Fundamentals of stochastic processes 048868

Home assignment 1: Probability and random variables

1. Read the material in the lecture notes on spaces and Hilbert space. Do excercise 10.8.

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For sets $A, B, A_i \in \mathcal{F}$ prove

   (a) Monotonicity: if $A \subset B$ then $\mathbb{P}\{A\} \leq \mathbb{P}\{B\}$,

   (b) Subadditivity: if $A \subset \cup_i A_i$ then $\mathbb{P}\{A\} \leq \sum_i \mathbb{P}\{A_i\}$.

   (c) Notation: if $A_i \subset A_{i+1}$ and $\cup_i A_i = A$ then we write $A_i \uparrow A$. If $A_{i+1} \subset A_i$ and $\cap_i A_i = A$ then we write $A_i \downarrow A$. Prove

      i. Continuity from below: if $A_i \uparrow A$ then $\mathbb{P}\{A_i\} \to \mathbb{P}\{A\}$.

      ii. Continuity from above: if $A_i \downarrow A$ then $\mathbb{P}\{A_i\} \to \mathbb{P}\{A\}$.

3. Probability spaces and random variables.

   (a) Let $\Omega = \{1, 2, 3\}$. Find a σ-field $\mathcal{F}$ such that $(\Omega, \mathcal{F})$ is a measuragble space, and a mapping X from Ω to $\mathbb{R}$ which is not a random variable.

   (b) Let $(\Omega, \mathcal{F})$ be a measuragble space and let X_n be a sequence of random variables. Assume that for each $\omega \in \Omega$ the limit $\lim_{n\to\infty} X_n(\omega)$ exists, and denote it by $X(\omega)$. Prove that X is a random variable.

   (c) Show that if f is Borel measurable (from $\mathbb{R}$ to $\mathbb{R}$) and X is a random variable then so is $Y = f(X)$.

   (d) Call a function g lower semi-continuous (l.s.c.) if $\liminf_{y\to x} g(y) \geq g(x)$ for all x (that is, there can only be jumps down). Call a function g upper semi-continuous (u.s.c.) if $(-g)$ is l.s.c.

      i. Show that g is l.s.c. iff $\{x : g(x) \leq a\}$ is closed for all a.

      ii. Conclude that l.s.c. functions are Borel measurable.

      iii. Conclude that continuous functions are Borel measurable.

   (e) Let h be an arbitrary real valued function on $\mathbb{R}$. Show that the set $\Delta \doteq \{x : h \text{ is discontinuous at } x\}$ is Borel measureable, as follows:

      i. Let $h^\delta \doteq \sup\{h(y) : |y - x| < \delta\}$ and $h_\delta \doteq \inf\{h(y) : |y - x| < \delta\}$. Show that h^δ is l.s.c. and h_δ is u.s.c.

      ii. Let $h^0 = \lim_{\delta\downarrow 0} h^\delta$ and $h_0 = \lim_{\delta\downarrow 0} h_\delta$. Show that $\Delta = \{x : h^0(x) \neq h_0(x)\}$.

      iii. Conclude that Δ is Borel measurable.