

Home Assignment 4

Submit by May 16.

I. Brownian motion

1. Let W_t be a Brownian motion (BM). Show that $W_t^2 - t$ is a martingale on $\sigma\{W_s, s \leq t\}$.
2. For the BM, show that if we choose a sequence of partitions so that $\delta_n = 2^{-n}$, then $S_n \rightarrow t$ a.s. (Use Borel-Cantelli and Chebychev).
3. Let W_t be a BM. Then so are $c^{1/2}W(t/c)$ and $tW(1/t)$.
4. $W_i(t)$ are independent BMs. Find a necessary and sufficient condition on α_i so that $\sum_{i=1}^N \alpha_i W_i(t)$ is a BM.
5. Let X be a Brownian motion with drift $-b$, i.e., $X_t = B_t - bt$. Let τ be the hitting time of level $a > 0$:

$$\tau = \inf\{t : X_t = a\}.$$

Prove that $P(\tau < \infty) = e^{-2ab}$, as follows. (i) Use the martingale (prove!) $\exp(\theta B_t - \theta^2 t/2)$ with $\theta = b + (b^2 + 2\lambda)^{1/2}$ to compute the moment generating function:

$$E \exp(-\lambda \tau) = \exp(-a\{b + (b^2 + 2\lambda)^{1/2}\}).$$

(ii) Send $\lambda \rightarrow 0$.

II. Stochastic integrals

1. For the stochastic integral, prove properties 1–6 (from the lectures) for processes in \mathcal{E} .
2. For processes in \mathcal{L}^2 : If $Y_n \rightarrow Y$ in L^2 , show that $EY_n \rightarrow EY$. Now prove property 1.
3. Show that if $X_n \rightarrow X$ in L^2 and $Y_n \rightarrow Y$ in L^2 then $X_n + Y_n \rightarrow X + Y$ in L^2 . Now prove property 2 in \mathcal{L}^2 .
4. Prove property 3 in \mathcal{L}^2 .