Efficient Rate-Constrained Nash Equilibrium in Collision Channels with State Information

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Abstract

We consider a wireless collision channel, shared by a finite number of users who transmit to a common base station. Users are self-optimizing, and each wishes to minimize its average transmission rate (or power investment), subject to minimum-throughput demand. The channel quality between each user and the base station is time-varying, and partially observed by the user in the form of channel state information (CSI) signals. We assume that each user can transmit at a fixed power level and that its transmission decision at each time slot is stationary in the sense that it can depend only on the current CSI. We are interested in properties of the Nash equilibrium of the resulting game between users.

We define the feasible region of user's throughput demands, and show that when the demands are within this region, there exist exactly two Nash equilibrium points, with one strictly better than the other (in terms of average power) for all users. We further address the performance benefits of improved CSI, and show that if even a single user obtains better CSI, the average power of *all* users is reduced. We then provide some lower bounds on the channel capacity that can be obtained, both in the symmetric and non-symmetric case. Finally, we show that a simple greedy mechanism converges to the best equilibrium point without requiring any coordination between the users.

1 Introduction

1.1 Background and Motivation

The emerging use of wireless technologies (such as WIFI and WIMAX) for data communication has brought to focus novel system characteristics which are of less importance in wireline platforms. Power control and the effect of mobility on network performance are good examples of topics which are prominent in the wireless area. An additional distinctive feature of wireless communications is the possible time variation in the channel quality between the sender and the receiver, an effect known as channel fading [1].

As wireless networks grew larger, it became evident that centralized control would be impractical for coordinating all elements of the network, and in particular end-user transmissions. The celebrated Aloha protocol was designed at the early 70's as a distributed mechanism which can allow efficient media sharing. This protocol and its variants, such as CSMA-CD and tree-algorithms [2], are *cooperative* in the sense that each user is committed to perform his part of the protocol. Modern wireless network protocols are often based on Aloha-related concepts (for example, the 802.11 standards [3]). The design of such protocols raises novel challenges and difficulties, as the wireless arena becomes more involved.

An additional consideration is the possibly selfish behavior of users, who can bias their transmission decisions to accommodate their own best interest. Such behavior is to be expected in wireless networks, considering the dynamic and ad-hoc nature of such networks, and the scarce resources of mobile terminals. In many cases, an individual user can momentarily improve its Quality of Service (QoS) metrics, such as delay and throughput, by accessing the shared channel more frequently. Aggressiveness of even a single user may lead to a chain reaction, resulting in possible throughput collapse. Significant research has been recently dedicated to analyzing wireless random access networks shared by self-interested agents, applying non-cooperative game theoretic tools for the analysis [4, 5, 6, 7].

Our work considers a shared uplink in the form of a collision channel, where a user's transmission can be successful only if no other user attempts transmission simultaneously. A basic assumption of our user model is that each user has some throughput requirement, which it wishes to sustain with a minimal power investment. The required throughput of each user may be dictated by its application (such as video or voice which may require fixed bandwidth), or mandated by the system. A distinctive feature of our model is that the channel quality between each user and the base station is stochastically varying. For example, the channel quality may evolve as a block fading process [1] with a general underlying state distribution (such as Rayleigh, Rice, and Nakagami-m, see [1]). A user may base its transmission decision upon available indications on the channel state, known as channel state information (CSI). This decision is selfishly made by the individual without any coordination with other users, giving rise to a non-cooperative game. Our focus in this paper is on *stationary* transmission strategies, in which the decision whether to transmit or not can depend (only) on the current CSI signal. Non-stationary strategies are naturally harder to analyze, and moreover, their advantage over stationary strategies is not clear in large, distributed and selfish environments[1].

The technological relevance for our work lies, for example, in WLAN systems, where underlying network users have diverse (application-dependent) throughput requirements. The leading standard, namely the 802.11x [3], employs a random access protocol, whose principles are based on the original Aloha. Interestingly, on-going IEEE standardization activity (the 802.11n standard) focuses on the incorporation of CSI for better network utilization. This last fact further motivates to study the use of CSI in distributed, self-optimizing user environments.

1.2 Related Work

Exploiting channel state information for increasing the network's capacity has been an ongoing research topic within the information theory community (see [1] for a survey). Recent research (see [8, 9] and references therein) is dedicated to uplink decentralized approaches, in which each station's transmission decision can be based on private CSI only. Nodes are assumed to operate in a cooperative manner, thus willing to accept a unified throughput-maximizing transmission policy.

Game theoretic tools have been widely applied to analyze selfish behavior in communication networks (see [10] for a survey). Recently, some papers have considered Aloha-like random access networks from a non-cooperative perspective [6, 5, 4, 11, 7]. Of specific relevance to our work is a paper by Jin and Kesidis [6], which considers a shared collision channel with users who have fixed throughput demands. Users dynamically adapt their transmission rates in order to obtain their required demands. Our work provides, as a spe-

[1] Accordingly, our study centers on equilibrium points that are obtained in stationary strategies. We note however that the Nash equilibrium in stationary strategies remains an equilibrium point even within the larger class of general strategies.

cial case, a comprehensive analysis of their model, and further extends it by incorporating channel state information as affecting the transmission policy.

Our user model that incorporates both channel-aware and self-interested mobiles is quite novel. A related model was considered in [11, 12], where users with long term power constraints determine the transmission power for given CSI to maximize their individual throughput. The reception rules which are considered are either single packet "capture" [11] or multi-reception [12]. These papers are mainly concerned with the existence of a Nash equilibrium point and some basic structural properties thereof.

1.3 Contribution and Paper Organization

This paper presents a comprehensive study of the non-cooperative game between the channel-aware, self-interested network users. The main contributions are summarized below.

- We provide a model for an uplink collision channel that incorporates stochastic channel variation and CSI, with which the interaction of selfish users may be studied.
- Our equilibrium analysis reveals that when the throughput demands are within the network capacity, there exist exactly two Nash equilibrium points in the resulting game.
- We show that one equilibrium is strictly better than the other in terms of power investment for all users. We further show that the performance gap (in terms of the total power investment) between the equilibrium points is potentially unbounded.
- We investigate the advantage of higher quality CSI and show that *all* users benefit from the ability of even a single user to obtain better CSI.
- A simple lower bound on the total channel throughput (or capacity) is provided. We relate this bound to the well-known result for the capacity of an Aloha network ($1/e$).
- We describe a fully distributed mechanism which converges to the better equilibrium point. The suggested mechanism is natural in the sense that it relies on the user's best response to given network conditions.

We emphasize that all our results are valid under general assumptions on the channel state distribution and CSI signals. We also note that our game model is related (but not identical) to S-modular games [13, 14]. However, the results we obtain here are stronger than these implied by the general theory.

The structure of the paper is as follows. We first present the general model (Section 2), and identify basic properties related to stationary transmission strategies. A detailed equilibrium analysis is provided in Section 3. The performance benefits of CSI are highlighted in Section 4. Section 5 focuses on the achievable network capacity. In Section 6 we present a mechanism which converges to the better equilibrium. We discuss several aspects of our results in Section 7. Conclusion and further research direction are drawn in Section 8[2].

2 The Model and Preliminaries

We consider a wireless network, shared by a finite set of mobile users $\mathcal{I} = \{1, \ldots, n\}$ who transmit at a fixed power level to a common base station over a shared collision channel. Time is slotted, so that each transmission attempt takes place within slot boundaries that are common to all. A transmission can be successful only if no other user attempts transmission simultaneously. Thus, at each time slot, at most one user can successfully transmit to the base station. To further specify our model, we start with a description of the channel between each user and the base station (Section 2.1), ignoring the possibility of collisions. In Section 2.2, we formalize the user objective and formulate the non-cooperative game which arises in a multi-user shared network.

2.1 The Single-User Channel

Our model for the channel between each user and the base station is characterized by two basic quantities.

a. Channel state information. At the beginning of each time slot k, every user i obtains a channel state information (CSI) signal $\zeta_{i,k} \in \mathcal{Z}_i \subset \mathbb{R}^+$, which provides an indication (possibly partial) of the quality of the current channel between the user and the base station (a larger number corresponds to a better channel quality). We assume that each set $\mathcal{Z}_i$ of possible CSI signals for user i is finite[3] and denote its elements by $\{z_{i1}, z_{i2}, \ldots, z_{ix_i}\}$, with $z_{i1} < z_{i2} < \cdots < z_{ix_i}$.

b. Expected data rate. We denote by $R_i(z_i) > 0$ the expected data rate (say, in bits

[2] A preliminary version of our work, which focuses on a simplified model with no CSI, was presented at the Net-Coop'07 workshop, Avignon [15].

[3] This is assumed for convenience only. Note that the channel quality may still take continuous value, which the user reasonably classifies into a finite number of information states.

per second) that user i can sustain at any given slot as a function of the current CSI signal $z_i \in \mathcal{Z}_i$. We assume that the function $R_i(z_i)$ strictly increases in z_i.

Throughout this paper we make the following assumption:

Assumption 1 *(i) $Z_i = \{\zeta_{i,k}\}_{k=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables; the probability of observing a particular CSI signal $z_i \in \mathcal{Z}_i$ in any given slot is denoted by $P_i(z_i) > 0$ (signals with zero probability are excluded from the set $\mathcal{Z}_i$). (ii) The sequences Z_i and Z_j are independent for $i \neq j$.*

The above model may be used to capture the following network scenario. The quality (or state) of the channel between user i and the base station may vary over time. Let w_i denote an actual channel state for user i at the beginning of some slot (time indexes are omitted here for simplicity). Instead of the exact channel state, user i observes a CSI signal z_i, which is some (possibly noisy) function of w_i. As already noted, larger z_i's indicate better channel conditions. After observing the CSI at the beginning of a slot, user i may respond by adjusting its coding scheme in order to maximize its data throughput on that slot. The expected data rate $R_i(z_i)$ thus takes into account the actual channel state distribution (conditioned on z_i), including possible variation within the slot duration, as well as the coding scheme used by the user. Specifically, let $\tilde{R}_i(w_i, z_i)$ be the expected data rate for channel state w_i, as determined by the coding scheme that corresponds to z_i. Then the expected data rate is given by $R_i(z_i) = \mathbb{E}(\tilde{R}_i(w_i, z_i)|z_i) = \int \tilde{P}_i(w_i|z_i)\tilde{R}_i(w_i, z_i)dw_i$, where $\mathbb{E}$ is the expectation operator and $\tilde{P}_i(w_i|z_i)$ is the conditional probability that the actual channel state is w_i when z_i is observed. Assuming that the actual channel quality forms an i.i.d. sequence across slots, this property is clearly inherited by the CSI sequence Z_i.

Our modeling assumptions accommodate, in particular, the so-called block-fading model, which is broadly studied in the literature (see [1, 8] and references therein). Note however that our model does not require the actual channel state to be fixed within each interval.

The following examples serve to illustrate certain aspects of our model.

1) *Partial vs. null CSI.* Let $\mathcal{Z}_i = \{z_{i1}, z_{i2}\}$. In this case, user i is able to roughly classify the channel quality as either "weak" or "good", denoted by the two signals z_{i1} and z_{i2} with $R_i(z_{i1}) < R_i(z_{i2})$. The maximal data rate which this user can obtain (in a collision free environment) is given by $P_i(z_{i1})R_i(z_{i1}) + P_i(z_{i2})R_i(z_{i2})$. Consider now the case where the same user is unable to obtain any CSI (so that the CSI set is formally a singleton $\tilde{\mathcal{Z}}_i = \{\tilde{z}_{i1}\}$). In this case, user i faces several options: (i) Coding its data as if the CSI signal is z_{i1}; (ii) Coding its data as if the CSI signal is z_{i2}, (iii) Optimizing its coding

scheme while taking into account the statistical properties of the unobserved CSI signals z_{i1} and z_{i2}. The first option leads to an expected rate which is around $R_i(z_{i1})$, as the better CSI signal is not exploited for higher data-rate transmissions. The second option risks data losses whenever the CSI signal is z_{i1}. The third option may lead to higher rates than the first two, for example, by applying broadcast techniques [16]. Regardless of the user's coding decision, it is expected that $P_i(z_{i1})R_i(z_{i1}) + P_i(z_{i2})R_i(z_{i2}) \geq R_i(\tilde{z}_{i1})$. We shall formally address the advantage of CSI acquisition in Section 4.

2) *Gaussian channel.* Consider the case where the channel quality between user i and the base station evolves as a block-fading process, with white Gaussian noise being added to the transmitted signal. Specifically, at each time t, the received signal $y_i(t)$ is given by $y_i(t) = \sqrt{w_i(t)}x_i(t) + n(t)$, where $x_i(t)$ and $w_i(t)$ are respectively the transmitted signal and channel gain (which is the physical interpretation for channel quality). Let T be the length of a slot. Then $w_i(t)$ remains constant within slot boundaries, i.e., $w_i(t) \equiv w_{i,k}, t \in [(k-1)T, kT)$, $k \geq 1$. Suppose user i is able to obtain the underlying channel quality with high precision, namely $\zeta_{i,k} \approx w_{i,k}$. Let S_i be the maximal energy per slot, and let $N_0/2$ be the noise power spectral density. Then if the user can optimize its coding scheme for rates approaching the Shannon capacity, the expected data rate which can be reliably transmitted is given by the well known formula $R_i(z_i) = B_i \log(1 + \frac{S_i}{N_0} z_i)$, where B_i is the bandwidth.

2.2 User Objective and Game Formulation

In this subsection we describe the user objective and the non-cooperative game which arises as a consequence of the user interaction over the collision channel. In Section 2.2.1 we define the Nash equilibrium of the game, and also characterize stationary transmission strategies, which are central in this paper. Some basic properties of these strategies are highlighted in Section 2.2.2.

2.2.1 Basic Definitions

We associate with each user i has a throughput demand ρ_i (in bits per slot) which it[4] wishes to deliver over the network. The objective of each user is to minimize its average transmission power (which is equivalent in our model to the average rate of transmission

[4]The *user* here should be interpreted as the algorithm that manages the transmission schedules, and is accordingly referred to in the third person neuter.

attempts, as users transmit at a fixed power level), while maintaining the effective data rate at (or above) this user's throughput demand. We further assume that users always have packets to send, yet they may postpone transmission to a later slot to accommodate their required throughput with minimal power investment.

A general transmission schedule, or strategy, π_i for user i specifies a transmission decision at each time instant, based on the available information, that includes the CSI signals and (possibly) the transmission history for that user. A transmission decision may include randomization (i.e., transmit with some positive probability). Obviously, each user's strategy π_i directly affects other users' performance through the commonly shared medium. The basic assumption of our model is that users are self-optimizing and are free to determine their own transmission schedule in order to fulfill their objective. We further assume that users are unable to coordinate their respective decisions. This situation is modeled and analyzed in our paper as a non-cooperative game [17] between the n users. In particular, we are interested in the Nash equilibrium point of the game, which we define below. Since the bulk of the paper focuses on stationary transmission strategies, we will not bother with a formal definition of a general strategy. For our purpose, it suffices to assume that the collection of user strategies $(\pi_i)_{i\in\mathcal{I}}$ together with the channel description, induce a well defined stochastic process of user transmissions.

We use the term multi-strategy when referring to a collection of user strategies, and denote by $\pi = (\pi_1, \ldots, \pi_n)$ the multi-strategy comprised of all users' strategies. The notation π_{-i} is used for the transmission strategies of all users but for the i-th one. For each user i, let $p_i(\pi)$ be the average transmission rate (or transmission probability), and let $r_i(\pi)$ be the expected average throughput, as determined by the user's own strategy π_i and by the strategies of all other users π_{-i}. Further denote by $c_{i,k}$ the indicator random variable which equals one if user i transmits at slot k and zero otherwise, and by $r_{i,k}$ the number of data bits *successfully* transmitted by user i at the same slot. Then

$$p_i(\pi) = \lim_{K\to\infty} \mathbb{E}^{\pi}\Big(\frac{1}{K}\sum_{k=1}^{K} c_{i,k}\Big), \tag{2.1}$$

and

$$r_i(\pi) = \lim_{K\to\infty} \mathbb{E}^{\pi}\Big(\frac{1}{K}\sum_{k=1}^{K} r_{i,k}\Big), \tag{2.2}$$

where $\mathbb{E}^{\pi}$ stands for the expectation operator under the multi-strategy π. If the limit in (2.1) does not exist we may take the $\limsup$ instead, and similarly the $\liminf$ in (2.2).

A Nash equilibrium point (NEP) is a multi-strategy $\pi = (\pi_1, \ldots, \pi_n)$, which is self-sustaining in the sense that all throughput constraints are met, and neither user can lower its transmission rate by unilaterally modifying its transmission strategy. Formally,

Definition 2.1 (Nash equilibrium point) *A multi-strategy $\pi = (\pi_1, \ldots, \pi_n)$ is a Nash equilibrium point if*

$$\pi_i \in \underset{\tilde{\pi}_i}{\operatorname{argmin}} \left\{ p_i(\tilde{\pi}_i, \pi_{-i}) : r_i(\tilde{\pi}_i, \pi_{-i}) \geq \rho_i \right\}. \tag{2.3}$$

The transmission rate p_i can be regarded as the cost which the user wishes to minimize. Using game-theoretic terminology, a Nash equilibrium is a multi-strategy $\pi = (\pi_1, \ldots, \pi_n)$ so that each π_i is a *best response* of user i to π_{-i}, in the sense that the user's cost is minimized.

Our focus in this paper is on *stationary* transmission strategies, in which the decision whether to transmit or not can depend (only) on the current CSI signal. A formal definition for a stationary strategy is provided below.

Definition 2.2 (stationary strategies) *A stationary strategy for user i is a mapping $\pi_i : \mathcal{Z}_i \to [0, 1]$. Equivalently, a stationary strategy will be represented by an x_i-dimensional vector $\mathbf{s}_i = (s_{i1}, \ldots, s_{ix_i}) \in [0, 1]^{x_i}$, where the m-th entry corresponds to the user i's transmission probability when the observed CSI signal is z_{im}.*

For example, the vector $(0, \ldots, 0, 1)$ represents the strategy of transmitting (w.p. 1) only when the CSI signal is the highest possible. Note that the transmission probability in a slot, which is a function of $\mathbf{s}_i$ only, is given by

$$p_i(\mathbf{s}_i) = \sum_{m=1}^{x_i} s_{im} P_i(z_{im}). \tag{2.4}$$

Let $\mathbf{s} \triangleq (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ denote a stationary multi-strategy for all users. Evidently, the probability that no user from the set $\mathcal{I} \backslash i$ transmits in a given slot is given by $\prod_{j \neq i}(1 - p_j(\mathbf{s}_j))$. Since the transmission decision of each user is independent of the decisions of other users, the expected average rate $r_i(\mathbf{s}_i, \mathbf{s}_{-i})$ is given by

$$r_i(\mathbf{s}_i, \mathbf{s}_{-i}) = \left[\sum_{m=1}^{x_i} s_{im} P_i(z_{im}) R_i(z_{im}) \right] \prod_{j \neq i} \big(1 - p_j(\mathbf{s}_j)\big), \tag{2.5}$$

where the expression $\sum_{m=1}^{x_i} s_{im} P_i(z_{im}) R_i(z_{im})$ stands for the average rate which is obtained in a collision-free environment under the same strategy $\mathbf{s}_i$.

2.2.2 Threshold Strategies

A subclass of stationary strategies which is central in our analysis is defined below.

Definition 2.3 (threshold strategies) *A threshold strategy is a stationary strategy of the form*
$\mathbf{s}_i = (0, 0, \ldots, 0, s_{im_i}, 1, 1 \ldots, 1)$*,* $s_{im_i} \in (0, 1]$*, where* z_{im_i} *is a threshold CSI level above which user* i *always transmits, and below which it never transmits.*

An important observation, which we summarize next, is that users should always prefer threshold strategies.

Lemma 1 *Assume that all users access the channel using a stationary strategy. Then a best response strategy of any user* i *is always a threshold strategy.*

Proof: For a given multi-strategy vector $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$, assume by way of contradiction that $\mathbf{s}_i$ is a best response strategy which is not a threshold strategy. Then there exist some indexes k and l with $l > k$, such that $s_{ik} > 0$ and $s_{il} < 1$. The key observation in establishing the claim, is that a lower-cost strategy can be obtained by increasing the transmission probability of the better CSI signal at the expense of decreasing the transmission probability of the lower quality signal.

Formally, we construct a lower-cost strategy $\tilde{\mathbf{s}}_i$ (derived from $\mathbf{s}_i$) via the two following steps: (i) Set $\tilde{s}_{im} = s_{im}$ for every $m \in \mathcal{Z}_i \backslash \{k, l\}$; and $\tilde{s}_{ik} = s_{ik} - \frac{\epsilon}{P_i(k)}$, $\tilde{s}_{il} = s_{il} + \frac{\epsilon}{P_i(l)}$, where $\epsilon > 0$ is a small constant. Note from (2.4) that this strategy maintains the same transmission rate, yet the average throughput in (2.5) increases (as R_i is an increasing function), thus it is strictly above ρ_i (as the original strategy $\mathbf{s}_i$ meets the throughput demand by definition). (ii) Pick an arbitrary m for which $s_{im} > 0$. By the last argument, there exists some $\delta > 0$ such that letting $\tilde{s}_{im} = s_{im} - \delta$ still maintains the throughput demand, yet with an overall lower transmission rate. This contradicts the optimality of $\mathbf{s}_i$.
□

As a result of the above lemma, we may analyze the non-cooperative game by restricting the strategies of each user i to the set of threshold strategies, denoted by T_i. We proceed by noting that every threshold strategy can be identified with a unique *scalar* value $p_i \in [0, 1]$, which is the transmission probability in every slot, i.e., $p_i \equiv p_i(\mathbf{s}_i)$. More precisely:

Lemma 2 *The mapping* $\mathbf{s}_i = (0, 0, \ldots, 0, s_{im_i}, 1, 1 \ldots, 1) \in T_i \mapsto p_i \equiv p_i(\mathbf{s}_i) \in [0, 1]$*, is* *a surjective (one-to-one and onto) mapping from the set of threshold strategies T_i to the interval* $[0,1]$.

Proof: The claim follows directly from (2.4), upon noting that $\sum_m P_i(z_{im}) = 1$ and $P_i(z_{im}) > 0$ by assumption. Note that under a threshold strategy, $p_i = s_{im_i} P_i(z_{im_i}) + \sum_{m=m_i+1}^{x_i} P_i(z)$. Indeed, $0 \leq p_i(\mathbf{s}_i) \leq \sum_m P_i(z_{im}) \leq 1$. Conversely, every $p_i \in [0,1]$ corresponds to a unique threshold strategy as follows: Given p_i, the corresponding is such that $\sum_{m=m_i+1}^{x_i} P_i(z_{im}) < p_i$ and $\sum_{m=m_i}^{x_i} P_i(z_{im}) \geq p_i$; the transmission probability for the threshold CSI is given by $s_{im_i} = \frac{p_i - \sum_{m=m_i+1}^{x_i} P_i(z_{im})}{P_i(z_{im_i})}$. $s_{im_i} = \frac{p_i - \sum_{m=m_i+1}^{x_i} P_i(z_{im})}{P_i(z_{im_i})}$. □

Given this mapping, the stationary policy of each user will be henceforth represented through a scalar $p_i \in [0,1]$,which uniquely determines the CSI threshold and its associated transmission probability, denoted by $z_{im_i}(p_i)$ and $s_{im_i}(p_i)$ respectively. Consequently, the user's expected throughput per slot in a collision free environment, denoted by H_i, can be represented as a function of p_i only, namely

$$H_i(p_i) \triangleq s_{im_i}(p_i) P_i(z_{im_i}(p_i)) R_i(z_{im_i}(p_i)) + \sum_{m=m_i(p_i)+1}^{x_i} P_i(z_{im}) R_i(z_{im}). \tag{2.6}$$

This function will be referred to as the *collision-free rate function.* Using this function, we may obtain an explicit expression for the user's average throughput, as a function of $\mathbf{p} = (p_1, \ldots, p_n)$, namely

$$r_i(p_i, \mathbf{p}_{-i}) = H_i(p_i) \prod_{j \neq i} (1 - p_j). \tag{2.7}$$

Example: No CSI. A special important case is when no CSI is available. This corresponds to $x_i = 1$ in our model. In this case the collision-free rate function is simply $H_i(p_i) = \bar{R}_i p_i$, where $\bar{R}_i = R_i(z_{i1})$ is the expected data rate that can be obtained in any given slot.

3 Equilibrium Analysis

In this section we analyze the Nash equilibrium point (2.3) of the network under stationary transmission strategies. For the analysis, we require the following properties of the rate function (2.6):

Lemma 3 *The collision-free rate function H_i satisfies the following properties.*

(i) $H_i(0) = 0$.

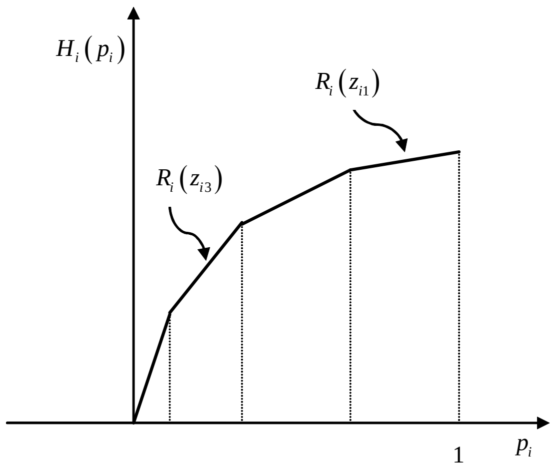

Figure 1: An example of the collision-free rate function $H_i(p_i)$. In this example there are four CSI signals. Note that the slope of $H_i(p_i)$ is exactly the rate of the threshold CSI which corresponds to p_i.

(ii) $H_i(p_i)$ *is a continuous and strictly increasing function over* $p_i \in [0, 1]$.

(iii) $H_i(p_i)$ *is concave.*

Proof: Noting (2.4), $p_i = 0$ means no transmission at all, thus an average rate of zero. It can be easily seen that $H_i(p_i)$ in (2.6) is a piecewise-linear (thus continuous), strictly increasing function. As to concavity, note that the slope of H_i is determined by $R_i(z_{im_i})$ which decreases with p_i (see Figure 1), as m_i decreases in p_i (from Eq. (2.6)) and z_{im} is increasing in m (by definition). □

A key observation which is useful for the analysis is that every Nash equilibrium point can be represented via a set of n equations in the n variables $\mathbf{p} = (p_1, \ldots, p_n)$. This is summarized in the next proposition.

Proposition 1 (The equilibrium equations) *A multi-strategy* $\mathbf{p} = (p_1, \ldots, p_n)$ *is a Nash equilibrium point if and only if it solves the following set of equations*

$$r_i(p_i, \mathbf{p}_{-i}) = H_i(p_i) \prod_{j \neq i} (1 - p_j) = \rho_i, \quad i \in \mathcal{I}. \tag{3.8}$$

Proof: Adapting the Nash equilibrium definition (2.3) to stationary threshold strategies, a NEP is a multi-strategy $\mathbf{p} = (p_1, \ldots, p_n)$ such that

$$p_i = \min \{\tilde{p}_i \in [0,1], \quad \text{subject to } r_i(\tilde{p}_i, \mathbf{p}_{-i}) \geq \rho_i\}, \quad i \in \mathcal{I}, \tag{3.9}$$

where r_i is defined in (2.7). Since $r_i(\tilde{p}_i, \mathbf{p}_{-i})$ is strictly increasing in $\tilde{p}_i$ (by Lemma 3), (3.9) is equivalent to $r_i(p_i, \mathbf{p}_{-i}) = \rho_i$, $i \in \mathcal{I}$, which is just (3.8). □

Due to the above result, we shall refer to the set of equations (3.8) as the *equilibrium equations*.

3.1 Two Equilibria or None

We next address the *number* of equilibrium points in our system. Obviously, if the overall throughput demands of the users are too high there cannot be an equilibrium point, since the network naturally has limited traffic capacity (the capacity of the network will be considered in Section 5). When throughput demands are within the feasible region, we establish that there are exactly *two* Nash equilibria.

Denote by $\rho = (\rho_1, \ldots, \rho_n)$ the vector of throughput demands, and let Ω be the set of feasible vectors ρ, for which there exists at least one Nash equilibrium point (equivalently, for which there exists a feasible solution to (3.8)). Figure 1 illustrates the set of feasible throughput demands for a simple two-user case, with $H_i(p_i) = p_i$.

To specify some structural properties of Ω, it is convenient to define the set of basis vectors $\hat{\Omega}$, where each $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_n) \in B_1^+$ is such that $\hat{\rho}_i > 0$ for every $i \in \mathcal{I}$ and $\|\hat{\rho}\|_2 = 1$, i.e., $\sum_i \hat{\rho}_i^2 = 1$. We also define the upper boundary of Ω, denoted Ω^+ as

$$\Omega^+ = \left\{\alpha(\hat{\rho})\hat{\rho} : \hat{\rho} \in B_1^+, \alpha(\hat{\rho}) = \sup\{\alpha \geq 0 : \alpha\hat{\rho} \in \Omega\}\right\}. \tag{3.10}$$

Proposition 2 *The feasible set Ω obeys the following properties.*
(i) Closed cone structure: For every $\hat{\rho} \in B^+$, $\alpha\hat{\rho} \in \Omega$ for all $\alpha \in [0, \alpha(\hat{\rho})]$.
(ii) Continuity of the upper boundary Ω^+: For every $\epsilon > 0$ there exists some $\delta > 0$ such that if $\|\hat{\rho}^1 - \hat{\rho}^2\|_2 < \delta$ then $|\alpha(\hat{\rho}^1) - \alpha(\hat{\rho}^2)| < \epsilon$.
(iii) Let $\rho \leq \tilde{\rho}$ be two throughput demand vectors. Then if $\tilde{\rho} \in \Omega$, it follows that $\rho \in \Omega$.

Proof: See Appendix A.

In particular, note that Ω is a closed set with nonempty interior. We can now specify the number of equilibrium points for any throughput demand vector $\rho = (\rho_1, \ldots, \rho_n)$.

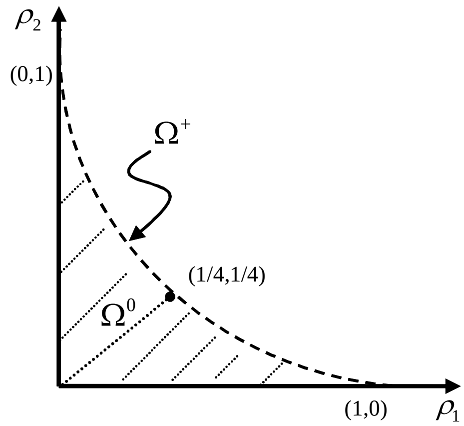

Figure 2: The set of feasible throughput demands for a two user network with $H_i(p_i) = p_i$, $i = 1, 2$.

Theorem 3 *Consider the non-cooperative game model under stationary transmission strategies. Let Ω be the set of feasible throughput demand vectors ρ, let Ω^+ be its upper boundary, and let Ω^0 be its interior. Then*
(i) For each $\rho \in \Omega^+$ there exists a unique Nash equilibrium point.
(ii) For each $\rho \in \Omega^0$ there exist exactly two Nash equilibria.

Proof: See Appendix A.

Cases where $\rho_i = 0$ for some $i \in \mathcal{I}$ are covered by a reduction to a smaller number of (active) users. Note that the case of a single equilibrium point is non-generic (i.e., occurs only for a set of throughput vectors ρ of measure zero). Accordingly, we shall can exclude the single equilibrium case from our discussion.

3.2 The Energy Efficient Equilibrium

Going beyond the basic questions of existence and number of equilibrium points, we wish to further characterize the properties of the equilibrium points. In particular, we are interested here in the following question: How do the two equilibrium points compare: is one "better" than the other? The next theorem shows that indeed one equilibrium point is power-superior for all users.

Theorem 4 *Assume that the throughput demand vector ρ is within the feasible region Ω^0, so that there exist two equilibria in stationary strategies. Let $\mathbf{p}$ and $\tilde{\mathbf{p}}$ be these two equilibrium points. If $p_i < \tilde{p}_i$ for some user i, then $p_j < \tilde{p}_j$ for every $j \in \mathcal{I}$.*

Proof: Define $a_{ik} \triangleq \frac{\rho_i}{\rho_k}$. For every user $k \neq i$ divide the ith equation in the set (3.8) by the kth one. We obtain

$$a_{ik} = \frac{H_i(p_i)(1-p_k)}{H_k(p_k)(1-p_i)} < \frac{H_i(\tilde{p}_i)(1-p_k)}{H_k(p_k)(1-\tilde{p}_i)}, \tag{3.11}$$

since H_i is increasing. Now since $\frac{H_i(\tilde{p}_i)(1-\tilde{p}_k)}{H_k(\tilde{p}_k)(1-\tilde{p}_i)} = a_{ik}$, it follows that $\frac{(1-\tilde{p}_k)}{H_k(\tilde{p}_k)} < \frac{(1-p_k)}{H_k(p_k)}$. Since H_k is increasing in p_k, we conclude from the last inequality that $p_k < \tilde{p}_k$. □

The last result is significant from the network point of view. It motivates the design of a network mechanism that will avoid the inferior equilibrium point, which is wasteful for *all* users. This will be our main concern in Section 6. Henceforth, we identify the better equilibrium point as the *Energy Efficient Equilibrium (EEE)*.

We now turn to examine the quality of the EEE relative to an appropriate social cost. Recall that each user's objective is to minimize its average transmission rate subject to a throughput demand. Thus, a natural performance criterion for evaluating any multi-strategy $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ (in particular, an equilibrium multi-strategy) is given by the sum of the user's average transmission rates induced by $\mathbf{s}$, namely

$$Q(\mathbf{s}) = \sum_i p_i(\mathbf{s}_i). \tag{3.12}$$

The next theorem addresses the quality of the EEE with respect to that criterion.

Theorem 5 *Let $\mathbf{p}$ be an EEE. Then $\sum_i p_i \leq 1$.*

Proof: See Appendix A.

An immediate conclusion from the above theorem is that the overall power investment at the EEE is bounded, as the sum of transmission probabilities is bounded. This means, in particular, that the average transmission power of all users is bounded by the maximal transmission power of a single station.

3.3 Social Optimality and Efficiency Loss

We proceed to examine the extent to which selfish behavior affects system performance. That it, we are interested to compare the quality of the obtained equilibrium points to the centralized, system-optimal solution (still restricted to stationary strategies). Recently, there has been much work in quantifying the "efficiency loss" incurred by the selfish behavior of users in networked systems (see [18] for a comprehensive review). The two concepts which are most commonly used in this context are the *price of anarchy* (PoA), which is (an upper bound on) the performance ratio (in terms of a relevant social performance measure) between the global optimum and the *worst* Nash equilibrium, and *price of stability* (PoS), which is (an upper bound on) the performance ratio between the global optimum and the *best* Nash equilibrium.

Returning to our specific network scenario, consider the case where a central authority, which is equipped with user characteristics $\mathbf{H} = (H_1, \ldots, H_n)$ and $\rho = (\rho_1, \ldots, \rho_n)$ can enforce a stationary transmission strategy for every user $i \in \mathcal{I}$. We consider (3.12) as the system-wide performance criterion, and compare the performance of this optimal solution to the performance at the Nash equilibria. A socially optimal multi-strategy denoted $\mathbf{s}^*(\mathbf{H}, \rho)$, is a strategy that minimizes (3.12), while obeying all user throughput demands ρ_i. Similarly, denote by $\mathbf{s}^{\mathbf{b}}(\mathbf{H}, \rho)$ and $\mathbf{s}^{\mathbf{w}}(\mathbf{H}, \rho)$ the multi-strategies at the better NEP and at the worse NEP, respectively. Then the PoA and PoS are given by

$$PoA = \sup_{\mathbf{H},\rho} \frac{Q\big(\mathbf{s}^{\mathbf{w}}(\mathbf{H}, \rho)\big)}{Q\big(\mathbf{s}^*(\mathbf{H}, \rho)\big)}, \quad PoS = \sup_{\mathbf{H},\rho} \frac{Q\big(\mathbf{s}^{\mathbf{b}}(\mathbf{H}, \rho)\big)}{Q\big(\mathbf{s}^*(\mathbf{H}, \rho)\big)}. \tag{3.13}$$

We next show that the PoA is generally unbounded, while the PoS is always one.

Theorem 6 *Consider the non-cooperative game, the NEP of which is defined in (2.3). Then (i) The PoS is always one, and (ii) The PoA is generally unbounded.*

Proof: (i) This claim follows immediately, noting that 1) the socially optimal stationary strategy is a threshold strategy (by applying a similar argument to the one used in Lemma 1), and 2) the socially optimal stationary strategy obeys the equilibrium equations (3.8) (following a similar argument to the one used in Proposition 1). Hence, by Proposition 1 the optimal solution is also an equilibrium point. Equivalently, this means that $PoS = 1$.

(ii) We establish that the price of anarchy is unbounded by means of an example. Consider a network with n identical users with $H_i(p_i) = \bar{R}p_i$ (this collision-free rate function corresponds to users who cannot obtain any CSI). Each user's throughput demand is $\rho_i = \epsilon \to 0$. Recall that the throughput demands are met with equality at every equilibrium point (Proposition 1). Then, by symmetry, we obtain a single equilibrium equation, namely $\bar{R}p(1-p)^{n-1} = \epsilon$. As ϵ goes to zero, the two equilibria are $p_a \to 1$ and $p_b \to 0$.

Obviously, the latter point is also a social optimum; it is readily seen that the price of anarchy here equals in the limit to $\frac{p_a}{p_b} \to \infty$. □

The above theorem clearly motivates the need for a mechanism that will induce the EEE, as this equilibrium point coincides with the socially-optimal solution, while the gap between the two equilibria could be arbitrarily large.

3.4 Computational Aspects

We conclude this section by briefly addressing the computational properties of an equilibrium point, which may be directly observed from the proof of Theorem 3. The proof of Theorem 3 relies on representing the equation set (3.8) through a single scalar equation $g_i(p_i)$, which was shown to be unimodal. Hence we may verify the existence of an equilibrium by finding the maximizer of $g_i(p_i)$. An equilibrium obviously exists if $\max_{p_i} g_i(p_i) \geq \log \rho_i$. The maximizer of $g_i(p_i)$ as a unimodal function could be efficiently obtained by standard search techniques (such as the bisection method or the golden section search method [19]). Subsequently, we may efficiently compute each of the two equilibria, by dividing the region $[0, 1]$ into two subregions, where the maximizer of $g_i(p_i)$ is the boundary point between the subregions. As a result of the unimodality $g_i(p_i)$, exactly one equilibrium point lies in each subregion. Each equilibrium can be efficiently calculated by finding the zero of the function $g_i(p_i) - \rho_i$, which is a monotonous function in each subregion.

An alternative way for verifying the existence of the equilibrium point, as well as calculating the EEE is obtained by simulating the best-response dynamics, which is considered in Section 6. Indeed, we show in that section that this dynamics either converges to the EEE, or obtains probabilities larger than 1 in case that no equilibrium exists.

4 Performance Benefits of Channel State Information

In this section we show that network performance is enhanced whenever the information quality is improved, that is, when users can obtain refined CSI. Generally, it is expected that under centralized control, system performance would improve when better quality information is available. However, a selfish-user environment can lead to various so-called paradoxes. A classic example is the Braess paradox [20], where the addition of a link in a transportation network increases the overall traffic delay. In [21] it was shown that users are sometimes better off without acquiring additional knowledge (such as the number of network users). We next show that in our context, CSI acquisition is beneficial even with self-optimizing users. We consider both the the quality of the EEE for given throughput demands and also the total throughput which can be supported by the network.

By enhanced channel state information, we mean that the user can distinguish between different underlying channel conditions which were originally classified identically. Consider Example 1 in Section 2.1. The partial CSI set with two signals is obviously of better quality than the null CSI set. An even better CSI set would be, for example, a four signal set, where the user gets different signals for "excellent" and "fair" conditions, for channel states which were originally interpreted only as "good". Similarly, the user may obtain two different CSI signals, indicating "no-connection" and "some-connectivity" channel conditions, for channel states which were originally classified as "weak" in the two-signal set. To formalize the notion of better quality (or refined) CSI sets, we use the next definition.

Definition 4.1 (CSI Refinement) *Consider two CSI sets $\mathcal{Z}_i$ and $\tilde{\mathcal{Z}}_i$ which correspond to the same underlying channel conditions. Let $P_i(z_i)$ be the probability of observing a CSI signal $z_i \in \mathcal{Z}_i$, and let $\tilde{P}_i(\tilde{z}_i)$ be the probability of observing a CSI signal $\tilde{z}_i \in \tilde{\mathcal{Z}}_i$. Then $\tilde{\mathcal{Z}}_i$ is said to be a refined CSI set with respect to $\mathcal{Z}_i$ if every $z_i \in \mathcal{Z}_i$ is either (i)* duplicated *in $\tilde{\mathcal{Z}}_i$ with $P_i(z_i) = \tilde{P}_i(z_i)$ or (ii)* refined *in $\tilde{\mathcal{Z}}_i$ into a subset of CSI signals $\tilde{z}_{i_1}, \tilde{z}_{i_1}, \ldots, \tilde{z}_{i_{M(z_i)}}$, such that $P_i(z_i) = \sum_m \tilde{P}_i(\tilde{z}_{i_m})$ and*

$$P_i(z_i)R_i(z_i) \leq \sum_m \tilde{P}_i(\tilde{z}_{i_m})R_i(\tilde{z}_{i_m}). \tag{4.14}$$

The set $\tilde{z}_{i_1}, \tilde{z}_{2_1}, \ldots, \tilde{z}_{i_{M(z_i)}}$ is referred to as the refinement of z_i.

Inequality (4.14) incorporates the basic assumption that a user can obtain higher data rates with refined CSI signals. The main consequence of (4.14) is that the collision-free rate functions H_i obtains higher values for refined CSI sets.

Lemma 4 *Consider two CSI sets $\mathcal{Z}_i$ and $\tilde{\mathcal{Z}}_i$, where $\tilde{\mathcal{Z}}_i$ refines $\mathcal{Z}_i$. Let H_i and $\tilde{H}_i$ be the collision-free rate functions for $\mathcal{Z}_i$ and $\tilde{\mathcal{Z}}_i$ respectively, defined in (2.6). Then $H_i(p_i) \leq \tilde{H}_i(p_i)$ for evert $p_i \in [0, 1]$.*

Proof: For a given threshold policy under $\mathcal{Z}_i$ (represented by the scalar p_i), denote by $z_{im_i}(p_i)$ and $s_{im_i}(p_i)$ the CSI threshold and its associated transmission probability, uniquely determined by p_i. The lemma essentially follows directly from (2.6) and (4.14). The only case which deserves a direct proof is the case where the only refined CSI signal is the threshold CSI z_{im_i}. To simplify notations, we write z_m instead of $z_{im_i}(p_i)$ and s_m instead of $s_{im_i}(p_i)$. Let $\tilde{z}_{m_1}, \tilde{z}_{m_2}, \ldots, \tilde{z}_{m_{M(z_m)}}$ be a refinement of z_m, and $\tilde{s}_{m_1}, \tilde{s}_{m_2}, \ldots, \tilde{s}_{m_{M(z_m)}}$ the respective transmission strategy for the refined CSI signals ($s_m = \sum_l \tilde{s}_{m_l}$). Then

$$s_m P_i(z_m) R_i(z_m) \leq \sum_l s_m \tilde{P}_i(\tilde{z}_{m_l}) R_i(\tilde{z}_{m_1}) \leq \sum_l \tilde{s}_{m_l} \tilde{P}_i(\tilde{z}_{m_l}) R_i(\tilde{z}_{m_1}),$$

where the first inequality follows by (4.14), and the second by the fact that the user applies a threshold strategy, putting more weight on CSI signals which lead to higher data rates (by the monotonicity of R_i). □

This lemma directly leads us to the main conclusions regarding the benefits of better CSI. That is: refined CSI is beneficial in terms of both the achievable throughput and the transmission rates at the energy efficient equilibrium.

Theorem 7 [5]*Consider two games instances G and $\tilde{G}$ with identical channel conditions. Denote by $\{\mathcal{Z}_i\}_{i\in\mathcal{I}}$ and $\{\tilde{\mathcal{Z}}_i\}_{i\in\mathcal{I}}$ the CSI sets under G and $\tilde{G}$ respectively, and assume that $\tilde{\mathcal{Z}}_i$ refines $\mathcal{Z}_i$ for every $i \in \mathcal{I}$. Let ρ be the throughput demand vector. Then*
(i) If ρ admits a NEP in G, so it does in $\tilde{G}$.
(ii) If ρ admits an equilibrium in both game instances, the EEE transmission rates in $\tilde{G}$ (denoted $\tilde{p}_i$) are lower than the ones in game G (denoted p_i) for every user i, i.e., $\tilde{p}_i \leq p_i$ for every $i \in \mathcal{I}$.
(iii) A strict *refinement (replace $\leq$ with $<$ in (4.14)) of a CSI signal which is in use at the EEE of G, leads to strictly lower transmission rates at the EEE for all users, i.e., $\tilde{p}_i < p_i$ for every $i \in \mathcal{I}$.*

Proof: The proof essentially follows from Lemma 4. yet, some proof techniques require the notion of best-response dynamics, which is considered in Section 6. Hence, a full proof of the theorem is deferred to Appendix C.

The theorem above implies that even if a single user obtains better CSI, the situation of all users is improved, in the sense that all users obtain lower transmission rates. Thus, CSI acquisition is beneficial from the viewpoint of both the individual and the entire network.

[5]Reference to Theorem 7 in [22] should be to Theorem 9.

5 Achievable Channel Capacity

The aim of this section is to provide explicit lower bounds for the achievable channel capacity. The term "capacity" is used here for the total throughput (normalized to successful transmission per slot) which can be obtained in the network. We focus here on the case where users have no CSI, and then relate our result to general CSI.

Consider the null-CSI model, where no user can observe any CSI (see the example at the end of Section 2). Recall that the collision-free rate in this case is given by $H_i(p_i) = \bar{R}_i p_i$, where $\bar{R}_i$ is the expected data rate in case of a successful transmission. Define $y_i \triangleq \frac{\rho_i}{\bar{R}_i}$, which we identify henceforth as the *normalized throughput* demand for user i: indeed, y_i stands for the required rate of successful transmissions. Then the equilibrium equations (3.8) become

$$p_i \prod_{j \neq i} (1 - p_j) = y_i, \quad 1 \leq i \leq n. \tag{5.15}$$

We shall first consider the symmetric case, i.e., $y_i = y$ for every user i, and then relate the results to the general non-symmetric case. The theorem below establishes the conditions for the existence of an equilibrium point in the symmetric null-CSI case.

Theorem 8 (Symmetric users) [6]*Let $y_i = y$ for every $1 \leq i \leq n$. Then (i) A Nash equilibrium exists if and only if*

$$ny \leq (1 - \frac{1}{n})^{n-1}. \tag{5.16}$$

(ii) In particular, a Nash equilibrium exists if $ny \leq e^{-1}$.

Proof: (i) By dividing the equilibrium equations (5.15) of any two users, it can be seen that every symmetric-users equilibrium satisfies $p_i = p_j = p$ ($\forall\, i, j$). Thus, the equilibrium equations (5.15) reduce to a single (scalar) equation:

$$h(p) \triangleq p(1-p)^{n-1} = y. \tag{5.17}$$

We next investigate the function $h(p)$. Its derivative is given as $h'(p) = (1-p)^{n-2}(1-np)$. It can be seen that the maximum value of the function $h(p)$ is obtained at $p = 1/n$. An equilibrium exists if and only if the maximal value of $h(p)$ is greater than y. Substituting the maximizer $p = 1/n$ in (5.17) implies the required result.

(ii) It may be easily verified that the right hand side of (5.16) decreases with n. Since $\lim_{n\to\infty}(1 - \frac{1}{n})^{n-1} = e^{-1}$, the claim follows from (i). □

[6] Reference to Theorem 8 in [22] should be to Theorem 10.

We now show that the simple bound obtained above holds for non-symmetric users as well, implying that the symmetric case is worst in terms feasible channel utilization.

Theorem 9 (Asymmetric users) *For any set of n null-CSI users with normalized throughput demands $\{y_i\}$, an equilibrium point exists if*

$$\sum_{i=1}^{n} y_i \leq (1 - \frac{1}{n})^{n-1}. \tag{5.18}$$

Proof: See Appendix B.

The quantity e^{-1} is also the well-known maximal throughput of a slotted Aloha system with Poisson arrivals and an infinite set of nodes [2]. In our context, if the normalized throughput demands do not exceed e^{-1}, an equilibrium point is guaranteed to exist. Thus, in a sense, we may conclude that noncooperation of users, as well restricting users to stationary strategies, do not reduce the capacity of the collision channel.

We conclude this section by noting that Equation (5.18) serves as a global sufficient condition for the existence of an equilibrium point, which holds for any level of channel observability. This observation follows by Theorem 7(i), which implies that the capacity can only increase when users obtain channel state information.

6 Best-Response Dynamics

stable working point, from which no user has incentive to deviate unilaterally. Still, the question of if and how the system arrives at an equilibrium remains open. Furthermore, since our system has two Nash equilibria with one (the EEE) strictly better than the other, it is of major importance (from the system viewpoint, as well as for each individual user) to employ mechanisms that converge to the better equilibrium rather than the worse.

The distributed mechanism we consider here relies on a user's best-response, which is generally the optimal user reaction to a given network condition (see [17]). Specifically, the best response of a given user is a transmission probability which brings the obtained throughput of that user (given other user strategies) to its throughput demand ρ_i. Accordingly, observing (2.7), the best response of user i for any multi-strategy $\mathbf{p} = (p_1, \ldots, p_n)$ is given by

$$p_i := H_i^{-1}\left(\frac{\rho_i}{\prod_{j \neq i}(1 - p_j)}\right), \tag{6.19}$$

where H_i^{-1} is the inverse function of the collision-free rate function H_i (if the argument of H_i^{-1} is larger than maximal value of H_i, p_i can be chosen at random). Note that H_i^{-1} is well defined, since H_i is continuous and monotone (Lemma 3). It is important to notice that each user is not required to be aware of the transmission probability of every other user. Indeed, only the overall idle probability of other users $\prod_{j\neq i}(1-p_j)$ is required in (6.19). Our mechanism can be described as follows. *Each user updates its transmission probability from time to time through its best response (6.19). The update times of each user need not be coordinated with other users.*

This mechanism reflects what greedy, self-interested users would naturally do: Repeatedly observe the current network situation and react to bring their costs to a minimum. For the analysis of best-response dynamics we assume the following.

Assumption 2
(i) The user population is fixed.
(ii) Users repeatedly update their transmission probabilities (i.e., an infinite number of updates for each user) using Eq. (6.19) .
(iii) The effective elements $\prod_{j\neq i}(1-p_j)$ and H_i^{-1} are perfectly estimated by the user before each update.
(iv) The transmission probabilities of each user are initialized to zero ("slow start").

Our convergence result is summarized below.

Theorem 10 (Convergence to the EEE) *Under Assumption 2, best response dynamics asymptotically converges to the EEE, in case that a Nash equilibrium point exists.*

The proof of the above result relies on showing that the vector of user probabilities $\mathbf{p}$ monotonously increases until convergence. See Appendix C for a detailed proof. We note that the initialization of the transmission probabilities to zero (or any other value smaller than the EEE) is essential for this result.

We briefly list here some important considerations regarding of the presented mechanism.
1) The slow start requirement (Assumption 2(ii)) is essential for preventing excessive transmissions which lead to the worst equilibrium.
2) It is important to notice that each user is not required to be aware of the transmission probability of every other user. Indeed, only the overall idle probability of other users

$\prod_{j\neq i}(1-p_j)$ is required in (6.19). This quantity could be estimated by each user by monitoring the channel utilization.
3) Assumption 2(iv) entails the notion of a quasi-static system, in which each user responses to the steady state reached after preceding user updates. This assumption approximates a natural scenario where users update their transmission probabilities at much slower time-scales than their respective transmission rates.

The result of Theorem 10 relates to a fixed user population. However, the user population (and the user throughput requirement) change over time. Hence, it is important to study the system dynamics when users join or leave the network. For our analysis, we assume that the network is at equilibrium, i.e., the required throughput demands are met for the present users, when new users join or leave. This assumption, is more acceptable in wireless systems, where the node population varies on a relatively slower time-scale than the convergence speed of the mechanism. This could be the case, for example, in wireless LAN networks. The next result shows that the network is resilient against a change in user population, in the sense that the best-response dynamics reconverge to the EEE.

Theorem 11 (Joining and Leaving) *Consider a network which is at its EEE, and the next two scenarios: (a) some users join the network (not necessarily at the same time slot). (b) some users leave the network (not necessarily at the same time slot). Then under Assumptions 2(i) and 2(ii), the best response dynamics will (asymptotically) re-converge to the EEE in either scenario.*

The case of joining users essentially follows from Theorem 10, as the transmission probabilities of all users monotonously increase. However, when users leave the network, present users would lower their transmission probabilities. Hence, in the latter case, different proof techniques are required for showing convergence to the EEE. A detailed proof of Theorem 11 is provided in Appendix C. The case where some users join and some abandon is more involved, and is under current investigation.

notice convergence results obtained in the section would still hold for a relaxed variation of (6.19), given by

$$p_i := \beta_i H_i^{-1}\left(\frac{\rho_i}{\prod_{j\neq i}(1-p_j)}\right) + (1-\beta_i)p_i, \tag{6.20}$$

where $0 \leq \beta_i \leq 1$. This update rule can be more robust against inaccuracies in the estimation of $\prod_{j\neq i}(1-p_j)$, perhaps at the expense of slower convergence to the desired equilibrium.

Our convergence results are obviously idealized and should be supplemented with further analysis of the effect of possible deviations from the model and possible remedies. In case that a worst equilibrium point does occur (or no equilibrium is obtained after a reasonably long time), users can reset their probabilities and restart the mechanism (6.19) for converging to the better equilibrium. This procedure resembles the basic ideas behind TCP protocols. The exact schemes for detecting operation at suboptimal equilibria, and consequently directing the network to the EEE are beyond the scope of the present paper.

7 Discussion

We briefly discuss here some consequences of our results, emphasizing network management aspects. Our equilibrium analysis has revealed that within the feasible region the system has two Nash equilibrium points with one strictly better than the other. The better equilibrium (the EEE) is socially optimal, hence the network should ensure that users indeed operate at that equilibrium. An important step in this direction is the above suggested distributed mechanism which converges to the EEE. It should be mentioned however that fluctuations in the actual system might clearly bring the network to an undesired equilibrium. Hence, centralized management (based on user feedbacks) may still be required to identify the possible occurrence of the worse equilibria, and then direct the network to the EEE. Possible mechanisms for this purpose remain a research direction for the future.

In this paper we mainly considered the throughput demands ρ_i as determined by the user itself. Alternatively, ρ_i may be interpreted as a bound on the allowed throughput which is imposed by the network (as part of a resource allocation procedure). The advantage of operating in this "allocated-rate" mode is twofold. First, the network can ensure that user demands do not exceed the network capacity (e.g., by restricting the allocated rate, or through call admission control). Second, users can autonomously reach an efficient working point without network involvement, as management overhead is reduced to setting the user rates only. The rate allocation phase (e.g., through service level agreements) is beyond the scope of the present model.

A final comment relates to elastic users that may lower their throughput demand based on a throughput–power tradeoff. An obvious effect of demand elasticity would be to lower the throughput at inefficient equilibria. It remains to be verified whether other properties established here remain valid in this case.

8 Conclusion

We have investigated in this paper the interaction between self-interested wireless users, each wishing to sustain a given throughput requirement, while making use of available CSI. We have characterized the set of feasible throughput requirements for which a Nash equilibrium exists, and shown that within the feasible region there exist two distinct NEPs, with one being power-superior for *all* users. We further demonstrated that the performance gap between these two equilibria (in terms of power investment) could be arbitrarily large. Consequently, network users should be willing to accept a mechanism which ensures convergence to the better equilibrium. We have suggested a simple and natural mechanism based on each user's best response. This mechanism is shown to converge to the better equilibrium point (within a simplified, dynamic model) without requiring any coordination between the users.

A reassuring result of our work is the utility of higher-quality channel state information, which leads to power savings for all users. Interestingly, better personal information is exploited by the individual user, yet not at the expense of other users' performance. This indicates that wireless platforms can benefit from technological enhancements which would lead to higher quality CSI, even when available to some users and not others, and under totally distributed and self-interested user environments.

The framework and results of this paper may be extended in several ways. One direction is to extend the reception model beyond the collision model studied in this paper. In particular, capture models (which sometimes better represent WLAN systems) and multi-packet reception models [23] (as in CDMA systems) are of obvious interest. Another extension of interest to the channel model is CSI signals that are *correlated* in time (i.e., subsequent CSI signals are statistically dependent) and/or in space (i.e., CSI signals of neighboring users are statistically dependent). Last, we intend to consider non-stationary user strategies. A central question is whether the system benefits from the use of more complex policies by selfish individuals. The incorporation of non-stationary strategies and correlated CSI seems to add considerable difficulty to the analysis, and may require more elaborate game theoretic tools than the ones used here.

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APPENDICES

A Proofs for Section 3

Proof of Proposition 2: The idea behind the proof is to reduce the equation set (3.8) to a scalar equation in a single variable p_i, for some arbitrarily chosen user i.

Construction of the scalar function. Consider an equilibrium point with throughput demands $\rho = \alpha\hat{\rho}$, where $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_n) \in B^+$ is some fixed basis vector. Then by dividing the ith equilibrium equation (3.8) by the jth one, we obtain

$$\frac{H_i(p_i)}{1-p_i} = \frac{\hat{\rho}_i}{\hat{\rho}_j} \frac{H_j(p_j)}{1-p_j}. \tag{A.21}$$

Note that for each $p_i \in [0,1]$ there exists a unique $p_j \in [0,1]$ such that (A.21) holds. This follows since the function $h_j(p_j) \triangleq \frac{H_j(p_j)}{1-p_j}$ is continuous, strictly increasing and ranges from 0 to ∞ over $p_j \in [0,1]$. Consequently, its inverse h_j^{-1} is a well defined, continuous increasing function, and

$$p_j(p_i) = h_j^{-1}\left(\frac{H_i(p_i)}{1-p_i}\frac{\hat{\rho}_i}{\hat{\rho}_j}\right). \tag{A.22}$$

By substitution, the ith equilibrium equation can be regarded as function of p_i only, namely

$$H_i(p_i)\prod_{j\neq i}(1-p_j(p_i)) = \alpha\hat{\rho}_i, \tag{A.23}$$

where $p_j(p_i)$ is defined in (A.22). It follows that an equilibrium exists for a given throughput vector $\rho = \alpha\hat{\rho}$ if and only if there exists some $p_i \in [0,1]$ such that

$$f_i(p_i, \hat{\rho}) \triangleq \frac{1}{\hat{\rho}_i} H_i(p_i)\prod_{j\neq i}(1-p_j(p_i)) = \alpha. \tag{A.24}$$

The key property which is required for the proof, is the continuity of the function $f_i(p_i, \hat{\rho})$, which is shown below.

Lemma 5 *The function $f_i(p_i, \hat{\rho})$ defined in (A.24) is continuous in p_i. Additionally, fixing p_i, it is continuous in $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_n)$.*

Proof: Continuity in p_i and $(\hat{\rho}_1, \ldots, \hat{\rho}_n)$ follows straightforwardly by the continuity of the functions H_i and $p_j(p_i)$. Note that $f_i(p_i, \hat{\rho}) = \frac{1}{\hat{\rho}_i} H_i(p_i)\prod_{j\neq i}\left(1 - h_j^{-1}\left(\frac{H_i(p_i)}{1-p_i}\frac{\hat{\rho}_i}{\hat{\rho}_j}\right)\right)$, hence continuity with respect to $\hat{\rho}$ follows by the continuity of $h_j^{-1}(\cdot)$. □

We are now ready to prove Proposition 2. Let $\alpha_{\max} = \max_{p_i \in [0,1]} f_i(p_i, \hat{\rho})$ (where the maximum is attained by continuity of f_i). Further note that $f_i(0, \hat{\rho}) = f_i(1, \hat{\rho}) = 0$ (by (A.21)) and that $f_i(p_i, \hat{\rho}) > 0$ for $p_i \in (0, 1)$. Thus, since $f_i(p_i, \hat{\rho})$ is continuous in p_i any value in the range $[0, \alpha_{\max}]$ is obtained for some $p_i \in [0, 1]$. This proves part (i) of the proposition. For part (ii), since $f_i(p_i, \hat{\rho})$ is continuous in $\hat{\rho}$, it immediately follows that the maximum value $\alpha_{\max}$ is continuous in $\hat{\rho}$ as well. Part (iii) is proved by techniques which are realted to best-response dynamics. Hence, we defer the proof to Appendix C □

Proof of Theorem 3: We analyze the same scalar function (A.24) which was used to prove Proposition 2. We show that this function is unimodal in p_i. Consequently, by continuity, every value in the range $[0, \alpha_{\max})$ corresponds to two equilibrium points, where the value $\alpha_{\max}$ corresponds to a single equilibrium. The details are provided below.

Fixing $\hat{\rho}$, let $g_i(p_i) \stackrel{\triangle}{=} \log\left(\hat{\rho}_i f_i(p_i)\right)$. Unimodality of $g_i(p_i)$ would clearly imply unimodality of $f_i(p_i, \hat{\rho})$. The function $g_i(p_i)$ is given by

$$g_i(p_i) = \log H_i(p_i) + \sum_{j \neq i} \log(1 - p_j(p_i)). \tag{A.25}$$

For simplicity of notations we shall henceforth write p_j instead of $p_j(p_i)$, yet recall that p_j is a function of p_i.

Step 1: The function $g_i(p_i)$ is continuous in p_i by the continuity of $f_i(p_i, \hat{\rho})$. Furthermore, $g_i(0) = g_i(1) = -\infty$, and $g_i(p_i) > -\infty$ for $p_i \in (0, 1)$ by the corresponding values of $f_i(p_i, \hat{\rho})$. In the sequel we claim that there exists a unique extremum point for the function $g_i(p_i)$. Furthermore this extremum lies in $(0, 1)$, hence it is the maximizer of $g_i(p_i)$. We proceed to compute the derivative $g'(p_i)$ (Step 2), and then show that $g'(p_i)$ changes sign exactly once (Step 3).

Step 2: Taking the logarithm from both sides of (A.21) we obtain

$$\log H_i(p_i) + \log(1 - p_j) = \tilde{a}_{ij} + \log H_j(p_j) + \log(1 - p_i) \tag{A.26}$$

(where $\tilde{a}_{ij} = \log \frac{\hat{\rho}_i}{\hat{\rho}_j}$). Denote by

$$H_i'(p_i) = \frac{dH_i(p_i)}{dp_i} \tag{A.27}$$

the derivative of H_i w.r.t. p_i. Recalling that H_i is piecewise-linear (see Theorem 3), at a finite set of points in which the standard derivative is undefined, we take the left derivative instead. Differentiating both sides of (A.26) yields the following equation

$$\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1 - p_i} = \left(\frac{H_j'(p_j)}{H_j(p_j)} + \frac{1}{1 - p_j}\right) \frac{dp_j}{dp_i}. \tag{A.28}$$

Thus,

$$\frac{dp_j}{dp_i} = \frac{\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}}{\frac{H_j'(p_j)}{H_j(p_j)} + \frac{1}{1-p_j}}. \tag{A.29}$$

The derivative of $g_i(p_i)$ w.r.t. p_i is given by

$$g_i'(p_i) = \frac{H_i'(p_i)}{H_i(p_i)} - \sum_{j \neq i} \frac{1}{1-p_j} \frac{dp_j}{dp_i}. \tag{A.30}$$

Using (A.29), it follows that

$$\frac{1}{1-p_j} \frac{dp_j}{dp_i} = \left(\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i} \right) \left[(1-p_j)\big(\frac{H_j'(p_j)}{H_j(p_j)} + \frac{1}{1-p_j} \big) \right]^{-1}. \tag{A.31}$$

Thus,

$$g_i'(p_i) = \frac{H_i'(p_i)}{H_i(p_i)} - \left(\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i} \right) \sum_{j \neq i} v_j(p_j), \tag{A.32}$$

where $v_j(p_j) \triangleq \frac{H_j(p_j)}{(1-p_j)H_j'(p_j) + H_j(p_j)}$.

Step 3: The function $g_i(p_i)$ increases if and only if $g_i'(p_i) > 0$. Equivalently,

$$\sum_{j \neq i} v_j(p_j) < \frac{\frac{H_i'(p_i)}{H_i(p_i)}}{\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}} \tag{A.33}$$

$$= 1 - \frac{\frac{1}{1-p_i}}{\frac{H_i'(p_i)}{H_i(p_i)} + \frac{1}{1-p_i}} = 1 - \frac{H_i(p_i)}{(1-p_i)H_i'(p_i) + H_i(p_i)} = 1 - v_i(p_i). \tag{A.34}$$

To summarize, $g_i(p_i)$ increases at p_i if and only if

$$\sum_{j \in \mathcal{I}} v_j(p_j) < 1. \tag{A.35}$$

Similarly g_i decreases if and only if $\sum_{j \in \mathcal{I}} v_j(p_j) > 1$.

Since H_i is concave increasing (Lemma 3), it may be verified that v_i is strictly increasing in p_i. This can be seen by noting that

$$\frac{1}{v_i(p_i)} = 1 + \frac{H_i'(p_i)(1-p_i)}{H_i(p_i)} \tag{A.36}$$

strictly decreases with p_i (by Lemma 3). Consequently, since p_j strictly increases with p_i due to (A.22), then $\sum_{j \in \mathcal{I}} v_j(p_j)$ strictly increases with p_i. Note further that if $p_i = 0$ then

$\sum_{j\in\mathcal{I}} v_j(p_j) = 0$; additionally, for $p_i = 1$, $p_j = 1$ for every j, hence $\sum_{j\in\mathcal{I}} v_j(p_j) = n$. Hence, the function $g_i(p_i)$ strictly increases up to some value $p_i^* \in (0,1)$ and then decreases.

Step 4: Due to the above and by the continuity of $f_i(p_i, \hat{\rho})$, the function $g_i(p_i)$ is a unimodal function. Hence $f_i(p_i, \hat{\rho})$ is unimodal and the result follows. □

Proof of Theorem 5: We adopt in this proof the notations used for the proof of Theorem 3. Consider a reference user i. Note that an equilibrium point is obtained as a solution to the equation $g_i(p_i) = \log \rho_i$. By the unimodality of $g_i(p_i)$ (see proof of Theorem 3), it follows that the better equilibrium point is obtained at some $p_i \in [0,1]$ for which $g_i(p_i)$ is increasing. Hence,

$$\sum_{j\in\mathcal{I}} v_j(p_j) \leq 1, \tag{A.37}$$

by (A.35) (equality holds when the two equilibria coincide). The proof will be completed by showing that $v_j(p_j) \geq p_j$ for all $p_j \in [0,1]$ and each j. For convenience, we omit the user index j in the sequel. To see the latter, note that by the concavity of H (Lemma 3) it follows that $H(0) \leq H(p) + H'(p)(0-p)$, using the gradient inequality ([24], p. 69), where H' is defined in (A.27). Hence $H(p) \geq pH'(p)$, or equivalently $H(p)(1-p) \geq pH'(p)(1-p)$. Thus, $H(p) \geq pH'(p)(1-p) + pH(p) = p\big[H(p) + H'(p)(1-p)\big]$, or $v(p) \equiv \frac{H(p)}{(1-p)H'(p)+H(p)} \geq p$. The result of the theorem is now established by summing the last inequality on all users, combined with (A.37). □

B Proof of Theorem 9

For the proof, we require the following lemma, which relates the user probabilities in equilibrium.

Lemma 6 *In every equilibrium point of null-CSI users, the following relation holds for every $i, j \in \mathcal{I}$.*

$$p_j = \frac{a_{ji} p_i}{1 - p_i + a_{ji} p_i}, \tag{B.38}$$

where $a_{ji} \triangleq y_j / y_i$.

Proof: Immediate by dividing the equilibrium equation of the ith user by the equation of the jth one. □

The idea behind the proof to fix p_i and to obtain an expression for the total normalized throughput by using (B.38). It can be shown that the minimal total normalized throughput is obtained when $a_{ji} = 1$ for every $j \in \mathcal{I}$, which is essentially the symmetric case. Since the above holds for every p_i, the maximal total normalized throughput is the lowest at the symmetric case.

Define $y_i^{\max}$ as the maximal normalized throughput which can be obtained by user i given the ratios $\{a_{ji}\}$. Noting (B.38),

$$y_i^{\max} \triangleq \max_{p_i \in [0,1]} p_i \prod_{j \neq i} \left(\frac{1-p_i}{1-p_i+a_{ji}p_i} \right). \tag{B.39}$$

The maximal normalized throughput for $\{a_{ji}\}$ is given by

$$(1 + \sum_{j \neq i} a_{ji}) y_i^{\max}. \tag{B.40}$$

Let $a \triangleq \frac{1}{n-1} \sum_{j \neq i} a_{ji}$ denote the mean of the sequence. We next show that for every *fix* $p_i \in [0,1]$ the following inequality holds.

$$p_i \prod_{j \neq i} \left(\frac{1-p_i}{1-p_i+a_{ji}p_i} \right) \geq p_i \left(\frac{1-p_i}{1-p_i+ap_i} \right)^{n-1}. \tag{B.41}$$

Define $b_j \triangleq 1 - p_i + a_{ji}p_i$. Noting that the nominators in (B.41) are all the same, we need to show that $\left(\prod_{j \neq i} b_j\right)^{\frac{1}{n-1}} \leq 1 - p_i + ap_i$. Since $1 - p_i + ap_i = \frac{\sum_{j \neq i}(1-p_i+a_{ji}p_i)}{n-1} = \frac{\sum_{j \neq i} b_j}{n-1}$, inequality (B.41) immediately follows (arithmetic average is greater or equal than geometric average). The proof of the theorem proceeds through the following two steps:

Step 1: *The maximal normalized throughput decreases when $\{a_{ji}\}$ are replaced with their mean.* Note first that the quantity $(1 + \sum_{j \neq i} a_{ji})$ in (B.40) remains constant when $\{a_{ji}\}$ are replaced with their mean. Thus, we only need to show that $y_i^{\max}$ decreases. This immediately follows since (B.41) is valid for every p_i (and thus the maximal value before the transformation is greater or equal than the maximal value after the transformation).

Step 2: Without loss of generality, let i be the user with the smallest normalized throughput demand ($y_j \geq y_i$, $j \neq i$). Assume $a_{ji} = a \geq 1$ are identical for every $j \neq i$. Fixing p_i, We next show that the normalized throughput strictly increases with a. This, combined with Step 1, would clearly indicate that the minimal normalized throughput is obtained when $a_{ji} = a = 1$. Applying (B.38), the total normalized throughput is given by

$$\begin{aligned} & p_i \left(\frac{1-p_i}{1-p_i+ap_i} \right)^{n-1} + (n-1) \frac{ap_i}{1-p_i+ap_i} (1-p_i) \\ & \times \left(\frac{1-p_i}{1-p_i+ap_i} \right)^{n-2} = p_i (1-p_i)^{n-1} \frac{1+(n-1)a}{(1-p_i+ap_i)^{n-1}}. \end{aligned} \tag{B.42}$$

Note that the denominator $(1 - p_i + ap_i)^{n-1}$ is positive. Differentiation of (B.42) w.r.t. a yields (after arranging terms) the following expression for the nominator

$$p_i(1 - p_i)^{n-1}(1 - p_i + ap_i)^{n-2}(n - 1)[1 - p_i + ap_i - p_i], \tag{B.43}$$

which is strictly positive for $a \geq 1$ and $p_i \in [0, 1)$ (when $p_i = 1$ all the obtained throughputs are zero, thus this value may be excluded from the analysis). This establishes the required result. $\square$

C Proofs for Best-Response Dynamics

For the proofs in this section we use the following notations. Let the update time-slots of each user i be given by an increasing sequence $\{t_i^k\}$, $k = 1, 2, 3, \ldots$. Also, let $\{t^k\} = \big\{\{t_1^k\} \cup \{t_1^k\} \cup \ldots \{t_n^k\}\big\}$, $k = 1, 2, \ldots$. Note that at each t^k at least a single user updates its transmission probability. We shall use the notation p_i^k for the transmission probability of user i at time t^k (similarly, $\mathbf{p}^k$ is the transmission probability vector at time t^k), with the convention of $p_i^0 = 0$ for every user i. The content of this section is as follows. We start with the proofs Section 6. We then complete the proofs of several results which required the notion of best-response dynamics.

Proof of Theorem 10: For the proof of the theorem we require the next lemma.

Lemma 7 *The sequence $\mathbf{p}^k$ is increasing.*

Proof: The result follows by induction. Obviously, $\mathbf{0} = \mathbf{p}^0 \leq \mathbf{p}^1$. Assume that $\mathbf{p}^0 \leq \mathbf{p}^1 \leq \ldots \mathbf{p}^{k-1}$. We next show that $\mathbf{p}^{k-1} \leq \mathbf{p}^k$. Denote by I^k the set of users who update their probabilities at time k (so that $p_i^{k-1} = p_i^k \; \forall i \notin I^k$). For each $i \in I^k$, let $k_i < k$ be the last time epoch at which user i updated its probability. Note that

$$r_i(p_i^{k-1}, \mathbf{p}_{-i}^{k_i - 1}) = r_i(p_i^k, \mathbf{p}_{-i}^{k-1}) = \rho_i, \tag{C.44}$$

as the best response probability (6.19) is such the the throughput demand is met with equality.

Since $r_i(\mathbf{p}) = p_i \prod_{j \neq i}(1 - p_j)$ is decreasing in $\mathbf{p}_{-i}$ and, by assumption, $\mathbf{p}_{-i}^{k_i-1} \leq \mathbf{p}_{-i}^{k-1}$, it follows that $p_i^{k-1} \leq p_i^k$ (as r_i is increasing in p_i). $\square$

It follows from the above lemma that either some component of $\mathbf{p}$ must exceed 1 at some iteration, or else $\mathbf{p}$ approaches a limit, say $\mathbf{p}^*$, and in this limit the equilibrium equations (3.8) are obviously satisfied (by continuity), i.e., it is an equilibrium point.

To conclude the proof, we now turn to show that if $\tilde{\mathbf{p}}$ is (another) equilibrium point, then $\mathbf{p}^* \leq \tilde{\mathbf{p}}$. To see this, we apply a similar induction as that of Lemma 7, and also use the notations thereof. Obviously $\mathbf{0} = \mathbf{p}^0 \leq \tilde{\mathbf{p}}$. Assume $\mathbf{p}^0 \leq \mathbf{p}^1 \leq \cdots \leq \mathbf{p}^{k-1} \leq \tilde{\mathbf{p}}$. Noting that $r_i(p_i^k, \mathbf{p}_{-i}^{k_i-1}) = r_i(\tilde{p}_i, \tilde{\mathbf{p}}_{-i}) = \rho_i$ and $\mathbf{p}_{-i}^{k_i-1} \leq \tilde{\mathbf{p}}_{-i}$ for every $i \in \mathcal{I}$, it follows that $p_i^k \leq \tilde{p}_i$, i.e., $\mathbf{p}^k \leq \tilde{\mathbf{p}}$. This argument also shows that if some component of $\mathbf{p}^k$ exceeds 1 for some k, then there is no equilibrium point (i.e., the set of required user throughputs $\{\rho_i\}$ is infeasible). □

Proof of Theorem 11: Convergence of case (a) (joining users) follows directly from the convergence property of the mechanism itself (Theorem 10), as joining users can be regarded as users who have been present at the network, yet decide to update their probabilities at late times.

For case (b), we will restrict ourselves to a *single* departure, for the simplicity of exposition. Results for multiple departures are obtained through the same arguments. We start our analysis with a lemma which compares the energy efficient equilibria for two throughput vectors, such that one is greater than the other.

Lemma 8 *Let ρ and $\tilde{\rho}$ be two throughput demand vectors such that $\tilde{\rho} \geq \rho$ (component-wise), and let $\mathbf{p}$ and $\tilde{\mathbf{p}}$ denote the respective EEEs. Then $\tilde{\mathbf{p}} \geq \mathbf{p}$. Consequently, fixing some ρ, the EEE transmission probabilities are lower (component-wise) with $n-1$ users present, in comparison to the EEE transmission probabilities with n users present.*

Proof: For the proof, we track the best response dynamics with parallel updates (where t_i^k does not depend on i), which are guaranteed to converge to an equilibrium point by Theorem 10. We next show that $\tilde{\mathbf{p}}^{\mathbf{k}} \geq \mathbf{p}^k$ for every k, thus also at the limit. Note that since $r_i(\tilde{p}_i^1, \mathbf{0}) = \tilde{\rho}_i \geq r_i(p_i^1, \mathbf{0}) = \rho_i$, then by the monotonicity of r_i, $\tilde{p}_i^1 \geq p_i^1$ for every i. At the next iteration, $r_i(\tilde{p}_i^2, \tilde{\mathbf{p}}_{-i}^1) = \tilde{\rho}_i \geq r_i(p_i^1, \mathbf{p}_{-i}^1) = \rho_i$. Since $\tilde{\mathbf{p}}_{-i}^1 \geq \mathbf{p}_{-i}^1$, it follows that $\tilde{p}_i^2 \geq p_i^2$ for every i. The same argument carries over to subsequent iteration, thus it is valid also at the limit. The case of $(n-1)$ users is obtained as a special case of the above, by setting $\rho_n = 0$. □

We are now ready to prove convergence for the case of a leaving user. The impact of an abandoning user (say the nth one) is equivalent to setting $p_n = 0$. Let $\hat{\mathbf{p}}$ denote the initial probability vector, representing the EEE when n users were present and let $\mathbf{p}^0$ denote the same vector, except that $p_n = 0$. For the proof of the convergence properties, we require the next two lemmas.

Lemma 9 *In case of an abandoning user, the sequence $\mathbf{p}^k$ is decreasing.*

Proof: Denote by I^1 the subset of users who update their probabilities at $k = 1$. For every $i \in I^1$, since $r_i(p_i^1, \mathbf{p}^0_{-i}) = r_i(p_i^0, \hat{\mathbf{p}}_{-i}) = \rho_i$, it follows by the monotonicity of r_i that $p_i^1 \le p_i^0$ Thus overall, $\mathbf{p}^1 \le \mathbf{p}^0$. The result of the lemma follows by proceeding similarly in subsequent iterations (see a similar proof idea in Theorem 10). □

Lemma 10 *The sequence* $\mathbf{p}^k$ *is bounded below by the EEE of the* $n-1$ *users.*

Proof: Denote by $\mathbf{p}^*$ the EEE with $n-1$ users. Then by Lemma 8 $\mathbf{p}^0 \ge \mathbf{p}^*$. Denote by I^1 the subset of users who update their probabilities at $k = 1$. For these users we have $r_i(p_i^1, \mathbf{p}^0_{-i} = r_i(p_i^*, \mathbf{p}^*{}_{-i})$. Since $\mathbf{p}^0_{-i} \ge \mathbf{p}^*{}_{-i}$ it follows that $p_i^1 \ge p_i^*$ for every i. This argument may be carried over to subsequent iterations ($\mathbf{p}^k_{-i} \ge \mathbf{p}^*{}_{-i}$ for every k) and the result follows. □

An immediate consequence of the last two lemmas is that the mechanism reobtains an equilibrium in the case that a user leaves the network. We now show that the mechanism converges to the EEE: We reuse some of the implications along the proof of Theorem 3. Using the notations within that proof, we observed that the following condition is valid at the EEE:

$$\sum_j v_j(p_j) < 1. \tag{C.45}$$

This is true, in particular, for the equilibrium point with n users. Since the sequence $\mathbf{p}^k$ decreases due to the abandonment of a single user and moreover since v_j is an increasing function, it follows that $\sum_{j=1}^{n-1} v_j(p_j^k) < 1$ for every k. Accordingly, the convergence of the sequence (guaranteed by the above two lemmas) must be to the EEE of the $(n-1)$ users. □

We now return to prove a couple of results which required the notion of best-response dynamics.

Proof of Proposition 2(iii): Since $\tilde{\rho}$ admits an equilibrium, the EEE probabilities obviously obey $\tilde{\mathbf{p}} \le 1$. It follows by Lemma 8 that the respective EEEs are such that $\tilde{\mathbf{p}} \ge \mathbf{p}$. Thus $\mathbf{p} \le 1$. □

Proof of Theorem 7: (i) Let H_i and $\tilde{H}_i$ be the collision-free rate functions for $\mathcal{Z}_i$ and $\tilde{\mathcal{Z}}_i$ respectively. Then a feasible throughput demand vector ρ for game G is such that the equilibrium equations (3.8) are satisfied, i.e., $H_i(p_i) \prod_{j \ne i}(1 - p_j) = \rho_i$ for every user i. Using the same probability vector $\mathbf{p} = (p_1, \ldots, p_n)$ with $\{\tilde{\mathcal{Z}}_i\}_{i \in \mathcal{I}}$, it follows by Lemma 4 that $\tilde{H}_i(p_i) \prod_{j \ne i}(1 - p_j) \stackrel{\triangle}{=} \tilde{\rho}_i \ge \rho_i$. Thus, a throughput demand vector $\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_n) \ge \rho$ is achievable under $\{\tilde{\mathcal{Z}}_i\}_{i \in \mathcal{I}}$. The result then immediately follows by Proposition 2(iii).

(ii) The proof is based on simultaneously tracking the best response dynamics under both $\{H_i\}$ and $\{\tilde{H}_i\}$. Specifically, we show that under synchronized best response dynamics (where all users update their probabilities at the same time-slots), the transmission probabilities are such that $\tilde{p}_i^k \leq p_i^k$ for every iteration k. Note that at the first iteration each user solves $H_i(p_i^1) = \rho_i$ (or $\tilde{H}_i(\tilde{p}_i^1) = \rho_i$). Then by Lemma 4, $\tilde{p}_i^1 \leq p_i^1$. At the next iteration, $\tilde{H}_i(\tilde{p}_i^2) = \frac{\rho_i}{\prod_{j \neq i}(1-\tilde{p}_j^1)} \leq \frac{\rho_i}{\prod_{j \neq i}(1-p_j^1)} = H_i(p_i^2)$, hence $\tilde{p}_i^2 \leq p_i^2$ by the same lemma. This inequality carries over to each $k \geq 1$, and by continuity holds in the limit as well, namely at the respective equilibrium points.

(iii) For lucidity, we shall consider the case where the CSI signal of only a single user i is active and refined, i.e., $H_i(p_i) < \tilde{H}_i(p_i)$. Consider the same iterative procedure of (ii) above. Then, noting (2.6) and by continuity, there exists some K for which the inequality $\tilde{p}_i^K < p_i^K$ becomes strict. Hence, it directly follows from the update rule (6.19) that $\tilde{p}_j^k < p_j^k$ for every user $j \in \mathcal{I}$ and $k > K$. $\square$