Incentive-Compatible Online Auctions for Digital Goods

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Abstract

Goldberg et al. [6] recently began the study of incentive-compatible auctions for digital goods, that is, goods which are available in unlimited supply. Many digital goods, however, such as books, music, and software, are sold continuously, rather than in a single round, as is the case for traditional auctions. Hence, it is important to consider what happens in the *online* version of such auctions. We define a model for online auctions for digital goods, and within this model, we examine auctions in which bidders have an incentive to bid their true valuations, that is, incentive-compatible auctions. Since the best offline auctions achieve revenue comparable to the revenue of the optimal fixed pricing scheme, we use the latter as our benchmark. We show that deterministic auctions perform poorly relative to this benchmark, but we give a randomized auction which is within a factor $O(\exp(\sqrt{\log\log h}))$ of the benchmark, where h is the ratio between the highest and lowest bids. As part of this result, we also give a new offline auction, which improves upon the previously best auction in a certain class of auctions for digital goods. We also give lower bounds for both randomized and deterministic online auctions for digital goods.

1 Introduction

Traditionally, auctions are useful mechanisms for allocating resources because they achieve a certain kind of economic efficiency: we would like to transfer resources to those who can make the most use of these resources. Auctions attempt to achieve this by giving bidders the opportunity to declare how much they are willing to pay. In a well-designed auction, these bids should be an indication of the value the bidders place on the resources, and these valuations are in turn taken to be a measure of the usefulness to the bidders. Of course, from the seller's perspective, what matters is not the relative usefulness of the item for sale, but rather the revenue generated by the sale. In general, it may not be possible to simultaneously maximize efficiency and revenue, but in many natural situations, the seller can extract a reasonable revenue while maximizing efficiency. (For example, see Ronen [12] on the single item case.) For more on the economic theory of auctions, see Klemperer [7].

Increasingly, however, we are interested in distributing not physical goods, but digital, or information, goods. This includes, for example, electronic books, software, and digital copies of music or movies. The key feature of such goods is that they can be duplicated with negligible marginal cost, and hence, we can think of them as being available in unlimited supply. With scarcity no longer an issue, there is no question of allocation: the economically efficient thing to do is to give a copy to anyone who asks. This course of action, though, would be unpalatable to the seller, who presumably incurred fixed costs in producing the initial copy of the digital good and would like to make a profit. Hence, we return to auctions, but this time, we use valuation information not to determine how to allocate goods per se, but to determine how to charge for them, with our only goal being to maximize revenue.

The study of revenue-maximizing auctions for goods available in unlimited supply was begun by Goldberg et al. [6], part of recent interest in using computational techniques to analyze economic problems such as auction design [10, 9, 3]. As with most work on designing auctions, they focus on auctions which are *incentive-compatible*, that is, auctions in which bidders always do best by placing bids which equal their valuations. Restricting to such auctions seems reasonable on a number of fronts (see [14, 13]): it relieves bidders of the need to make assumptions about the behavior of other bidders, and it gives the seller the most information about the bidders. In what follows, we too will be examining only incentive-compatible auctions.

In the model of Goldberg et al., they assume that bids are presented to the auction mechanism as a set, and that the mechanism makes all pricing decisions at once, after all bids have been received. Such an assumption might be reasonable for selling a digital broadcast, for instance. In this case, the item is transmitted all at once to the buyers, and hence it might be reasonable to have a cutoff time for the bidding. However, most digital goods, for instance, books, music, or software, are sold continuously, without any point at which sales end.

Furthermore, since digital goods can be delivered

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immediately, buyers would presumably expect a fairly rapid, if not immediate, response to a bid for digital goods. Even in the case of a digital broadcast, where the delivery is not immediate, buyers want to know quickly whether or not their offers have been accepted. In auctioning a pay-per-view broadcast, for instance, we might want to allow bids to come in mere minutes before the broadcast, or maybe even after the broadcast has begun. It would be clearly unacceptable to delay all decisions about who receives the broadcast until such a point.

What this suggests is that, particularly for the case of digital goods, it is important to design auctions which work in an online model. That is, we would like incentive-compatible auction mechanisms which decide whether to satisfy a bid and at what price *before the next bid arrives*. As is usual in the study of online algorithms, we want our auction to perform well relative to the best offline algorithm, on any possible sequence of bids.

In this setting, some care needs to be taken in defining what we mean by the best offline algorithm. One possible view is that the offline algorithm also makes decisions one-by-one, but that it knows the sequence in advance. Note that since we are dealing with incentive-compatible auctions, knowing the sequence of bids is the same as knowing the sequence of valuations. Hence, the offline algorithm can simply offer the item to each bidder for a price which is exactly equal to that bidder's valuation. Such an algorithm achieves a revenue equal to the sum of the valuations, $\mathcal{T}$, which is the maximum possible under standard assumptions.

It may seem unfair, however, to allow the offline algorithm to know the valuations, since this defeats the purpose of restricting to incentive-compatible auctions. Instead, we can view the offline algorithm as receiving the entire sequence before deciding who to sell to and at what price. Restricting the offline algorithm to be incentive-compatible yields the model of Goldberg et al. [6]. They give a randomized algorithm which achieves revenue at least a constant fraction of the optimal fixed price revenue, $\mathcal{F}$ (the maximum revenue one could achieve by setting a single price for all bidders). Hence, we are primarily interested in how well our online algorithms do relative to $\mathcal{F}$.

As in the offline case, deterministic algorithms do poorly, so we concentrate on randomized algorithms. The prior work of Goldberg et al. [6, 4] suggests that random sampling techniques work well in the design of auctions which are competitive relative to the optimal fixed price revenue. However, in our setting, the requirement that we make decisions one by one makes it difficult to apply sampling techniques; instead, we look for randomized functions which we can apply retrospectively to the sequence of bids received so far. Previously, the best such function yielded an *offline* auction with competitive ratio $O(\sqrt{\log h})$ relative to $\mathcal{F}$, where h is the ratio between the highest and lowest bids. We present an improved offline auction, which achieves a constant competitive ratio, under slightly stronger assumptions, or a competitive ratio of $O(\exp(\sqrt{\log\log h}))$ under standard assumptions. Note that this competitive ratio is $o(log^{\epsilon} h)$, for every constant $\epsilon > 0$. We then use this as the basis for an online auction which has competitive ratio $O(\exp(\sqrt{\log\log h}))$. (Note that the constants hidden by the asymptotic notation are all reasonable; the largest is 30.) We also give a lower bound on the competitive ratio of any incentive-compatible online auction of $\Omega(1)$ relative to $\mathcal{F}$.

Lavi and Nisan [8] previously considered online auctions for goods in limited supply. We note that their results do not directly apply to our setting, for a number of reasons. Most importantly, when there is a limited supply, a competitive auction needs to balance the revenue gained by a current sale with the revenue lost from not being able to sell the same item to a later bidder. This is reflected in nondecreasing prices for many natural situations, which is not the case when the supply is unlimited. Furthermore, Lavi and Nisan use as a benchmark the Vickrey auction [14], which Goldberg et al. [6] show is not competitive in the unlimited supply case. Bagchi et al. [1] also consider the limited supply case, though in a model where incentive- compatibility is not required.

In what follows, we begin in Section 2 by describing our model. In Section 3, we show that any deterministic auction performs poorly in our model. In Section 4, we give an extremely simple algorithm and show that it performs optimally relative to $\mathcal{T}$. Since this algorithm is almost completely independent of the bids, this result suggests that $\mathcal{T}$ is a less useful benchmark. In the remainder of Section 4, we give our improved offline auction, show its competitiveness relative to $\mathcal{F}$, and show how to use it in the online setting.

2 Preliminaries

2.1 The Model

We consider auctions for goods available in unlimited supply. In particular, the auctioneer can sell the good to each one of the bidders. We assume each bidder can win at most one item of the good.

Let n be the number of bidders. Each bidder i has a *valuation* v_i for the auctioned good, which is known only to himself and stays fixed throughout the auction. By normalizing the valuations, we can assume that each v_i is a real number in the interval $[1, h]$ (we do not permit bidders to have null valuations). Notice that h is the ratio between the highest and lowest valuations. The *utility* of bidder i, u_i, then is $v_i - p_i$ if the bidder wins the good at price p_i, and 0 if he does not win the good. The goal of each bidder is to maximize his utility.

An *online auction* receives bids from the n bidders one by one in a sequence, $\overline{b} = b_1, \ldots, b_n$. The auctioneer, when presented with the bid b_i of bidder i, determines whether to sell the good to bidder i and if so, at what price. This determination must be made *before any further bids are presented*. In order to ensure that the bidders have an incentive to participate in the auction, we assume that the price p_i charged to bidder i is never more than b_i.

An online auction A can then be viewed as a collection of n pricing functions $s_1, \ldots, s_n$, where $s_i(b_1, \ldots, b_i)$ is a function of the bids received up to step i. The interpretation of these functions is that s_i sets a "selling price" for bidder i. If $s_i \leq b_i$, bidder i wins the good and pays s_i; otherwise, bidder i does not win the good and pays nothing. The functions s_i may be deterministic or randomized, and if randomized, they may be correlated. The *total revenue* of an auction, $R_A(\overline{b})$, is the sum of the payments received. When A is randomized, the total revenue is a random variable, and thus we speak about the *expected total revenue*. The goal of the auctioneer is to maximize the (expected) total revenue.

A bidder, in general, has access to all prior bids and is allowed to apply an arbitrary bidding strategy to determine his bid. This strategy is a (possibly randomized) function of the bidder's valuation, of the bids seen so far, and of the responses to those bids. A *truthful* strategy is one that always bids the true valuation: $b_i = v_i$. We concentrate on *incentive-compatible* auctions, that is, auctions for which the truthful strategy is a dominant one, for every bidder. In other words, given the previous bids and previous responses, u_i is maximized by setting $b_i = v_i$. For randomized auctions, we use a strong definition of incentive-compatibility and require that u_i be maximized at $b_i = v_i$ for every fixed set of random choices. Notice that for incentive-compatible auctions, we can view s_i and R_A as functions of either the bids or the valuations, interchangeably.

An auction is called *bid-independent* if the pricing function for bidder i depends only on the previous bids, and not on b_i. That is, for any sequence of bids, $b_1, \ldots, b_{i-1}$, and for any two choices for the ith bid, b_i and b'_i, $s_i(b_1, \ldots, b_{i-1}, b_i) = s_i(b_1, \ldots, b_{i-1}, b'_i)$. (For a randomized A, we again fix the random choices.) Note that our definition of bid-independence is more general than the one used by Goldberg et al. They call a randomized auction bid-independent when the pricing functions use independent random coins, and hence they distinguish between bid-independent auctions and random sampling auctions, which can be thought of as using bid-independent pricing functions with access to common random coins.

PROPOSITION 2.1. *An online auction is incentive-compatible if and only if it is bid-independent.*

To see that bid-independent auctions are incentive-compatible, note that given a fixed set of random choices, s_i has some fixed value just before bidder i arrives (although, of course, bidder i may only know that s_i has been drawn from some particular distribution). Now if bidder i bids $b_i < v_i$, then if $b_i < s_i < v_i$, the bidder loses, when she could have had positive utility by bidding v_i. Similarly, if $b_i > v_i$, then if $b_i > s_i > v_i$, the bidder has negative utility, when she could have had zero utility. For other values of s_i, the utility is unaffected, and hence, utility is maximized at $b_i = v_i$.

The converse claim that any incentive-compatible auction is bid-independent is given by Goldberg et al. [6] for the deterministic offline case. Their proof extends naturally to the online setting, and because we require conditions to hold for every fixed set of random choices, their proof also holds in the randomized online case.

2.2 Competitive Analysis

Following previous work [8, 6], we adopt worst-case *competitive analysis* from the study of online algorithms (see [2]) as our means of evaluating the performance of online auctions. We compare the revenue of the online auction on the "worst" sequence of valuations $\overline{v} = v_1, \ldots, v_n$ against a "benchmark". For the case of unlimited supply, Goldberg et al. [6] propose two possible benchmarks:

1. Total revenue: $\mathcal{T}(\overline{v}) = \sum_{i=1}^{n} v_i$; this is the maximum revenue any auction can hope to achieve from the given set of bidders.
2. Optimal fixed price revenue: $\mathcal{F}(\overline{v}) = \max_{i \in [n]} \{v_i n_i\}$, where $n_i = |\{j \in [n] \mid v_j \geq v_i\}|$; this is the maximum revenue that can be achieved using a single fixed price for all bidders.

An incentive-compatible online auction A is said to be *c-competitive* relative to a benchmark B, if for any sequence of valuations $\overline{v}$, $R_A(\overline{v}) \geq B(\overline{v})/c$ (or $E(R_A(\overline{v})) \geq B(\overline{v})/c$ for a randomized A). We call c the *competitive ratio* of A relative to B.

As previously pointed out [6], no auction does well when there is a single high bidder. Hence, it is natural to restrict our attention to valuation sequences for which $\mathcal{F} \geq \alpha h$, for some $\alpha \geq 1$. Most of our upper bounds on competitive ratios apply only with such a restriction. All of our lower bounds apply even when so restricted.

2.3 Valuation Buckets

A notion that will play an important role in the auctions we present in the sequel is the partitioning of the valuation range $[1, h]$ into $\ell = \lfloor \log h \rfloor + 1$ sub-intervals, $I_0, \ldots, I_{\ell-1}$, where $I_k = [2^k, 2^{k+1})$. Given a valuation sequence $\overline{v}$, we define the *kth bucket* of $\overline{v}$ to be the collection of valuations that fall in the kth interval: $B_k(\overline{v}) \stackrel{\text{def}}{=} \{i \in [n] \mid v_i \in I_k\}$. The *weight* of the kth bucket is the sum of the valuations in the bucket: $w_k(\overline{v}) \stackrel{\text{def}}{=} \sum_{i \in B_k(\overline{v})} v_i$. We omit the dependence on $\overline{v}$ when it is clear from context.

3 Deterministic Auctions

We first show that no deterministic online auction is competitive relative to the optimal fixed price revenue. Note that the proof of this fact given by Goldberg et al. [6] for offline auctions does not carry over into our setting. Their proof assumes that a single, symmetric bid-independent function is used for all bidders i, whereas in the online setting, each function $s_i(b_1, \ldots, b_{i-1})$ takes a different number of input values.

THEOREM 3.1. *Any deterministic incentive-compatible online auction A is $\Omega(h)$-competitive relative to the optimal fixed price revenue, even when restricting to valuations $\overline{v}$ with $\mathcal{F}(\overline{v}) \geq \alpha h$ for any constant $\alpha \geq 1$.*

Proof. Let A be any deterministic incentive-compatible online auction; by Proposition 2.1, A is bid-independent. Therefore, for all i, the price for bidder i depends only on $b_1, \ldots, b_{i-1}$.

We will construct a valuation sequence $\overline{v}$, for which $R_A(\overline{v}) \leq \mathcal{F}(\overline{v})/h$. $\overline{v}$ will be a *bipolar sequence*, that is, consisting of valuations only in $\{1, h\}$. Note that for such a sequence, we can assume without loss of generality that A sets prices only in $\{1, h\}$.

Fix any constant $\alpha \geq 1$. We choose the valuation sequence $\overline{v}$ as follows: if $s_i(b_1, \ldots, b_{i-1}) = 1$, pick $v_i = h$, and if $s_i(b_1, \ldots, b_{i-1}) = h$, pick $v_i = 1$. We stop when either the number of 1-valuations in $\overline{v}$ exceeds αh, or the number of h-valuations exceeds α. Note that the revenue of A from bidder i is 0 if $v_i = 1$, and 1 if $v_i = h$. Therefore, if we denote by n_h the number of h-valuations in $\overline{v}$ and by n_ℓ the number of 1-valuations in $\overline{v}$, we have that $R_A(\overline{v}) = n_h \cdot 1 + n_\ell \cdot 0 = n_h$. On the other hand, $\mathcal{F}(\overline{v}) \geq \max\{hn_h, n_\ell\}$, implying that $R_A(\overline{v}) \leq \mathcal{F}(\overline{v})/h$, with $\mathcal{F}(\overline{v}) \geq \alpha h$.

Note that a competitive ratio of h is trivially achievable by an auction which sells to every bidder at price 1.

4 Randomized Auctions

4.1 Lower Bounds

We first give lower bounds on the competitive ratio of any randomized incentive-compatible online auction.

THEOREM 4.1. *Any (randomized) incentive-compatible online auction A is $\Omega(\log h)$-competitive relative to the total revenue, even when restricting to valuations with $\mathcal{F}(\overline{v}) \geq \alpha h$ for any constant $\alpha \geq 1$.*

Proof. Let A be any incentive-compatible online auction; by Proposition 2.1, A is bid-independent. Therefore, for all i, given bids $b_1, \ldots, b_{i-1}$, $s_i(b_1, \ldots, b_{i-1})$ generates some fixed distribution just before bidder i arrives.

We construct a valuation sequence as follows: at step i, based on the distribution for s_i, for each $k \in \{0, \ldots, \lfloor \log h \rfloor\}$, compute the expected revenue from a bidder i with valuation $v_i = 2^k$. (This is $r_{ik} = \int_1^{2^k} \Pr[s_i = x] \cdot x \, dx$.) Now set $v_i = 2^k$ for the k which minimizes the ratio $r_{ik}/2^k$. Continue this until $\mathcal{F}$ is sufficiently large.

For the analysis, consider the intervals $I_k = [2^k, 2^{k+1})$, as defined above, and recall that $\ell = \lfloor \log h \rfloor + 1$ is the number of such intervals. Let $q_{ik} = \Pr[s_i \in I_k]$. Note that $r_{ik} \leq q_{ik} \cdot 2^k + \sum_{r=0}^{k-1} q_{ir} \cdot 2^{r+1}$, and hence, $r_{ik}/2^k \leq q_{ik} + \sum_{r=0}^{k-1} q_{ir}/2^{k-r-1}$. Then the ratios $r_{ik}/2^k$ sum to:

$$\begin{aligned}
\sum_{k=0}^{\ell-1} \frac{r_{ik}}{2^k} &\leq \sum_{k=0}^{\ell-1} \left(q_{ik} + \sum_{r=0}^{k-1} \frac{q_{ir}}{2^{k-r-1}} \right) \\
&= 1 + \sum_{r=0}^{\ell-1} q_{ir} \cdot \left(\sum_{k=r+1}^{\ell-1} \frac{1}{2^{k-r-1}} \right) \\
&\leq 1 + \sum_{r=0}^{\ell-1} q_{ir} \cdot 2 \ \leq \ 3
\end{aligned}$$

Thus, there must be some k such that $r_{ik}/2^k \leq 3/\ell$, and in particular, $v_i = 2^k$ must have the property that $r_{ik}/v_i \leq 3/\ell$. Summing over i, we get that $R_A(\overline{v}) \leq (3/\ell)\mathcal{T}(\overline{v})$, which gives the claimed competitive ratio.

Using the fact proven in [6] that $\mathcal{F} \geq \mathcal{T}/(2 \log h)$, we get the following corollary.

COROLLARY 4.1. *Any (randomized) incentive-compatible online auction A is $\Omega(1)$-competitive relative to the optimal fixed price revenue, even when restricting to valuations with $\mathcal{F}(\overline{v}) \geq \alpha h$ for any constant $\alpha \geq 1$.*

4.2 Simple Randomized Auction

We now give an extremely simple incentive-compatible randomized online auction that achieves an optimal competitive ratio of $O(\log h)$ relative to the total revenue.

To begin with, assume we know a priori the bound h on the range of valuations. Consider the following simple auction S: For each bidder i, pick a random $k \in \{0, \ldots, \lfloor \log h \rfloor\}$, and set $s_i = 2^k$. Clearly, S is bid-independent, and hence incentive-compatible.

THEOREM 4.2. *The auction S is $\Theta(\log h)$-competitive relative to the total revenue and relative to the optimal fixed price revenue.*

Proof. We show that S is $O(\log h)$-competitive relative to $\mathcal{T}$. This immediately implies that it is also $O(\log h)$-competitive relative to $\mathcal{F}$.

Consider any valuation sequence $\overline{v}$. For each valuation v_i, denote by k_i the maximal integer k for which $2^k \le v_i$. Note that $v_i/2 \le 2^{k_i} \le v_i$, and thus if S sets $s_i = 2^{k_i}$, it is guaranteed to earn a payment of at least $v_i/2$ from bidder i. Since S picks k uniformly at random, so $E(R_S(\overline{v})) \ge \sum_{i=1}^{n} \frac{v_i}{2} \cdot \Pr[s_i = 2^{k_i}]$, and thus

$$E(R_S(\overline{v})) \ge \sum_{i=1}^{n} \frac{v_i}{2}\left(\frac{1}{\log h + 1}\right) = \frac{1}{2(\log h+1)}\mathcal{T}(\overline{v})$$

We now show that S is $\Omega(\log h)$-competitive relative to $\mathcal{F}$. This implies that it is also $\Omega(\log h)$-competitive relative to $\mathcal{T}$. Fix some constant $\alpha \ge 1$, and let $\overline{v}$ be the valuation sequence consisting of $\lceil \alpha \rceil$ h's. Clearly, $\mathcal{F}(\overline{v}) \ge \alpha h$. However, the expected revenue of S from every bidder i is $O(h/\log h)$, and thus the total expected revenue is $O(\alpha h/\log h) \le \mathcal{F}(\overline{v})/\log h$.

If h is not known in advance, the auction S starts by assuming $h = 1$, and continuously updates h according to the maximum bid seen so far. Note that the modified S might miss additional revenue from at most one bid in each bucket. However, the sum of these bids cannot be more than $2h$, and thus, assuming $\mathcal{T}(\overline{v}) \ge 4h$, S can still extract an $O(\log h)$ fraction of $\mathcal{T}(\overline{v})$.

4.3 Offline Weighted Buckets Auction

In this section, we give the offline auction which forms the basis for the online auction to be described next. Our offline auction W_d uses the valuation buckets we previously defined, as well as a parameter $d \ge 1$, which we will tune later. The idea is that for the ith bidder, we construct the valuation buckets for $\overline{v}_{-i}$, which is the set of valuations of all bidders except the ith bidder. We then pick a bucket at random, with the probabilities weighted according to the weight of the buckets raised to the power d. The bucket we choose is essentially our guess for the next bid, and we set the selling price to be the lower limit of that bucket. More formally,

$$\Pr[s_i = 2^k] = \frac{(w_k(\overline{v}_{-i}))^d}{\sum_{r=0}^{\ell-1}(w_r(\overline{v}_{-i}))^d}$$

Note that W_d is bid-independent, and therefore incentive-compatible.

THEOREM 4.3. *Restricting to valuation sequences with $\mathcal{F}(\overline{v}) \ge 9\alpha h$, for any $\alpha \ge 2$, auction W_d with any $d \ge 1$ is $O(e^{2d/(\alpha+1)}(\log h)^{1/(d+1)})$-competitive relative to the optimal fixed price revenue.*

Proof. We begin by characterizing $\mathcal{F}$ in terms of the maximum weight bucket. Let $w_*(\overline{v}) = \max_k w_k(\overline{v})$.

LEMMA 4.1. $w_*(\overline{v})/2 \le \mathcal{F}(\overline{v}) \le 3w_*(\overline{v})$

Proof. Fix a valuation sequence $\overline{v}$. For the lower bound, let k_* be (any) bucket with maximum weight. The optimal fixed price revenue is at least the revenue obtained by setting the price at 2^{k_*}. Note that for any $i \in B_{k_*}$, $v_i/2 \le 2^{k_*} \le v_i$. Thus a fixed price of 2^{k_*} yields the following revenue:

$$\mathcal{F}(\overline{v}) \ \ge\ 2^{k_*}|B_{k_*}| \ \ge\ \sum_{i\in B_{k_*}} \frac{v_i}{2} \ =\ \frac{w_*(\overline{v})}{2}$$

For the upper bound, let p be the optimal fixed price, and let k be the integer such that $2^k \le p < 2^{k+1}$. In setting the price to be p, we obtain revenue from bids in the kth bucket, as well as from successive buckets. Note that for any $r > k$, $w_r \ge 2^r|B_r| \ge 2^{r-k-1}\cdot p\cdot |B_r|$. Therefore:

$$\begin{aligned}\mathcal{F}(\overline{v}) &\le w_k(\overline{v}) + p\cdot\sum_{r=k+1}^{\ell-1}|B_r| \ \le\ w_k(\overline{v}) + \sum_{r=k+1}^{\ell-1}\frac{w_r(\overline{v})}{2^{r-k-1}} \\ &\le w_*(\overline{v})\cdot\left(1+\sum_{r=k+1}^{\ell-1}\frac{1}{2^{r-k-1}}\right) \ \le\ 3w_*(\overline{v})\end{aligned}$$

Because of the lemma, we can equivalently show a competitive ratio relative to w_*. Fix some valuation sequence $\overline{v}$, for which $\mathcal{F}(\overline{v}) \ge 9\alpha h$. Consider any $v_i \in \overline{v}$. Let B_k be the bucket such that $i \in B_k$. The expected revenue from this ith bidder is at least $2^k \cdot \Pr[s_i = 2^k]$. Using the definition of the auction and summing over the bidders, we get that

$$\begin{aligned}E(R_{W_d}) &\ge \sum_{k=0}^{\ell-1}\sum_{i\in B_k} 2^k\cdot\frac{(w_k(\overline{v}_{-i}))^d}{\sum_{r=0}^{\ell-1}(w_r(\overline{v}_{-i}))^d} \\ &\ge \sum_{k=0}^{\ell-1}\sum_{i\in B_k}\frac{v_i}{2}\cdot\frac{(w_k(\overline{v})-v_i)^d}{\sum_{r=0}^{\ell-1}(w_r(\overline{v}))^d}\end{aligned}$$

Let K_1 be the set of buckets k for which $|B_k| \geq \alpha$, and let K_2 be the rest of the buckets. For any $i \in B_k$, where $k \in K_1$, $w_k(\overline{v}) - v_i \geq (1 - \frac{2}{\alpha+1})w_k(\overline{v})$. Hence,

$$E(R_{W_d}) \geq \frac{\sum_{k \in K_1} w_k(\overline{v})((1-\frac{2}{\alpha+1})w_k(\overline{v}))^d}{2 \cdot \sum_{r=0}^{\ell-1} (w_r(\overline{v}))^d}$$

Next, we argue that the weight contributed by the buckets in K_2 is no more than that contributed by the buckets in K_1. First, we lower bound the contribution from the buckets in K_1. Note that the maximum weight bucket must be in K_1, since by Lemma 4.1, $w_* \geq \mathcal{F}/3 \geq 3\alpha h$, so $\sum_{k \in K_1} (w_k(\overline{v}))^d \geq (w_*(\overline{v}))^d \geq (3\alpha h)^d$.

Next, we upper bound the contribution from the buckets in K_2. Note that if $i < \lfloor \log h \rfloor$ and bucket i is in K_2, then $w_i(\overline{v}) < 2^{i+1}\alpha \leq (h\alpha)/2^{\lfloor \log h \rfloor - i}$, so $(w_i(\overline{v}))^d < ((h\alpha)/2^{\lfloor \log h \rfloor - i})^d$. This means that $\sum_{k \in K_2} (w_k(\overline{v}))^d < (h\alpha)^d + \sum_{i=0}^{\ell-2} (h\alpha)^d 2^{-id} < 3(h\alpha)^d$. Combining the two bounds, we get that $\sum_{r=0}^{\ell-1} (w_r(\overline{v}))^d \leq \sum_{k \in K_1} (w_k(\overline{v}))^d$.

Also, let $u_k(\overline{v}) = w_k(\overline{v})/w_*(\overline{v})$. We now have

$$E(R_{W_d})/w_* \geq \frac{1}{4} \cdot \left(1 - \frac{2}{\alpha+1}\right)^d \cdot \frac{\sum_{k \in K_1} (u_k(\overline{v}))^{d+1}}{\sum_{k \in K_1} (u_k(\overline{v}))^d}$$

We then apply the following lemma, which is proven using Hölder's inequality.

LEMMA 4.2. *Let $u_1, \ldots, u_m$ be real values in the interval $[0,1]$, with at least one $u_i = 1$. Then, for any $d \geq 1$, $\frac{\sum_{i=1}^m u_i^d}{\sum_{i=1}^m u_i^{d+1}} \leq m^{1/(d+1)}$.*

Proof. If $\sum u_i^d \leq m^{1/(d+1)}$, then the proof follows immediately from the fact that at least one $u_i = 1$ and hence $\sum u_i^{d+1} \geq 1$.

Otherwise, $\sum u_i^d \geq m^{1/(d+1)}$, and we use Hölder's inequality [11, page 241], which states that for $p, q \geq 1$ with $p + q = pq$, and two vectors $\overline{x}, \overline{y}$, $|\overline{x} \cdot \overline{y}| \leq \|\overline{x}\|_p \|\overline{y}\|_q$. In our case, we take $x_i = u_i^d$, $y_i = 1$, $p = (d+1)/d$, and $q = (d+1)$ to get $\sum u_i^d \leq \left(\sum u_i^{d+1}\right)^{d/(d+1)} m^{1/(d+1)}$ Rearranging, we get

$$\sum u_i^{d+1} \geq \left(\sum u_i^d\right)^{(d+1)/d} (1/m)^{1/d}$$

Hence,

$$\frac{\sum u_i^d}{\sum u_i^{d+1}} \leq \left(\frac{m}{\sum u_i^d}\right)^{1/d} \leq \left(\frac{m}{m^{1/(d+1)}}\right)^{1/d} = m^{1/(d+1)}$$

Recall again that the maximum weight bucket must be in K_1, and so Lemma 4.2 applies, and together with a bound on $(1 - \frac{2}{\alpha+1})^d$, this completes the proof.

The appropriate choice of parameters yields the following corollaries.

COROLLARY 4.2. *Restricting to valuation sequences with $\mathcal{F}(\overline{v}) \geq 9h \log\log h$, W_d with $d = \log\log h$ is $O(1)$-competitive relative to the optimal fixed price revenue.*

COROLLARY 4.3. *Restricting to valuation sequences with $\mathcal{F}(\overline{v}) \geq 18h$, W_d with $d = \sqrt{\log\log h}$ is $O(\exp(\sqrt{\log\log h}))$-competitive relative to the optimal fixed price revenue.*

One can view the parameter d as essentially controlling the amount of randomization in the auction W_d. At one extreme, setting $d = 0$ gives an auction which behaves much like the auction S above. W_0 selects a bucket uniformly at random from all buckets which have any weight in them at all. Although the proof as given does not apply when $d = 0$, one can show that the theorem still holds and the competitive ratio of W_0 is $\Theta(\log h)$. At the other extreme, as d tends to ∞, W_d approaches an auction which deterministically selects the bucket of maximum weight, using randomness only to break ties. Such an auction has competitive ratio $\Theta(h)$. Note that these extreme auctions perform well on different inputs: the deterministic auction performs well when the bucket weights are imbalanced; W_0 performs well when the weights are balanced. Setting d somewhere between these extremes gives an auction which can handle all types of inputs. (As an aside, note that setting $d = 1$ gives an auction which is related to the weighted pairing auction of [6], with the same competitive ratio.)

We note that it is possible to give a lower bound of $\Omega((\log h)^{1/(d+1)})$ on the competitive ratio of W_d. This ratio is achieved using an extension of the bad example given by Goldberg et al. [5] for their weighted pairing auction. In our example, the valuation sequence contains $\alpha(\log h)^{1/(d+1)}$ valuations at h, with the remaining valuations such that each bucket $B_0, \ldots, B_{\ell-2}$ has weight h. The analysis proceeds similarly to that in [5].

4.4 Online Weighted Buckets Auction

We can transform W_d into an online auction W'_d by replacing $\overline{v}_{-i}$ with $v_1, \ldots, v_{i-1}$. That is, we bucket all of the valuations we have seen so far, except the current one, and again randomly select a bucket according to its weight raised to the dth power.

THEOREM 4.4. *Restricting to valuation sequences with $\mathcal{F}(\overline{v}) \geq 9h$, auction W'_d with any $d \geq 1$ is $O(3^d(\log h)^{1/(d+1)})$-competitive relative to the optimal fixed price revenue.*

Proof. Fix a valuation sequence $\overline{v}$ satisfying the condition of the theorem. As before, we will ignore the small buckets. Let K_1 be the set of buckets with $|B_k| \geq 2$, and let K_2 be the remaining buckets.

Consider some bucket $k \in K_1$, and let B'_k be the first $\lceil |B_k|/2 \rceil$ bids in B_k to arrive, with B''_k the remaining bids. We will show that the expected revenue from B''_k is high relative to the total weight of the bucket.

Consider some $i \in B''_k$. When bidder i arrives, W'_d has already seen the bids in B'_k. Note that since all the valuations in the kth bucket are at most a factor of two away from each other, the sum of any $\lceil |B_k|/2 \rceil$ of them cannot be less than 1/3 of their total sum. Thus we have:

$$\begin{aligned}
\Pr[s_i = 2^k] &= \frac{(w_k(v_1, \ldots, v_{i-1}))^d}{\sum_{r=0}^{\ell-1} (w_r(v_1, \ldots, v_{i-1}))^d} \\
&\geq \frac{\left(\sum_{i \in B'_k} v_i\right)^d}{\sum_{r=0}^{\ell-1} (w_r(\overline{v}))^d} \geq \frac{(w_k(\overline{v})/3)^d}{\sum_{r=0}^{\ell-1} (w_r(\overline{v}))^d}
\end{aligned}$$

Summing over the buckets, we get a total expected revenue of

$$\begin{aligned}
E(R_{W'_d}) &\geq \sum_{k \in K_1} \sum_{i \in B''_k} 2^k \cdot \frac{(w_k(\overline{v})/3)^d}{\sum_{r=0}^{\ell-1} (w_r(\overline{v}))^d} \\
&\geq \sum_{k \in K_1} \sum_{i \in B''_k} \frac{v_i}{2} \cdot \frac{(w_k(\overline{v})/3)^d}{\sum_{r=0}^{\ell-1} (w_r(\overline{v}))^d} \\
&\geq \frac{\sum_{k \in K_1} (w_k(\overline{v})/20)(w_k(\overline{v})/3)^d}{\sum_{k \in K_1} (w_k(\overline{v}))^d}
\end{aligned}$$

where the last inequality follows from the fact that the $\lfloor |B_k|/2 \rfloor$ valuations in B''_k sum to at least 1/5 of the total weight $w_k(\overline{v})$.

We can now apply Lemmas 4.1 and 4.2, using the argument above to derive the desired bound.

COROLLARY 4.4. *Restricting to valuation sequences with $\mathcal{F}(\overline{v}) \geq 9h$, W'_d with $d = \sqrt{\log \log h}$ is $O(\exp(\sqrt{\log \log h}))$-competitive relative to the optimal fixed price revenue.*

5 Future Work

A gap remains between our upper and lower bounds on the competitive ratio of a randomized incentive-compatible online auction. Our analysis is relatively pessimistic with respect to which valuations have or have not been seen at step i; a finer analysis might be able to show that W'_d has a constant competitive ratio for an appropriate choice of d.

There are also a number of extensions to the model which could be considered. In our current results, we assume that bidders bid only once, and that they bid independently of each other. In real life, collusion, at least on a small scale, is a definite possibility, and we would like to ensure that our auction mechanisms are at least somewhat collusion-resistant. Furthermore, although we might find ways to prevent individual bidders from flooding the auction with low bids, it seems unlikely that we can prevent the submission of any false bids at all. In addition, there could be legitimate reasons for a buyer to submit multiple bids, for example, if his valuation suddenly increases. Handling this situation, while maintaining incentive-compatibility, would be a challenge.

References

[1] Amitabha Bagchi, Amitabh Chaudhary, Rahul Garg, Michael T. Goodrich, and Vijay Kumar. Seller-focused algorithms for online auctioning. In *Proceedings of the 7th International Workshop on Algorithms and Data Structures (WADS 2001)*, volume 2125. Springer Verlag LNCS, 2001.

[2] Allan Borodin and Ran El-Yaniv. *Online Computation and Competitive Analysis*. Cambridge University Press, 1998.

[3] Joan Feigenbaum, Christos Papadimitriou, and Scott Shenker. Sharing the cost of multicast transmissions. In *Proceedings of the 32nd ACM Symposium on Theory of Computing (STOC 00)*, pages 218–226, 2000.

[4] Andrew Goldberg and Jason Hartline. Competitive auctions for multiple digital goods. Technical Report STAR-TR-00-01, STAR Laboratory, Intertrust, 2000.

[5] Andrew Goldberg, Jason Hartline, and Andrew Wright. Competitive auctions and digital goods. Technical Report STAR-TR-99-01, STAR Laboratory, Intertrust, 1999.

[6] Andrew Goldberg, Jason Hartline, and Andrew Wright. Competitive auctions and digital goods. In *Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 735–744, 2001.

[7] Paul Klemperer. Auction theory: A guide to the literature. *Journal of Economic Surveys*, 13(3):227–286, 1999.

[8] Ron Lavi and Noam Nisan. Competitive analysis of incentive compatible on-line auctions. In *Proceedings of the 2nd ACM Conference on Electronic Commerce (EC-00)*, pages 233–241, 2000.

[9] Daniel Lehmann, Liadan O'Callaghan, and Yoav Shoham. Truth revelation in rapid, approximately efficient combinatorial auctions. In *Proceedings of the 1st ACM Conference on Electronic Commerce (EC 99)*, pages 96–102, 1999.

[10] Noam Nisan and Amir Ronen. Algorithmic mechanism design. In *Proceedings of the 31st ACM Symposium on Theory of Computing (STOC 99)*, pages 129–140, 1999.

[11] Steven Roman. *Advanced Linear Algebra*. Springer-Verlag, 1992.

[12] Amir Ronen. On approximating optimal auctions. In *Proceedings of the 3rd ACM Conference on Electronic Commerce (EC 01)*, 2001.

[13] Tuomas Sandholm and Fredrik Ygge. On the gains and losses of speculation in equilibrium markets. In *Proceedings of the 15th International Joint Conference on Artificial Intelligence (IJCAI 97)*, 1997.

[14] William Vickrey. Counterspeculations, auctions, and competitive sealed tenders. *Journal of Finance*, 16:8–37, 1961.