thus follows from (5) that $z - r > 0$ if $z^2 > N(N-1)(\lambda_l)^2 + (\lambda_l + (p(l)-1)\lambda_l)^2$, i.e. if

$$(N+1+p(l))^2 > N(N-1+p(l)^2), \tag{6}$$

which is equivalent to

$$\begin{aligned}
0 &< 1 + 3N + 2(N+1)p(l) + (1-N)p(l)^2 \\
( &= -(p(l)+1)((N-1)p(l) - (3N+1))\ ).
\end{aligned}$$

For any $p(l) \in (0, p^*)$, (6) indeed holds.

Hence, by Theorem 2 of [18], the Nash equilibrium is indeed unique. $\diamond$

## 4 Efficient Nash equilibria

We consider in this section a general topology with the following cost:

$$f_l(\lambda_l) = a(l)\lambda_l^p, \qquad p > 0.$$

This is the BPR cost which has been considered earlier in [7] (see also [3]). It was shown there that with this cost, the Wardrop Equilibrium and the globally optimal solution coincide.

We assume here that all users have the same source and the same destination. We show that, the Nash equilibrium is a globally optimal solution, and that the link flows of different users are proportional to their total input traffic.

The global optimization problem is to determine the link flows $\overline{\lambda}$ that minimize the global cost $J(\lambda)$ given by

$$J(\lambda) = \sum_l J_l(\lambda_l),$$

where $J_l(\lambda_l) = \lambda_l f_l(\lambda_l)$ and $\lambda_l = \sum_i \lambda_l^i$. In other words, we weigh the link costs $f_l(\lambda_l)$ by the flow $\lambda_l$ that uses that link. The global optimization problem is thus to find $\overline{\lambda}$ that achieves

$$\min_\lambda J(\lambda) \quad s.t. \quad (1) \text{ holds.} \tag{7}$$

A Nash equilibrium can be obtained from the globally optimal solution as follows:

**Theorem 2** *Let $\overline{\lambda}$ be the globally optimal solution and define $\alpha_i = \Lambda_i / \sum_{j=1}^N \Lambda_j$. Then $\hat{\lambda}_l^i = \alpha_i \overline{\lambda}_l$ is a Nash equilibrium, with the corresponding cost for user $i$ being $J^i(\overline{\lambda}) = \alpha_i J(\overline{\lambda})$. In particular, $\hat{\lambda}_l^i$ is an efficient equilibrium.*

**Proof:** $\hat{\lambda}$ is a Nash equilibrium if and only if it is feasible and the following holds for all $i$:

$$\sum_{l \in L} \frac{\partial J_l^i(\hat{\lambda})}{\partial \lambda_l^i} (\hat{\lambda}_l^i - \lambda_l^i) \leq 0 \tag{8}$$

for any feasible $\lambda^i$ (see e.g. [8]).

In particular, $\overline{\lambda} = (\overline{\lambda}^1, ..., \overline{\lambda}^N)$ is globally optimal if and only if it is feasible (i.e. it satisfies the traffic constraints (1)) and the following holds:

$$\sum_{l \in L} \frac{\partial J_l(\overline{\lambda}_l)}{\partial \lambda_l} (\overline{\lambda}_l - \lambda_l) \leq 0 \tag{9}$$

for any feasible $\lambda$.

Under the cost adopted, the partial derivative in (8) reduces to

$$\frac{\partial J_l^i(\lambda_l)}{\partial \lambda_l^i} = a(l)(p(\lambda_l)^{p-1}\lambda_l^i + (\lambda_l)^p). \tag{10}$$

Let $\hat{\lambda}_l^i = \alpha_i \overline{\lambda}_l$ where $\overline{\lambda}_l$ is the unique globally optimal solution (the uniqueness of the globally optimal solution is well known, see e.g. [1]). Then we have by (10)

$$\begin{aligned} &\sum_{l\in L} \frac{\partial J_l^i(\hat{\lambda})}{\partial \lambda_l^i}(\hat{\lambda}_l^i - \lambda_l^i) \qquad (11)\\ &= \alpha_i(p\alpha_i + 1)\sum_{l\in L} a(l)(\overline{\lambda}_l)^p(\overline{\lambda}_l - \tilde{\lambda}_l), \end{aligned}$$

where $\tilde{\lambda}_l \stackrel{\text{def}}{=} \lambda_l^i/\alpha_i$. Note that in the global optimization problem

$$\begin{aligned} &\sum_{l\in L} \frac{\partial J_l(\overline{\lambda}_l)}{\partial \lambda_l}(\overline{\lambda}_l - \lambda_l)\\ &= (p+1)\sum_{l\in L} a(l)(\overline{\lambda}_l)^p(\overline{\lambda}_l - \lambda_l) \le 0, \end{aligned}$$

where the above inequality follows from (9). By combining this with (11), we conclude that (8) holds with $\hat{\lambda} = \overline{\lambda}$. The last relation in the Theorem immediately follows, which establishes the proof. $\diamond$

# 5 Computation of the Nash equilibrium: Affine costs

We consider in this section the case of a network consisting of $M$ parallel links and with the affine cost function for all links:

$$f_l(\lambda_l) = a_l\lambda_l + b_l.$$

This coincides with the BPR cost for $p(l) = 1$. A possible interpretation for this type of cost in the context of telecommunication systems is that it is the expected delay of a packet in a light traffic regime. For example, the expected value of the delay $D_l$ in link $l$ for an M/G/1 queue is given by

$$ED_l = \frac{\beta_l^{(2)}}{2(1-\lambda_l\beta_l)}\lambda_l + \beta_l.$$

Under the light traffic regime, i.e., $\lambda_l\beta_l << 1$, the above can be approximated by $a_l\lambda_l + \beta_l$, where $a_l = \beta_l^{(2)}/2$.

It follows from standard nonlinear (convex) programming theory that the following Kuhn-Tucker conditions are both necessary and sufficient for a solution $\lambda = \{\lambda_l^i\}_{l,i}$ to constitute a Nash equilibrium:

$$K_l^i(\lambda) = a_l\lambda_l + b_l + a_l\lambda_l^i = \mu^i \quad \text{if } \lambda_l^i > 0, \forall i, l, \tag{12}$$

$$K_l^i(\lambda) = a_l\lambda_l + b_l \geq \mu^i \quad \text{if } \lambda_l^i = 0, \forall i, l, \tag{13}$$

$$\sum_{l=1}^{M} \lambda_l^i = \Lambda^i, \ \forall i, \quad \lambda_l^i \geq 0 \quad \forall i, l. \tag{14}$$

Since (12)-(14) constitute a system of linear equalities and inequalities, they can be solved by standard LP methods. We present below an alternative approach that yields explicitly the equilibrium when the bias terms or their variability are relatively small.

**Lemma 1** *(i) Assume that at equilibrium, each user sends a positive amount of flow to each link. Then, the unique Nash equilibrium is given by*

$$\lambda_l^i = \frac{\frac{1}{a_l}}{\sum_{j=1}^{M} \frac{1}{a_j}} \left( \Lambda^i + \frac{1}{N+1} \sum_{j=1}^{M} \frac{b_j - b_l}{a_j} \right), \ \forall l, i. \tag{15}$$

*(ii) At equilibrium, each user sends a positive amount of flow to each link if and only if the following condition holds:*

$$\min_i R(i) > \frac{\max_l b_l}{N+1} \tag{16}$$

*where*

$$R(i) \stackrel{\text{def}}{=} \frac{1}{\sum_{j=1}^{M} \frac{1}{a_j}} \left( \Lambda^i + \frac{1}{N+1} \sum_{j=1}^{M} \frac{b_j}{a_j} \right).$$

**Proof:** We shall initially assume that in equilibrium all users send positive flows on all the links. This will allow us to compute an equilibrium which is consistent with this assumption, which in turn will be the unique equilibrium by the result of the previous section.

By summing (12) over all users, we have:

$$\begin{aligned} & N(a_l\lambda_l + b_l) + a_l\lambda_l \\ = \ & (N+1)(a_l\lambda_l) + Nb_l = \sum_{i=1}^{N} \mu^i \end{aligned} \tag{17}$$

Thus

$$a_l\lambda_l + b_l = \frac{\sum_i \mu^i + b_l}{N+1} \tag{18}$$

Substituting in the Kuhn-Tucker condition, we obtain:

$$a_l\lambda_l^i + \frac{b_l}{N+1} = \left( \mu^i - \frac{\sum_i \mu^i}{N+1} \right) \tag{19}$$

Dividing by $a_l$ and summing over all the links, we get

$$\begin{aligned}\Lambda^i + \frac{1}{N+1}\left(\sum_{l=1}^{M}\frac{b_l}{a_l}\right) & \\ = \left(\mu^i - \frac{\sum_i \mu^i}{N+1}\right)\left(\sum_{l=1}^{M}\frac{1}{a_l}\right). &\end{aligned} \tag{20}$$

Hence

$$\begin{aligned}\left(\mu^i - \frac{\sum_i \mu^i}{N+1}\right) & \\ = \frac{1}{\sum_l \frac{1}{a_l}}\left(\Lambda^i + \frac{1}{N+1}\sum_l \frac{b_l}{a_l}\right) = R(i). &\end{aligned} \tag{21}$$

Then, from (19),

$$a_l\lambda_l^i = R(i) - \frac{b_l}{N+1} \tag{22}$$

Thus, if $\min_i R(i) > \max_l b_l/(N+1)$, the equilibrium is given by

$$\lambda_l^i = \frac{R(i) - b_l/(N+1)}{a_l} > 0,$$

which is compatible with our initial assumption on the positivity of the flows. This establishes the Lemma. $\diamond$

**Corollary 1** *If all $b_l$'s are equal, then the Nash equilibrium is given by*

$$\lambda_l^i = \Big(\frac{1}{a_l} \Big/ \sum_{j=1}^{M}\frac{1}{a_j}\Big)\Lambda^i$$

We next consider some further properties of the Nash equilibrium.

**Lemma 2** *(i) Consider a Nash equilibrium which dictates a particular user to send all its traffic over a single link. If there is some other user who sends all its traffic over a single link, then these two links are the same.*
*(ii) All users send their traffic over a particular link $m$ if and only if*

$$a_l\Lambda + b_l + \max_i a_l\Lambda^i \geq \min_{n\neq m} b_n.$$

**Proof:** (i) Consider a Nash equilibrium where player $i$ uses only link $m$ and player $j$ uses only link $l$, with $l \neq m$. From (12), we get

$$\mu^i = a_l \lambda_l + b_l + a_l \Lambda^i,$$
$$\mu^j = a_n \lambda_n + b_n + a_n \Lambda^j.$$

From (13) we get:

$$\mu^j \leq a_l \lambda_l + b_l, \qquad \mu^i \leq a_n \lambda_n + b_n.$$

Combining the last two relationships, we obtain the following contradiction:

$$\begin{aligned} a_l \lambda_l + b_l + a_l \Lambda^i &= \mu^i \\ &\leq a_n \lambda_n + b_n = \mu^j - a_n \Lambda^j \\ &\leq a_l \lambda_l + b_l - a_n \Lambda_j. \end{aligned}$$

(ii) By (12)-(13), all users send their traffic to link $l$ if and only if

$$\mu^i = a_l \Lambda + b_l + a_l \Lambda^i, \; \mu^i \leq b_n, \; \forall n \neq m.$$

This then implies the desired result. ◇

**Lemma 3** *Consider the case of two links.*
*Let $\mathcal{I}$ be a nonempty set of players who at equilibrium send all their traffic to link 2. Then the equilibrium for the remaining players $i \notin \mathcal{I}$ is given by*

$$\begin{aligned} \lambda_1^i &= \\ & \frac{{a_1}^{-1}}{\sum_{j=1}^{2} {a_j}^{-1}} \left( \Lambda^i + \frac{1}{N+1} \frac{b_2 - b_1 + a_2 \Lambda(\mathcal{I})}{a_2} \right), \\ \lambda_2^i &= \\ & \frac{{a_2}^{-1}}{\sum_{j=1}^{M} {a_j}^{-1}} \left( \Lambda^i + \frac{1}{N+1} \frac{b_1 - b_2 - a_2 \Lambda(\mathcal{I})}{a_2} \right), \end{aligned} \tag{23}$$

*where $\Lambda(\mathcal{I}) \stackrel{\text{def}}{=} \sum_{i \in \mathcal{I}} \Lambda^i$.*

**Proof:** Let

$$\lambda_l(\mathcal{I}^c) \stackrel{\text{def}}{=} \sum_{i \notin \mathcal{I}} \lambda_l^i, \qquad b_2^{\mathcal{I}} \stackrel{\text{def}}{=} b_2 + a_2 \Lambda(\mathcal{I}).$$

Each player $i \notin \mathcal{I}$ faces the problem of minimizing the cost

$$\begin{aligned} J_i(\lambda) &= \lambda_1^i (a_1 \lambda_1 + b_1) + \lambda_2^i (a_2 \lambda_2 + b_2) \\ &= \lambda_1^i (a_1 \lambda_1(\mathcal{I}^C) + b_1) + \lambda_2^i (a_2 \lambda_2(\mathcal{I}^C) + b_2^{\mathcal{I}}). \end{aligned}$$

This is equivalent to the original game in which users from the set $\mathcal{I}$ do not participate, and in which the corresponding value of $b_2$ is changed to $b_2^{\mathcal{I}}$. It follows from Lemma 2 that each user in this new game sends positive flow over both links at equilibrium. The proof is now completed by using Lemma 1. $\diamond$

By comparing (15) and (23) we arrive at:

**Corollary 2** *Consider a two-link problem, and assume that $b_1 > b_2$. Let $\hat{\mathcal{I}}$ be the set of users for which $\lambda_1^j$, computed in (15), is negative. Then $\mathcal{I} \subset \hat{\mathcal{I}}$, where $\mathcal{I}$ is as defined in Lemma 3.*

# 6 Conclusion

We have investigated in this paper the problem of static competitive routing to parallel queues with linear link holding costs. We have established the uniqueness of the Nash equilibrium for a general topology with the BPR cost [19] and have obtained a simple relationship with the globally optimal solution. We have further obtained some explicit results for the special case of affine link costs.

The results of this paper should prove useful for the analysis of networks with source-determined routing, when the link cost functions can be approximated by polynomial (and in particular affine) costs of the type considered here. The fact that the Nash equilibrium was shown to be efficient (that is, to coincide with the globally optimal solution) under a class of nonlinear costs can be used *as a starting point* for designing pricing mechanisms so as to obtain a socially optimal use of the network.

**Acknowledgement:** The work of the first, third and fourth co-authors was partially supported by a Grant from the France-Israel Scientific Cooperation (in Computer Science and Engineering) between the French Ministry of Research and Technology and the Israeli Ministry of Science and Technology, on information highways. The work of the first and second authors was partially supported by an INRIA/NSF grant. We wish to thank Thomas Boulogne for his suggestions and comments on the paper.

# References

[1] E. Altman and H. Kameda, “Uniqueness of solutions for optimal static routing in multi-class open networks”, submitted, Research report ISE-TR-98-156, Institute of Information Sciences and Electronics University of Tsukuba, Japan.

[2] T. Başar and G. J. Olsder, “Dynamic Noncooperative Game Theory,” SIAM Series in Classics in Applied Mathematics, Philadelphia, 1999.

[3] L. D. Bennett, "The existence of equivalent mathematical programs for certain mixed equilibrium traffic assignment problems", *European J. of Operational Research* **71**, pp. 177-187, 1993.

[4] D. E. Boyce, B. N. Janson and R. W. Eash, "The effect on equilibrium trip assignment of different congestion functions", *Transportation Research A*, **15**, pp. 223-232, 1981.

[5] J. E. Cohen and C. Jeffries, "Congestion resulting from increased capacity in single-server queueing networks" *IEEE/ACM Trans. on Networking*, Vol 5, April 1997, pp. 305-310.

[6] S. Dafermos and A. Nagurney, "On some traffic equilibrium theory paradoxes", *Transportation Research B*, **18**, pp. 101-110, 1984.

[7] S. Dafermos and F. T. Sparrow, "The traffic assignment problem for a general network", *Journal of Research of the National Bureau of Standards-B. Mathematical Sciences* **73B**, No. 2, pp. 91-118, 1969.

[8] A. Haurie and P. Marcotte, "On the relationship between Nash-Cournot and Wardrop Equilibria", *Networks*, **Vol. 15**, pp. 295-308, 1985.

[9] H. Kameda, E. Altman, and T. Kozawa, "Braess-like Paradoxes of Nash Equilibria for Load Balancing in Distributed Computer Systems", *Proc. of the 38th IEEE Conference on Decision and Control*, Phoenix, Arizona, USA, Dec. 1999. Full paper in Research report ISE-TR-98-157, Institute of Information Sciences and Electronics University of Tsukuba, Japan.

[10] H. Kameda, T. Kozawa and J. Li, "Anomalous relations among various performance objectives in distributed computer systems", *Proc. World Congress on Systems Simulation*, pp. 459-465, Sept, 1997.

[11] H. Kameda and Y. Zhang, "Uniqueness of the solution for optimal static routing in open BCMP queueing networks", *Math. Comput. Modelling* **22**, No. 10-12, pp. 119-130, 1995.

[12] Y. A. Korilis, A. A. Lazar and A. Orda, "Architecting noncooperative networks", *IEEE J. on Selected Areas in Communications*, **13**, No. 7, sept 1995, pp. 1241-1251.

[13] R. J. La and V. Anantharam, "Optimal routing control: game theoretic approach", *Proc. of the 36th IEEE Conference on Decision and Control*, San Diego, California, Dec. 1997.

[14] P. Marcotte and S. Nguyen (eds.), *Equilibrium and Advanced Transportation Modelling,* Kluwer Academic, 1998.

[15] A. Orda, R. Rom and N. Shimkin, "Competitive routing in multiuser communication networks", *IEEE/ACM Transactions on Networking*, **1**, pp. 614-627, 1993.

[16] M. Patriksson *The Traffic Assignment Problem: Models and Methods* VSP BV, P.O. Box 346, 3700 AH Zeist, The Netherlands, 1994.

[17] G. Rose, M. A. P. Taylor and P. Tisato, “Estimating travel time functions for urban roads: options and issues”, *Transportation Planning and Technology*, **14**, pp. 63-82, 1989.

[18] J. B. Rosen, “Existence and uniqueness of equilibrium points for concave N-person games”, *Econometrica*, **33**, pp. 153-163, 1965.

[19] US Bureau of Public Roads, “Traffic assignment manual”, US Department of Commerce, Washington DC, 1964.

[20] J.G. Wardrop, Some Theoretical Aspects of Road Traffic Research, *Proc. Inst. Civ. Eng.* Part 2, 1, 325–378, 1952.

[21] W. I. Zangwill and C. B. Garcia, *Pathways to Solutions, Fixed Points and Equilibria* Englewood Cliffs, NJ: Prentice Hall, Inc., 1981.

# 7 Appendix

We show here that the two nonzero eigenvalues ($x_1$ and $x_2$) of $\mathcal{G}_l + \mathcal{G}_l^T$ are given by $z + r - 2\lambda_l$ and $z - r - 2\lambda_l$ where $z$ and $r$ are given in (4) and where

$$\mathcal{G}_l = \begin{bmatrix} q_l^1 & q_l^1 & \cdots & q_l^1 \\ q_l^2 & q_l^2 & \cdots & q_l^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_l^N & q_l^N & \cdots & q_l^N \end{bmatrix}$$

Let $\underline{1}$ denote a column vector of dimension $N$ whose entries are all 1, and let $q_l$ be the column vector given by $q_l^1, q_l^2, \cdots, q_l^N$. Then we have

$$\mathcal{G}_l + \mathcal{G}_l^T = q_l \underline{1}^T + \underline{1} q_l^T.$$

Since $\mathcal{G}_l + \mathcal{G}_l^T$ is a symmetric matrix, all its (columns) eigenvectors, which we denote by $y_i$, $i = 1, ..., N$, are orthogonal. On the other hand, the columns of $\mathcal{G}_l + \mathcal{G}_l^T$ are orthogonal to $y_3, ..., y_N$ since the latter are the eigenvectors corresponding to the 0 eigenvalues. The eigenvectors $y_1$ and $y_2$ are therefore linear combinations of the columns of the matrix $\mathcal{G}_l + \mathcal{G}_l^T$ and therefore have the form

$$\alpha q + \beta \underline{1}$$

(where $\alpha$ and $\beta$ are some constants). We can now obtain the two possible values of $\alpha$, $\beta$, and of the eigenvalues. Since $y_i$ is an eigenvector, we have

$$(q_l \underline{1}^T + \underline{1} q_l^T)(\alpha q_l + \beta \underline{1}) = x_l(\alpha_i q_l + \beta_i \underline{1}),$$

or equivalently,

$$q_l \Big( \alpha [(\underline{1}^T q_l) - x] + \beta \underline{1}^T \underline{1} \Big)$$

$$+\underline{1}\Big(\alpha(q_l^T q + l) + \beta[(q_l^T \underline{1}) - x]\Big) = 0.$$

The solutions for $x$ are obtained by equating the coefficients $q_l$ and $\underline{1}$ to 0. This yields the pair of equations:

$$\alpha[(\underline{1}^T q_l) - x] + \beta \underline{1}^T \underline{1} = 0,$$

$$\alpha(q_l^T q + l) + \beta[(q_l^T \underline{1}) - x] = 0.$$

In order for a nontrivial solution to exist for $\alpha$ and $\beta$, the determinant of their coefficient matrix in the above equations have to be zero, i.e.

$$(\underline{1}^T q_l - x)(q_l^T \underline{1} - x) - (\underline{1}^T \underline{1})(q_l^T q_l) = 0.$$

This is a quadratic equation in terms of the unknown eigenvalues, whose solutions are indeed $z + r - 2\lambda_l$ and $z - r - 2\lambda_l$. $\diamond$